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ON THE EXISTENCE OF LINEAR AND BILINEAR MULTIPLIERS ON LORENTZ SPACES

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Abstract. First we show that any translation invariant bounded linear operator from $L^{p,t}(\mathbf{R})$ the Lorentz space on $\mathbf{R}$ to $L^{p,s}(\mathbf{R})$ $(1 < p < \infty,\ 1 \leqslant s < t < \infty)$ is trivial, whose result improves Blozinski's result [4]. Next let ϕ be a bounded continuous function on $\mathbf{R}^2$, and

$$T_\phi(f,g)(x) = \int\int \phi(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix(\xi+\eta)}d\xi d\eta$$

the bilinear operator on Lorentz spaces. Then, we prove that the bounded bilinear operator T_ϕ is trivial in some cases of Lorentz spaces.

1. Introduction

Let (X,ν) be a measure space. Given a complex valued measurable function f we shall denote the distribution function of f by $\nu_f(t) = \nu(E_t)$ for $t > 0$, where $E_t = \{x :| f(x) |> t\}$. The nonincreasing rearrangement of f is denoted by $f^*(t) = \inf\{y > 0 : \nu_f(y) \leqslant t\}$.

The Lorentz space $L^{p,q}(X)$ consists of those measurable function f such that $\| f \|_{pq} < \infty$, where

$$L^{p,q}(X) = \{f : \ f \text{ is measurable}, \ \| f \|^*_{p,q(X)} < \infty\},$$

and

$$\| f \|^*_{p,q(X)} = \begin{cases} \{\frac{q}{p}\int_0^\infty (t^{1/p}f^*(t))^q \frac{dt}{t}\}^{1/q} & 0 < p < \infty,\ 0 < q < \infty \\ \sup_{t>0} t^{1/p}f^*(t) & 0 < p \leqslant \infty, q = \infty. \end{cases}$$

It is well known that if $0 < p,q < \infty$ and f a measurable function, then

$$\| f \|^*_{p,q(X)} = \left(q\int_0^\infty (t\nu_f(t)^{1/p})^q \frac{dt}{t}\right)^{1/q}.$$

Let recall some facts about these spaces for $(X,\nu) = (\mathbf{R}^n, \mu = \frac{dx}{(2\pi)^n})$ or $(X,\nu) = (\mathbf{T}^n, m = \frac{dx}{(2\pi)^n})$. Simple functions are dense in $L^{p,q}(X)$ for $q \neq \infty$, and we have

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$(L^{p,1})^* = L^{p',\infty}$ for $1 \leqslant q < \infty$, and $(L^{p,q})^* = L^{p',q'}$ for $1 < p,q < \infty$. The reader is referred to [9] for basic imformation on Lorentz spaces. When $p = q$, we denote $||f||_{p(X)} = ||f||^*_{p,q(X)}$, since $L^{p,q}(X) = L^p(X)$.

Now let T be any bounded linear translation invariant operator from $L^{p,t}(\mathbf{R})$ to $L^{p,s}(\mathbf{R})$ for $1 < p < \infty$ and $1 \leqslant s < t < \infty$. Then, Blozinski [4] proved that $T = 0$ if $Tf \geqslant 0$ for $f \geqslant 0$. We will remove the condition $Tf \geqslant 0$ for $f \geqslant 0$ by the idea of Kaneko-Sato [10]. Our result is as follows:

THEOREM 1.1. *Let $1 < p < \infty$, $1 \leqslant s < t < \infty$, and T be a bounded linear translation invariant operator from $L^{p,t}(\mathbf{R})$ to $L^{p,s}(\mathbf{R})$. Then we have $T = 0$.*

Also let $\phi(\xi,\eta)$ be a bounded continuous function on $\mathbf{R}^2$, and

$$T_\phi(f,g)(x) = \int\int \phi(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix(\xi+\eta)}d\xi d\eta \ (f,g \in C_c^\infty(\mathbf{R})), \tag{1}$$

where $C_c^\infty(\mathbf{R})$ consists of all infinitely differentiable functions with compact support. Lacey and Thiele [11] studied those operators when they solved A. P. Calderon's conjecture. They showed that $\| T_\phi(f,g) \|_q \leqslant C \| f \|_p \| g \|_r$ for $\phi(\xi,\eta) = \mathrm{sign}(\xi + \alpha\eta)$ $(\alpha \in \mathbf{R}\backslash\{0,1\})$ $(q > 2/3,\ 1/p + 1/r = 1/q,\ 1 < p,r < \infty)$, where C is a constant. After that, there are many papers with respect to the multilinear operators ([5], [6], etc.). Let $0 < p_j, q_j < \infty$ $(j = 1,2,3)$, and $T_\phi(f,g)$ a bilinear operator from $L^{p_1,q_1}(\mathbf{R}) \times L^{p_2,q_2}(\mathbf{R})$ to $L^{p_3,q_3}(\mathbf{R})$ such that $\| T_\phi(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| f \|^*_{p_1,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})}$, where C is a constant. Then, Grafakos and Torres [6] proved that $T_\phi = 0$ when we have $1/p_3 > 1/p_1 + 1/p_2$, $1 < p_1, p_2 < \infty$ with $p_j = q_j$ $(j = 1,2,3)$. This result is an analogy of Hörmander's result [8], in which it is proved that any translation invariant bounded linear operator from $L^p(\mathbf{R})$ to $L^q(\mathbf{R})$ is trivial for $1 \leqslant q < p < \infty$. In this paper, we will show a generalization of [6; Proposition 5] which we stated before (cf. [14]). Our results are as follows:

THEOREM 1.2. *Let $0 < p_j, q_j < \infty$ $(j = 1,2,3)$ such that $1/p_1 + 1/p_2 < 1/p_3$, and ϕ be a bounded continuous function on $\mathbf{R}^2$. If we have*

$$\| T_\phi(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| f \|^*_{p_1,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})} \ (f,g \in C_c^\infty(\mathbf{R})), \tag{2}$$

where C is a constant, then $T_\phi = 0$.

THEOREM 1.3. *Let $1 < p_j, q_j < \infty$ $(j = 1,2)$ such that $1/p_1 + 1/p_2 = 1/p_3$, $1/q_1 + 1/q_2 < 1/q_3$, and ϕ be a bounded continuous function on $\mathbf{R}^2$. If we have*

$$\| T_\phi(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| f \|^*_{p_1,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})} \ (f,g \in C_c^\infty(\mathbf{R})), \tag{3}$$

where C is a constant, then $T_\phi = 0$.

In §2, we will give the proof of Theorem 1.1 by Kaneko-Sato [10] to Fourier multiplier operators for Lorentz spaces on the real line. This result is a generalization of Blozinski [4]. In §3, we will give the proof of Theorem 1.2 and Theorem 1.3 by applying the idea of Kaneko-Sato [10] (cf. [15]). Also we will show Theorem 1.3 by using Blasco-Villarroya [3] (cf. [5], [10]) and Hare-Sato [7]. Throughtout this paper, we may use varying a constant C.

2. On Blozinski's result

Let $1 \leqslant p_j, q_j \leqslant \infty$ $(j = 1,2)$, and T a bounded linear operator from $L^{p_1,q_1}(\mathbf{R})$ to $L^{p_2,q_2}(\mathbf{R})$, and $M(p_1,q_1;p_2,q_2)(\mathbf{R})$ the set of all translation invariant bounded linear operator T. Then, it is called Fourier multiplier operator, and it is known that there exist $\varphi \in L^\infty(\mathbf{R})$ such that $\hat{Tf}(\xi) = \hat{\varphi}(\xi)\hat{f}(\xi)(f \in C_c^\infty(\mathbf{R}))$ ([12]). Kaneko-Sato [10] remarks that $T = 0$, if $p_1 > p_2$. In 1972, Blozinski [4] showed that $T = 0$, if a Fourier multiplier operator T from $L^{p_1,q_1}(\mathbf{R})$ to $L^{p_2,q_2}(\mathbf{R})(p_1 = p_2,\ q_2 < q_1)$ has $Tf(x) \geqslant 0$ for $f(x) \geqslant 0$. In this section, we show that we can remove the condition $Tf \geqslant 0$ $(f \geqslant 0)$ in Blozinski's result by using the idea of Kaneko-Sato [10].

DEFINITION 2.1. For $\phi \in L^\infty(\mathbf{R})$, we define

$$Tf(x) = \int \phi(\xi)\hat{f}(\xi)e^{ix\xi}\frac{d\xi}{2\pi}\ (f \in C_c^\infty(\mathbf{R})). \tag{4}$$

Also for a bounded continuous function ϕ, $\widetilde{T}_\varepsilon F$ is defined by

$$\widetilde{T}_\varepsilon F(x) = \sum_{n,m} \phi(\varepsilon k)\hat{F}(k)e^{ikx}\ (\varepsilon > 0,\ F \in C^\infty(\mathbf{T})), \tag{5}$$

where $\hat{F}(k) = \int_0^{2\pi} F(x)e^{-ikx}\frac{dx}{2\pi}$ and

$$\| T \|_{M(p,t;p,s)(\mathbf{R})} = \sup_{\|f\|^*_{p,t(\mathbf{R})} \leqslant 1} \| Tf \|^*_{p,s(\mathbf{R})}\ . \tag{6}$$

Similarly, we define

$$\| \widetilde{T}_\varepsilon \|_{M(p,t;p,s)(\mathbf{T})} = \sup_{\|F\|^*_{p,t(\mathbf{T})} \leqslant 1} \| TF \|^*_{p,s(\mathbf{T})}\ . \tag{7}$$

Then, by Kaneko-Sato [10] (cf. [1], [13], [15]), we can show the following:

PROPOSITION 2.2. *Let $1 \leqslant p < \infty$, $1 \leqslant s < t \leqslant \infty$, and ϕ be a bounded continuous function on $\mathbf{R}$. Then we have*

$$\| \tilde{T}_\varepsilon \|_{M(p,t;p,s)(\mathbf{T})} \leqslant C \| T \|_{M(p,t;p,s)(\mathbf{R})}\ (\varepsilon > 0),$$

where C is a constant.

It is easy to see the following proposition, and we omit the proof (cf. [2]).

PROPOSITION 2.3. *Let $1 < p < \infty$, $1 \leqslant s < t \leqslant \infty$, and $\mu \in M(\mathbf{R})$, where $M(\mathbf{R})$ is the bounded regular Borel measures on $\mathbf{R}$. Then we have*

$$\| T_{\mu*\phi} \|_{M(p,t;p,s)(\mathbf{R})} \leqslant C \| T_\phi \|_{M(p,t;p,s)(\mathbf{R})} \| \mu \|$$

for $T_\phi \in M(p,t;p,s)(\mathbf{R})$, where $\| \mu \|$ is the total variation of μ, and C is a constant.

The proof of Theorem 1.1. First suppose that ϕ is bounded continuous. Then we assume that there exists $\xi_0 \in \mathbf{R}$ such that $\phi(\xi_0) \neq 0$, and define

$$T_0 f(x) = \int \phi(\xi + \xi_0)\hat{f}(\xi)e^{i\xi x}dx\ (f \in C_c^\infty(\mathbf{R})).$$

Since $\| T_0 \|_{M(p,t;p,s)(\mathbf{R})} = \| T \|_{M(p,t;p,s)(\mathbf{R})}$, we may assume $\phi(0) \neq 0$. Moreover, we may assume that *supp* ϕ is a compact subset. Then, by Proposition 2.2 there exists a constant C such that

$$\| \tilde{T}_\varepsilon \|_{M(p,t;p,s)(\mathbf{T})} \leqslant C \| T \|_{M(p,t;p,s)(\mathbf{R})} \ (\varepsilon > 0).$$

Now let P, Q be any trigonometric polynomials on $\mathbf{T}$, and $N = \max(\text{deg of } P, \text{ deg of } Q)$. Since

$$\int \tilde{T}_\varepsilon P(\theta)\bar{Q}(\theta)\frac{d\theta}{2\pi} = \Sigma_{-N}^{N}\phi(\varepsilon n)\hat{P}(n)\hat{\bar{Q}}(n), \tag{8}$$

we have

$$\lim_{\varepsilon \to 0} \int \tilde{T}_\varepsilon P(\theta)\bar{Q}(\theta)\frac{d\theta}{2\pi} = \phi(0) \int P(\theta)\bar{Q}(\theta)\frac{d\theta}{2\pi} \tag{9}$$

by Parseval's equality. By (8), (9) and the duality, we have

$$\begin{aligned} | \phi(0) \int P(\theta)\bar{Q}(\theta)\frac{d\theta}{2\pi} | &\leqslant C \limsup_{\varepsilon \to 0} \| \tilde{T}_\varepsilon P \|^*_{p,s(\mathbf{T})} \| Q \|^*_{p',s'(\mathbf{T})} \\ &\leqslant C \| T \|_{M(p,t;p,s)(\mathbf{R})} \| P \|^*_{p,t(\mathbf{T})} \| Q \|^*_{p',s'(\mathbf{T})} . \end{aligned}$$

Then, for any trigonometric polynomial P we get

$$| \phi(0) | \| P \|^*_{p,s(\mathbf{T})} \leqslant C \| T \|_{M(p,t;p,s)(\mathbf{R})} \| P \|^*_{p,t(\mathbf{T})} .$$

Therefore, we obtain $L^{p,s}(\mathbf{T}) = L^{p,t}(\mathbf{T})$ by $\phi(0) \neq 0$. This is a contradiction. So we get $\phi(x) = 0$. Next let $\phi(x)$ be in $L^\infty(\mathbf{R})$. Also let

$$K_N(x) = \frac{1}{2\pi(N+1)} \left(\frac{\sin(\frac{(N+1)x}{2})}{\frac{x}{2}} \right)^2 \tag{10}$$

the Fejer kernel of degree N on $\mathbf{R}$. Then we have that $T_{\phi * K_N}$ is a bounded linear operator from $L^{p,t}(\mathbf{R})$ to $L^{p,s}(\mathbf{R})$ by Proposition 2.3. Here, we may assume that *supp* ϕ is a compact subset. Hence, for $f, g \in C_c^\infty(\mathbf{R})$ we have

$$\begin{aligned} | T_{\phi * K_N}(f)(x) - T_\phi(f)(x) | &\leqslant \left(\int | \phi * K_N(\xi) - \phi(\xi) |^2 \, d\xi \right)^{1/2} \| f \|_2 \\ &\longrightarrow 0 \ (N \to \infty). \end{aligned}$$

On the other hand, we remark that $\phi * K_N$ is bounded continuous. Therefore, by the former half we obtain that $T_{\phi * L_N} = 0$, and the desired result $T_\phi = 0$. □

3. The proofs of Theorems 1.2 and 1.3

The proof of Theorem 1.2. According to Kaneko-Sato [10] (cf. [5]), we introduce some notations. Let $\widetilde{T_\phi}$ define

$$\widetilde{T_\phi}(P,Q)(x) = \sum_{n,m} \phi(n,m)\hat{P}(n)\hat{Q}(m)e^{ix(m+n)} \tag{11}$$

for trigonometric polynomials P and Q on $\mathbf{T}$, where χ a non-negative smooth even function bounded by 1 from above on $\mathbf{R}$ with compact support such that $\chi(x)=1$ ($|x|\leqslant 2\pi$), $\chi(x)=0$ ($|x|\geqslant 4\pi$), and $\chi^{\varepsilon}(x)=\chi(\varepsilon x)$. Then we set

$$\gamma_\varepsilon(x)=(\chi^{\varepsilon}(x))^2\widetilde{T_\phi}(P,Q)(x)-T_\phi(\chi^{\varepsilon}P,\chi^{\varepsilon}Q)(x) \tag{12}$$

for trigonometric plynomials P and Q on $\mathbf{T}$. By the Fourier transform, we have

$$\begin{aligned}\gamma_\varepsilon(x)=&\sum_{n,m}\hat{P}(n)\hat{Q}(m)\int\int\frac{1}{\varepsilon}\hat{\chi}\left(\frac{\xi}{\varepsilon}\right)\frac{1}{\varepsilon}\hat{\chi}\left(\frac{\eta}{\varepsilon}\right)\\&\times e^{ix(\xi+n)}e^{ix(\eta+m)}(\phi(n,m)-\phi(\xi+n,\eta+m))d\xi d\eta,\end{aligned}$$

and

$$\|\gamma_\varepsilon\|_\infty\leqslant\sum_{n,m}|\hat{P}(n)||\hat{Q}(m)||\int\int\hat{\chi}(\xi)\hat{\chi}(\eta)(\phi(n,m)-\phi(\varepsilon\xi+n,\varepsilon\eta+m))d\xi d\eta|. \tag{13}$$

Here, for any $\delta'>0$ we choose the compact interval I such that

$$\int\int_{(I\times I)^c}2\|\phi\|_\infty|\hat{\chi}(\xi)||\hat{\chi}(\eta)|d\xi d\eta<\delta'.$$

Then, we get

$$\begin{aligned}\|\gamma_\varepsilon\|_\infty\leqslant&\sum_{n,m}|\hat{P}(n)||\hat{Q}(m)|\delta'\\&+\sum_{n,m}\int\int_{I\times I}|\hat{\chi}(\xi)||\hat{\chi}(\eta)||\phi(n,m)-\phi(\varepsilon\xi+n,\varepsilon\eta+m))|d\xi d\eta.\end{aligned}$$

By the continuity of ϕ on $\mathbf{R}^2$, for any $\delta>0$ we get the compact interval I and $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$ we have $\|\gamma_\varepsilon\|_\infty<\delta$. So when we put $\Delta_\varepsilon=\|\gamma_\varepsilon\|_\infty$, we have $\lim_{\varepsilon\to 0}\Delta_\varepsilon=0$.

Now let $\Lambda_F(t)=m(\{x\in\mathbf{T}:|F(x)|>t\})$ be a distribution function of a function F on $\mathbf{T}$, and $\lambda_f(t)=\mu(\{x\in\mathbf{R}:|f(x)|>t\})$ a distribution function of a function f on $\mathbf{R}$. For $t>0$, suppose $\varepsilon>0$ such that $\Delta_\varepsilon<t$. By (12), we have

$$\{x\in\mathbf{R}:|T_\phi(\chi^{\varepsilon}P,\chi^{\varepsilon}Q)(x)|>t-\Delta_\varepsilon\}\supset\{x\in\mathbf{R}:|(\chi^{\varepsilon}(x))^2\widetilde{T_\phi}(P,Q)(x)|>t\}. \tag{14}$$

Therefore,

$$\begin{aligned}&\left(\int_a^\infty(t\lambda_{(\chi^{\varepsilon})^2\widetilde{T_\phi}(P,Q)}(t)^{1/p_3})^{q_3}\frac{dt}{t}\right)^{1/q_3}\\&\leqslant\left(\int_a^\infty(t\lambda_{T_\phi(\chi^{\varepsilon}P,\chi^{\varepsilon}Q)}(t-\Delta_\varepsilon)^{1/p_3})^{q_3}\frac{dt}{t}\right)^{1/q_3}\\&\leqslant\left(\int_{a-\Delta_\varepsilon}^\infty 2(t\lambda_{T_\phi(\chi^{\varepsilon}P,\chi^{\varepsilon}Q)}(t)^{1/p_3})^{q_3}\frac{dt}{t}\right)^{1/q_3}\\&\leqslant 2\left(\int_0^\infty(t\lambda_{T_\phi(\chi^{\varepsilon}P,\chi^{\varepsilon}Q)}(t)^{1/p_3})^{q_3}\frac{dt}{t}\right)^{1/q_3}\end{aligned}$$

for any a and ε such that $\frac{a}{2} > \Delta_\varepsilon$, and

$$\left(q_3 \int_a^\infty (t\lambda_{(\chi_k^\varepsilon)^2 \widetilde{T_\phi}(P,Q)}(t)^{1/p_3})^{q_3} \frac{dt}{t}\right)^{1/q_3} \leqslant 2 \parallel T_\phi(\chi^\varepsilon P, \chi^\varepsilon Q) \parallel^*_{p_3,q_3(\mathbf{R})}$$
$$\leqslant 2C \parallel \chi^\varepsilon P \parallel^*_{p_1,q_1(\mathbf{R})} \parallel \chi^\varepsilon Q \parallel^*_{p_2,q_2(\mathbf{R})}$$

by the assumption. By the way, we have $0 \leqslant \chi^\varepsilon \leqslant 1$ and $supp\ \chi^\varepsilon \subset [-4\pi/\varepsilon, 4\pi/\varepsilon]$. The number of the indices m satisfying $[2m\pi, 2(m+1)\pi] \cap [-4\pi/\varepsilon, 4\pi/\varepsilon] \neq \phi$ is less or equal than $2[2/\varepsilon+1]$. Hence we obtain $\{x \in \mathbf{R} : |\chi^\varepsilon(x)P(x)| > t\} \subset \cup_m \{x \in [2m\pi, 2(m+1)\pi) : |P(x)| > t, [2m\pi, 2(m+1)\pi) \cap [-4\pi/\varepsilon, 4\pi/\varepsilon] \neq \phi\}$, and, we have

$$\begin{aligned} \parallel \chi^\varepsilon P \parallel^*_{p_1,q_1(\mathbf{R})} &= \left(q_1 \int_0^\infty (t\lambda_{\chi^\varepsilon P}(t)^{1/p_1})^{q_1} \frac{dt}{t}\right)^{1/q_1} \\ &\leqslant \left\{2\left(\frac{2}{\varepsilon}+1\right)\right\}^{1/p_1} \left(q_1 \int_0^{2\pi} (t\Lambda_P(t)^{1/p_1})^{q_1} \frac{dt}{t}\right)^{1/q_1} \\ &= \left\{2\left(\frac{2}{\varepsilon}+1\right)\right\}^{1/p_1} \parallel P \parallel^*_{p_1,q_1(\mathbf{T})}, \end{aligned}$$

and

$$\parallel \chi^\varepsilon Q \parallel^*_{p_2,q_2(\mathbf{R})} \leqslant \left\{2\left(\frac{2}{\varepsilon}+1\right)\right\}^{1/p_2} \parallel Q \parallel^*_{p_2,q_2(\mathbf{T})}, \tag{15}$$

similarly. On the other hand, since $\chi^\varepsilon(x) = 1$ on $[-2\pi/\varepsilon, 2\pi/\varepsilon]$, the number of the indices m satisfying $[2m\pi, 2(m+1)\pi) \subset [-2\pi/\varepsilon, 2\pi/\varepsilon]$ is at least $(2/\varepsilon - 2)$. Therefore, by the periodicity of $\widetilde{T_\phi}(P,Q)$ we have

$$\left\{\left(\frac{2}{\varepsilon}-2\right)\right\}^{1/p_3} q_3^{1/q_3} \left(\int_a^{2\pi} (t\Lambda_{\tilde{T}_\phi(P,Q)}(t)^{1/p_3})^{q_3} \frac{dt}{t}\right)^{1/q_3}$$
$$\leqslant \{2(2/\varepsilon+1)\}^{1/p_1} \{2(2/\varepsilon+1)\}^{1/p_2} 2C \parallel P \parallel^*_{p_1,q_1(\mathbf{T})} \parallel Q \parallel^*_{p_2,q_2(\mathbf{T})} .$$

After we divide the both side of the above inequality by $\{2(1/\varepsilon - 1)\}^{1/q_3}$, we get

$$\left(q_3 \int_a^{2\pi} (t\Lambda_{\tilde{T}_\phi(P,Q)}(t)^{1/p_3})^{q_3} \frac{dt}{t}\right)^{1/q_3}$$
$$\leqslant \{2(2+\varepsilon)\}^{1/p_1} \{2(2+\varepsilon)\}^{1/p_2} \{2(1-\varepsilon)\}^{-1/p_3} \varepsilon^{1/p_3 - 1/p_1 - 1/p_2}$$
$$\times 2C \parallel P \parallel^*_{p_1,q_1(\mathbf{T})} \parallel Q \parallel^*_{p_2,q_2(\mathbf{T})} .$$

Here, by the assumption $1/p_3 > 1/p_1 + 1/p_2$, we have

$$\int_a^{2\pi} (t\Lambda_{\tilde{T}_\phi(P,Q)}(t)^{1/p_3})^{q_3} \frac{dt}{t} = 0$$

for any $a > 0$, and $\widetilde{T_\phi}(P,Q) = 0$ for trigonometric polynomials P and Q on $\mathbf{T}$. Therefore, we have $\phi(n,m) = 0$ on $\mathbf{Z}^2$.

Now let ϕ_ε define $\phi_\varepsilon(\xi,\eta)=\phi(\varepsilon\xi,\varepsilon\eta)$, and

$$T_\varepsilon(f,g)(x)=\int\int \phi_\varepsilon(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix(\xi+\eta)}d\xi d\eta. \tag{16}$$

Then, by the change of variable we have

$$\| T_\varepsilon(f,g) \|^*_{p_3,q_3(\mathbf{R})}\leqslant C\varepsilon^{1/p_3-1/p_1-1/p_2} \| f \|^*_{p_1,q_1(\mathbf{R})}\| g \|^*_{p_2,q_2(\mathbf{R})} \ \ (f,g\in C_c^\infty(\mathbf{R})),$$

where C is a constant which depends only on T_ϕ. Hence, after the same process as we showed $\phi=0$ on $\mathbf{Z}^2$ before, we have

$$\phi(\varepsilon n,\varepsilon m)=\phi_\varepsilon(n,m)=0$$

on $\mathbf{Z}^2$ for any $\varepsilon>0$. Then, we get $\phi(\xi,\eta)=0$ on $\mathbf{R}^2$ by the continuity of ϕ on $\mathbf{R}^2$. $\square$

Next we show Theorem 1.3. Before the proof, we prove some Lemmas, and introduce some notations for using the idea of Hare-Sato [7], according to [7].

DEFINITION 3.1. Let λ be a large integer ($\lambda=1000$ will suffice) and for convinenience set $M_N=2\lambda^N+1$. Let D_N be the Dirichlet kernel of degree λ^N. Set $x_j=2(j-1)/\sqrt{M_N}$ for $j=1,\ldots,2^N$ and set $z_k=3^Nk/\sqrt{M_N}$ for $k=1,\ldots,N$. Define $D_{j,k}(x)=D_N(x-(x_j+z_k))$ and

$$\widetilde{D_{j,k}}(x)=\begin{cases} D_{j,k}(x) & \text{if} \in[-2/M_N,-2/M_N]+x_j+z_k \\ 0 & \text{else.}\end{cases}$$

Notice that if N is sufficiently large than the functions $\widetilde{D_{j,k}}(x)$ are disjointly supported. F_N will be defined by

$$F_N(x)=\frac{1}{M_N}\Sigma_{k=1}^N 2^{-k/p}\Sigma_{j=1}^{2^k}\widetilde{D}_{j,k}(x),$$

and for $0<p<\infty$, $0<s\leqslant\infty$

$$h(x)=\frac{M_N^3}{2}\chi_{[-1/M_N^3,1/M_N^3]},\quad G_N(x)=F_N*h(x)\in C(\mathbf{T}).$$

Then, we have the estimate of the quasi-norm

$$\| G_N \|^*_{p,r(\mathbf{T})}\sim M_N^{-1/p}N^{1/r} \tag{17}$$

like [7; Proposition 3.3] and [7; Proposition 3.6].

LEMMA 3.2. *Let* $0<p_j,q_j<\infty$ $(j=1,2,3)$ *such that*

$$\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2},\quad \frac{1}{q_3}>\frac{1}{q_1}+\frac{1}{q_2}. \tag{18}$$

Then, there is no constant $C > 0$ such that

$$\| PQ \|^*_{p_3,q_3(\mathbf{T})} \leqslant C \| P \|^*_{p_1,q_1(\mathbf{T})} \| Q \|^*_{p_2,q_2(\mathbf{T})} \tag{19}$$

for all trigonometric polynomials P and Q on $\mathbf{T}$.

Proof. Assume (19) holds for some constant $C > 0$. Then, we have

$$\| FG \|^*_{p_3,q_3(\mathbf{T})} \leqslant C \| F \|^*_{p_1,q_1(\mathbf{T})} \| G \|^*_{p_2,q_2(\mathbf{T})} \quad (F, G \in C(\mathbf{T})), \tag{20}$$

where $C(\mathbf{T})$ denotes the set of all continuous functions on $\mathbf{T}$. In fact, for $F, G \in C(\mathbf{T})$ and $\varepsilon > 0$, there exist trigonometric polynomials P and Q on $\mathbf{T}$ such that $\| F - P \|_\infty < \varepsilon$ and $\| G - Q \|_\infty < \varepsilon$. Since we have

$$\begin{aligned} \| FG \|^*_{p_3,q_3(\mathbf{T})} &\leqslant C(\| (F-P)G \|^*_{p_3,q_3(\mathbf{T})} + \| P(G-Q) \|^*_{p_3,q_3(\mathbf{T})} + \| PQ \|^*_{p_3,q_3(\mathbf{T})}) \\ &\leqslant C\varepsilon(||G||_\infty + \varepsilon + ||F||_\infty + \varepsilon) + C(||F||^*_{p_1,q_1(\mathbf{T})} + \varepsilon)(||G||^*_{p_2,q_2(\mathbf{T})} + \varepsilon), \end{aligned}$$

we get the above result. On the other hand, by the definition of G_N and (17) we obtain

$$\| G_N^2 \|^*_{p_3,q_3(\mathbf{T})} \sim M_N^{-1/p_3} N^{1/q_3}, \tag{21}$$

since we have $\| G_N^2 \|^*_{q,q_3(\mathbf{T})} = \| G_N \|^{*2}_{2q,2q_3(\mathbf{T})}$. By $G_N \in C(\mathbf{T})$ and (20), we have

$$\| G_N \cdot G_N \|^*_{p_3,q_3(\mathbf{T})} \leqslant C \| G_N \|^*_{p_1,q_1(\mathbf{T})} \| G_N \|^*_{p_2,q_2(\mathbf{T})}$$

for some constant C, and

$$M_N^{-1/p_3} N^{1/q_3} \leqslant CC' M_N^{-1/p_1} N^{1/q_1} M_N^{-1/p_2} N^{1/q_2}$$

for some $C' > 0$. This contradicts to $1/p_3 = 1/p_1 + 1/p_2$ and $1/q_3 > 1/q_1 + 1/q_2$. Therefore, we get the desired results. □

Now let $\widetilde{T}_\varepsilon(P,Q)$ define

$$\widetilde{T}_\varepsilon(P,Q) = \sum_{n,m} \phi(n,m) \hat{P}(n) \hat{Q}(m) e^{i(n+m)x} \tag{22}$$

for trigonometric polynomials P and Q on $\mathbf{T}$.

Blasco-Villarroya [3] proved the following result (cf. [10]).

PROPOSITION 3.3. *Let $0 < p_j, q_j < \infty$ $(j = 1,2,3)$, and $1/p_3 = 1/p_1 + 1/p_2$. Also let ϕ be a bounded continuous function on $\mathbf{R}^2$. If we have*

$$\| T_\phi(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| f \|^*_{p_1,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})} \quad (f, g \in C_c^\infty(\mathbf{R})) \tag{23}$$

for some constant $C > 0$, then we get

$$\| \widetilde{T}_\varepsilon(P,Q) \|^*_{p_3,q_3(\mathbf{T})} \leqslant C \| P \|^*_{p,q_1(\mathbf{T})} \| Q \|^*_{p_2,q_2(\mathbf{T})} \tag{24}$$

for trigonometric polynomials P *and* Q *on* $\mathbf{T}$.

The proof of Theorem 1.3. We assume $\phi \neq 0$ for getting the contradiction. First let ϕ be a bounded continuous function on $\mathbf{R}^2$. We define

$$\tilde{T}_\varepsilon(f,g)(\theta) = \Sigma_{n,m}\phi(\varepsilon n, \varepsilon m)\hat{f}(n)\hat{g}(m)e^{i\theta(n+m)} \ (\varepsilon > 0).$$

Then by the assumption of T_ϕ and Proposition 3.3, we have

$$\| \tilde{T}_\varepsilon(P,Q) \|^*_{p_3,q_3(\mathbf{T})} \leqslant C \| P \|^*_{p_1,q_1(\mathbf{T})} \| Q \|^*_{p_2,q_2(\mathbf{T})}$$

for trigonometric polynomials P and Q on $\mathbf{T}$, where C is a positive constant which is independent from $P,Q,$ and ε. We may assume that $\phi(0,0) \neq 0$ and *supp* $\phi \subset (-\delta,\delta)\times(-\delta,\delta)$ for some $\delta > 0$, since we have that

$$\| T_{\phi_0}(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| f \|^*_{p_1,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})} \ \ (f,g \in C_c^\infty(\mathbf{R})) \tag{25}$$

for $\phi_0(\xi,\eta) = \phi(\xi-\xi_0, \eta-\eta_0)$ $((\xi_0,\eta_0) \in \mathbf{R}^2)$ and

$$\| T_{\phi\varphi}(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| \hat{\varphi} \|_1 \| f \|^*_{p_1,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})} \ \ (f,g \in C_c^\infty(\mathbf{R})) \tag{26}$$

for $\varphi(\xi,\eta) = \varphi_1(\xi)\varphi_2(\eta)$ $(\varphi_j \in C_c^\infty(\mathbf{R}) \ \ j = 1,2)$. Now let P,Q be trigonometric polynomials on $\mathbf{T}$, and $N = \max(\textit{deg of } P, \textit{ deg of } Q)$. Then there exists $\varepsilon_0 > 0$ such that $\{(\varepsilon n, \varepsilon m) \mid n,m = 0 \pm 1, \cdots, \pm N\} \subset (-\delta,\delta)\times(-\delta,\delta)$ for $0 < \varepsilon < \varepsilon_0$. So we obtain that

$$\begin{aligned}\lim_{\varepsilon\to 0} \tilde{T}_\varepsilon(P,Q)(\theta) &= \phi(0,0)\Sigma_{n,m=-N}^N \hat{P}(n)\hat{Q}(m)e^{i\theta(n+m)} \\ &= \phi(0,0)P(\theta)Q(\theta),\end{aligned}$$

and by Fatou's lemma we have

$$| \phi(0,0) | \| PQ \|^*_{p_3,q_3(\mathbf{T})} \leqslant C \| P \|^*_{p_1,q_1(\mathbf{T})} \| Q \|^*_{p_2,q_2(\mathbf{T})} . \tag{27}$$

On the other hand, we remark that $\phi(0,0) \neq 0$, $C > 0$, $1/p_3 = 1/p_1 + 1/p_2$, and $1/q_3 > 1/q_1 + 1/q_2$. This contradicts to Lemma 3.2. We get the desired result. □

Next proposition is similar to Proposition 2.3. we omit the proof.

PROPOSITION 3.4. *Let* $1 < p_j, q_j < \infty$ $(j = 1,2,3)$ *and* μ *be in* $M(\mathbf{R}^2)$, *where* $M(\mathbf{R}^2)$ *denotes the set of all bounded regular Borel measures on* $\mathbf{R}^2$, *and* $\| \mu \|$ *the total variation of* μ. *If for* $\phi \in L^\infty(\mathbf{R}^2)$, *we have*

$$\| T_\phi(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| f \|^*_{p_1,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})} \ \ (f,g \in C_c^\infty(\mathbf{R})) \tag{28}$$

for some constant $C > 0$, *then we obtain*

$$\| T_{\mu*\phi}(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| \mu \| \| f \|^*_{p,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})} \ \ (f,g \in C_c^\infty(\mathbf{R})). \tag{29}$$

COROLLARY 3.5. *Let $1 < p_j, q_j < \infty$ $(j = 1,2,3)$ such that $1/p_3 > 1/p_1 + 1/p_2$ or $1/p_1 + 1/p_2 = 1/p_3$ with $1/q_3 > 1/q_1 + 1/q_2$. Also let ϕ be in $L^\infty(\mathbf{R}^2)$, and*

$$T_\phi(f,g)(x) = \int\int \phi(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{ix(\xi+\eta)}d\xi d\eta \ (f,g \in C_c^\infty(\mathbf{R})). \tag{30}$$

If we have

$$\| T_\phi(f,g) \|^*_{p_3,q_3(\mathbf{R})} \leqslant C \| f \|^*_{p_1,q_1(\mathbf{R})} \| g \|^*_{p_2,q_2(\mathbf{R})} \ \ (f,g \in C_c^\infty(\mathbf{R})), \tag{31}$$

where C is a constant, then $T_\phi = 0$.

Proof. Let

$$K_N(x) = \frac{1}{2\pi(N+1)} \left(\frac{\sin\left(\frac{(N+1)x}{2}\right)}{\frac{x}{2}} \right)^2 \tag{32}$$

be the Fejer kernel of degree N on $\mathbf{R}$, and $L_N(\xi,\eta) = K_N(\xi)K_N(\eta)$. Then we have that $T_{\phi * L_N}$ is a bounded bilinear operator from $L^{p_1,q_1}(\mathbf{R}) \times L^{p_2,q_2}(\mathbf{R})$ to $L^{p_3,q_3}(\mathbf{R})$ by Proposition 3.4. Here, we may assume that *supp* ϕ is a compact subset as we showed it in Theorem 1.2. Then, for $f,g \in C_c^\infty(\mathbf{R})$ we have

$$\begin{aligned} &| T_{\phi * L_N}(f,g)(x) - T_\phi(f,g)(x) | \\ &\quad \leqslant \left(\int\int | \phi * L_N(\xi,\eta) - \phi(\xi,\eta) |^2 \, d\xi d\eta \right)^{1/2} \| f \|_2 \| g \|_2 \\ &\quad\quad \longrightarrow 0 \ (N \to \infty). \end{aligned}$$

On the other hand, we remark that $\phi * L_N$ is bounded continuous. Therefore, by Theorems1.2 and 1.3 we obtain that $T_{\phi * L_N} = 0$, and the desired result $T_\phi = 0$. □

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REFERENCES

[1] N. H. ASMAR, E. BERKSON AND J. BOURGAIN, *Restrictions from $\mathbf{R}^n$ to $\mathbf{Z}^n$ of weak type $(1,1)$ multipliers*, Studia Math., **108** (1994), 291–299.

[2] O. BLASCO, *Notes on the spaces of bilinear multipliers*, Advanced courses of mathematical analysis III, 28–38, World Sci. Publ.,Hackensack, NJ, 2008.

[3] O. BLASCO AND F. VILLARROYA, *Transference of bilinear multilinear operators on Lorentz spaces,* Illinois J. Math., **47** (2003), 1327–1343.

[4] A. P. BLOZINSKI, *Convolution of $L(p,q)$ functions*, Proc. Amer. Math. Soc., **32** (1972), 237–240.

[5] D. FAN AND S. SATO, *Transference on certain multilinear operators*, J. Aust. Math. Soc., **70** (2001), 37–55.

[6] L. GRAFAKOS AND R. H. TORRES, *Multilinear Calderon-Zygmund Theory,* Advances in Math., **165** (2002), 124–164.

[7] K. E. HARE AND E. SATO, *Spaces of Lorentz multipliers*, Canad. J. Math., **53** (2001), 565–591.

[8] L. HÖMANDER, *Estimates for translation invariant operators in L^p spaces*, Acta Math., **104** (1960), 93–140.
[9] R. HUNT, *On L(p,q) spaces*, Enseign. Math., **12** (1966), 249–276.
[10] M. KANEKO AND E. SATO, *Notes on transference of continuity from maximal Fourier multiplier operators on $\mathbf{R}^n$ to those on $\mathbf{T}^n$*, Interdiscip. Inform. Sci., **4** (1998), 97–107.
[11] M. LACEY AND C. THIELE, *L^p estimates on the bilinear Hilbert transform,* Ann. of Math., **146** (1997), 693–724.
[12] R. LARSEN, *An introduction to the thory of multipliers*, Springer-Verlag, Berlin–Heidelberg–New York, 1971.
[13] J. RAPOSO, *Weak type $(1,1)$ multipliers on LCA groups*, Studia Math., **122** (1997), 123–130.
[14] F. VILLARROYA, *Bilinear multipliers on Lorentz spaces*, Czechoslovak Math. J., **58**, 4 (2008), 1045–1057.
[15] K. WOZNIAKOWSKI, *A new proof of the restriction theorem for weak type $(1,1)$ multipliers on $\mathbf{R}^n$*, Illinois J. Math., **40** (1996), 479–483.

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