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BERNSTEIN–DOETSCH TYPE RESULTS FOR h–CONVEX FUNCTIONS

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Abstract. In this paper we introduce a class of h-convex functions which is a common generalization of the convexity, s-convexity, the Godunova-Levin functions and the P-functions. Namely, an h-convex function is defined as a function $f : D \to \mathbb{R}$ (where D is an open, convex, nonempty subset of a linear space) which satisfies

$$f(\lambda x+(1-\lambda)y) \leqslant h(\lambda)f(x)+h(1-\lambda)f(y),$$

for all $\lambda \in [0,1]$ and $x,y \in D$, where h is a given real function.

In this paper some regularity and Bernstein-Doetsch type results for h-convex functions are presented.

1. Introduction

The concept of h-convexity was introduced by Varošanec [26] in the following way:

DEFINITION 1. Let I and J are real intervals, $(0,1) \subseteq J$ and $h : J \to \mathbb{R}$ be a nonnegative function. We say that $f : I \to \mathbb{R}$ is an h-convex function, if f is nonnegative and for all $x,y \in I$ and $\lambda \in (0,1)$ we have

$$f(\lambda x+(1-\lambda)y) \leqslant h(\lambda)f(x)+h(1-\lambda)f(y).$$

This type of h-convexity is a common generalization of the usual convexity, the Godunova-Levin functions, the Breckner s-convex functions and the so called P-functions, when the function f is nonnegative. The term on the left-hand side of the inequality are the same in all definitions while the right-hand side of all inequalities has a similar form.

In the paper of Varošanec [26] was investigated the nonnegative h-convex functions on real intervals. However, in that case the h-convexity is not a real generalization of the usual one since the extra assumptions allow only nonnegative functions.

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The Godunova-Levin functions was investigated in [7]. We say that $f : I \to \mathbb{R}$ (where I is a real interval) is a Godunova-Levin function, if f is nonnegative and for all $x, y \in I$ and $\lambda \in (0,1)$ we have

$$f(\lambda x + (1-\lambda)y) \leqslant \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.$$

Some properties of this type of functions are given in [6], [15], [16]. Among others, it is proved that nonnegative monotone and nonnegative convex functions belong to this class of functions. The Godunova-Levin functions are h-convex, with $h(\lambda) = \frac{1}{\lambda}$.

The concept of s-convexity was introduced by Breckner [4]. A real valued function $f : D \to \mathbb{R}$ (where D is a convex, open, nonempty subset of a real (complex) linear space X) is called *Breckner s-convex* (or briefly *s-convex*), if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda^s f(x) + (1-\lambda)^s f(y)$$

for every $x, y \in D$ and $\lambda \in [0,1]$, where $s \in]0,1]$ is a fixed number. The case $s = 1$ means the usual convexity of f. In [4] and [5] Berstein-Doetsch type results were proved on rationally s-convex functions, moreover, for the s-Hölder property of s-convex functions. Pycia [23] gives a new proof of the latter statement, when f is defined on a nonempty, convex subset of a finite dimensional vector space. In [13] the authors collect some properties of s-convex functions defined on the nonnegative reals. In [2] there are some Berstein-Doetsch type result on (H,s)-convex functions. The s-convex functions are h-convex with $h(\lambda) = \lambda^s$.

The P-functions was investigated in [6]. A real valued function $f : D \to \mathbb{R}$ (where D is a convex, open, nonempty subset of a real (complex) linear space X) is called P-function, if for every $x, y \in D$ and $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \leqslant f(x) + f(y).$$

Some results about the P-functions there are in [22], [25]. The P-functions are h-convex, with $h(\lambda) = 1$.

Bernstein and Doetsch in [1] proved that if a function $f : D \to \mathbb{R}$ (where D is a convex, open, nonempty subset of a real (complex) linear space X) is locally bounded from above at a point of D, then the Jensen-convexity of the function yields its local boundedness and continuity as well, which implies the convexity of the function f (see [14] for further references). This result has been generalized by several authors. The first such type results are due to Nikodem and Ng [18] for the approximately Jensen-convex functions (the so-called ε-Jensen-convexity), which was extended by Páles ([19], [20]) to approximately t-convex functions. Further generalizations can be found in papers of Mrowiec [17], Házy ([9], [10]), Házy and Páles ([11], [12]). In the paper of Gilányi, Nikodem and Páles [8] there are some Bernstein-Doetsch type results for quasiconvex functions.

In this paper we introduce a more general concept of the h-convexity, and the concept of the so called (H,h)-convexity.

The main goal of the paper is to prove some regularity and Bernstein-Doetsch type result for h-convex and (H,h)-convex functions. Besides we also collect some facts on such functions. Finally we collect some interesting, easily-proved properties of h-convex functions.

2. Definition and basic results

In the sequel, let D be a nonempty, convex, open subset of a real (complex) linear space X.

DEFINITION 2. Let $h:[0,1]\to\mathbb{R}$ be a given function. We say that $f:D\to\mathbb{R}$ is an h-convex function if, for all $x,y\in D$ and $\lambda\in[0,1]$, we have

$$f(\lambda x+(1-\lambda)y)\leqslant h(\lambda)f(x)+h(1-\lambda)f(y). \tag{1}$$

In this case $h(\lambda)=\lambda$ means the usual convexity of f, without any further assumptions.

Let $H\subseteq[0,1]$ be a nonempty set. A real valued function $f:D\to\mathbb{R}$ is called *(H,h)-convex* if it fulfills (1) for all $\lambda\in H$.

In the special cases when $H=\left\{\frac{1}{2}\right\}$, $H=\{\lambda\}$ or $H=\mathbb{Q}\cap[0,1]$, the corresponding (H,h)-convex functions are said to be *(Jensen, h)-convex, (λ,h)-convex* and *(rationally, h)-convex.*

The following property shows that the nonnegativity and nonpositivity of (λ,h)-functions depends only the sign of $h(\lambda)+h(1-\lambda)-1$, therefore we do not assume the nonnegativity of f in the definition.

PROPOSITION 1. *Let $\lambda\in[0,1]$, $h:[0,1]\to\mathbb{R}$ be a given function and $f:D\to\mathbb{R}$ be a (λ,h)-convex function. Then*

(i) *if $h(\lambda)+h(1-\lambda)>1$ then f is nonnegative.*

(ii) *if $h(\lambda)+h(1-\lambda)<1$ then f is nonpositive.*

Proof. Let x be an arbitrary element of D. Using (λ,h)-convexity of f

$$f(x)=f(\lambda x+(1-\lambda)x)\leqslant h(\lambda)f(x)+h(1-\lambda)f(x)=(h(\lambda)+h(1-\lambda))f(x),$$

which implies

$$0\leqslant(h(\lambda)+h(1-\lambda)-1)f(x).$$

If $h(\lambda)+h(1-\lambda)-1>0$, then we have $f(x)\geqslant 0$ and if $h(\lambda)+h(1-\lambda)-1<0$, then we have $f(x)\leqslant 0$. $\square$

Let us note that if $h(\lambda)+h(1-\lambda)=1$, then, similarly the usual convexity and Jensen-convexity, there is no such type result. An easy consequence of the previous proposition is the following:

COROLLARY 1. *Let $h:[0,1]\to\mathbb{R}$ be a given function and let $f:D\to\mathbb{R}$ be an h-convex function. Then* (*i*) *if* $h(\lambda)+h(1-\lambda)\geqslant 1$ *for all* $\lambda\in[0,1]$ *and there exists* $\lambda_1\in[0,1]$ *such that* $h(\lambda_1)+h(1-\lambda_1)>1$*, then* f *is nonnegative.*

(*ii*) *if* $h(\lambda)+h(1-\lambda)\leqslant 1$ *for all* $\lambda\in[0,1]$ *and there exists* $\lambda_1\in[0,1]$ *such that* $h(\lambda_1)+h(1-\lambda_1)<1$*, then* f *is nonpositive.*

(*iii*) *if there exists* $\lambda_1,\lambda_2\in[0,1]$ *such that* $h(\lambda_1)+h(1-\lambda_1)>1$ *and* $h(\lambda_2)+h(1-\lambda_2)<1$ *then* f *is identically zero.*

The proof of this Corollary is similar to the proof of Proposition 1.

REMARK 1. Analogue statement remains true for (H,h)-convex functions, whenever H fulfils any of the properties (i), (ii), (iii) respectively.

3. Regularity properties of (λ,h)-convex functions

In this section we assume that $(X,\|\cdot\|)$ is a real (complex) normed space. We recall that a function $f:D\to\mathbb{R}$ is called locally bounded from above on D if, for each point of $p\in D$, there exist $\rho>0$ and a neighborhood $U(p,\rho):=\{x\in X\,:\,\|x-p\|<\rho\}$ such that f is bounded from above on $U(p,\rho)$.

We assume that $h:[0,1]\to\mathbb{R}$ is nonnegative, furthermore $h(\lambda)$ and $h(1-\lambda)$ are not zero simultaneously. Since, if $h(\lambda)=h(1-\lambda)=0$, then the (λ,h)-convexity does not imply the boundedness of f, only the upper boundedness of f. Indeed, in this case we get $f(x)\leqslant 0$ from the inequality (1).

THEOREM 1. *Let* $D\subset X$ *be convex, open, nonempty, let* $\lambda\in]0,1[$ *be fixed, let* $h:[0,1]\to\mathbb{R}$ *be a given nonnegative function such that* $h(\lambda)$ *and* $h(1-\lambda)$ *are not zero simultaneously, and let* $f:D\to\mathbb{R}$ *be* (λ,h)*-convex. Then*

(*i*) *if* $h(\lambda)+h(1-\lambda)<1$ *or*

(*ii*) *if* $h(\lambda)+h(1-\lambda)\geqslant 1$ *and* f *is locally bounded from above at a point* $p\in D$

then f *is locally bounded at every point of* D.

Proof. Since $h(\lambda)$ and $h(1-\lambda)$ are not zero simultaneously, therefore, without loss generality, we may assume that $h(\lambda)>0$. In the case (i) the local upper boundedness yields from Proposition 1. So we prove that f is locally bounded from above on D in the case (ii).

First we prove that f is locally bounded from above on D. Define the sequence of sets D_n by

$$D_0:=\{p\},\qquad\qquad D_{n+1}:=\lambda D_n+(1-\lambda)D.$$

Using induction on n, we prove that f is locally upper bounded at each point of D_n. By assumption, f is locally bounded from above at $p\in D_0$. Assume that f is locally upper bounded at each point of D_n. For $x\in D_{n+1}$, there exist $x_0\in D_n$ and $y_0\in D$ such that $x=\lambda x_0+(1-\lambda)y_0$. By the inductive assumption, there exist $r>0$ and a constant

$M_0 \geqslant 0$ such that $f(x') \leqslant M_0$ for $\|x_0 - x'\| < r$. Then, by the (λ, h)-convexity of f, for $x' \in U_0 := U(x_0, r)$ we have

$$f(\lambda x' + (1-\lambda)y_0) \leqslant h(\lambda)f(x') + h(1-\lambda)f(y_0) \leqslant h(\lambda)M_0 + h(1-\lambda)f(y_0) =: M.$$

Therefore, for

$$y \in U := \lambda U_0 + (1-\lambda)y_0 = U(\lambda x_0 + (1-\lambda)y_0, \lambda r) = U(x, \lambda r),$$

we get that $f(y) \leqslant M$. Thus f is locally bounded from above on D_{n+1}.

On the other hand, we show that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

From the definition of D_n, it follows by induction that $D_n = \lambda^n p + (1-\lambda^n)D$. For fixed $x \in D$, define the sequence x_n by

$$x_n := \frac{x - \lambda^n p}{1 - \lambda^n}.$$

Then $x_n \to x$ if $n \to \infty$. As D is open, $x_n \in D$ for some n. Therefore

$$x = \lambda^n p + (1-\lambda^n)x_n \in \lambda^n p + (1-\lambda^n)D = D_n.$$

Thus f is locally bounded from above on D.

Now, we prove that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q, there exist $\rho > 0$ and $M > 0$ such that

$$\sup_{U(q,\rho)} f \leqslant M.$$

Let $x \in U(q, (1-\lambda)\rho)$ and $y := \dfrac{q - \lambda x}{1-\lambda}$. Then y is in $U(q, \rho)$. By (λ, h)-convexity,

$$f(q) \leqslant h(\lambda)f(x) + h(1-\lambda)f(y),$$

which implies

$$f(x) \geqslant \frac{f(q) - h(1-\lambda)f(y)}{h(\lambda)} \geqslant \frac{f(q) - h(1-\lambda)M}{h(\lambda)} =: M'.$$

Therefore f is locally bounded from below at any point of D. $\square$

As an immediate consequence of the previous theorem we obtain the following generalization of the celebrated theorem of Bernstein and Doetsch:

COROLLARY 2. *Let $f : D \to \mathbb{R}$ be a (Jensen, h)-convex function. If either $h(1/2) < 1/2$ or $h(1/2) \geqslant 1/2$ with the additional assumption on the locally boundedness of f from above at a point of D, then f is locally bounded at every point of D.*

The next theorem essentially weakens the local boundedness assumption if the underlying space is of finite dimension. The proof is analogous what was followed in [11] (that is based on Steinhaus' and Piccard's theorems (cf. [24], [21])).

THEOREM 2. *Let D be an open convex subset of $\mathbb{R}^n$, let $h:[0,1]\to\mathbb{R}$ be a given nonnegative function such that $h(\lambda)$ and $h(1-\lambda)$ are not zero simultaneously, and let $f:D\to\mathbb{R}$ be (λ,h)-convex function with a fixed $0<\lambda<1$. Assume that there exist a Lebesgue-measurable set of positive measure (or a Baire-measurable set of second category) $S\subseteq D$ and a Lebesgue-measurable (resp. Baire-measurable) function $g:S\to\mathbb{R}$ such that $f\leqslant g$ on S. Then f is locally bounded on D.*

REMARK 2. It is a well-known fact that if a Jensen-convex function f is locally bounded above at a point of its domain (see [1], [14]), then it is continuous on its domain. This is not true for (Jensen,h)-convex functions. Indeed, in the case $h(\lambda)=\lambda^s$ (where $0<s<1$ is a fixed number), in [2] we give an example which is (Jensen,h)-convex, bounded and nowhere continuous.

Next theorem gives a sufficient condition when local boundedness implies continuity.

THEOREM 3. *Let $D\subset X$ be a nonempty, convex, open set and let $h:[0,1]\to\mathbb{R}$ be a given nonnegative function satisfying the limit conditions*

$$\lim_{x\to 0} h(x)=0 \qquad \text{and} \qquad \lim_{x\to 1} h(x)=1.$$

Let the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ be such that $\lambda_n\in]0,1]$ and λ_n tends to 0 (when $n\to\infty$) and assume that $h(\lambda_n)$ and $h(1-\lambda_n)$ not simultaneously zero. If $f:D\to\mathbb{R}$ is $(\{\lambda_n\}_{n\in\mathbb{N}},h)$-convex and f is locally bounded from above at a point $x_0\in D$, then f is continuous at x_0.

Proof. Since $h(\lambda_n)$ and $h(1-\lambda_n)$ are not zero simultaneously, therefore, without loss generality, we may assume that $h(1-\lambda_n)>0$.

Since f is locally bounded from above at a point $x_0\in D$, there exists a neighborhood U at x_0 and a constant $K\geqslant 0$ such that $f(x)\leqslant K$ for every $x\in U$. Let ε be an arbitrary nonnegative constant. Then there exists $n_0\in\mathbb{N}$ such that if $n\geqslant n_0$, then

$$h(\lambda_n)K+[h(1-\lambda_n)-1]f(x_0)<\varepsilon,$$

whence

$$\frac{h(\lambda_n)}{h(1-\lambda_n)}K+\left[1-\frac{1}{h(1-\lambda_n)}\right]f(x_0)<\varepsilon.$$

Let V be a neighborhood of 0 such that $x_0+V\subseteq U$, and let $U'=x_0+\lambda_n V$. We prove that

$$|f(x)-f(x_0)|<\varepsilon \qquad (x\in U').$$

For $x \in U'$ there exist $y, z \in x_0 + V$ such that

$$x = \lambda_n y + (1-\lambda_n)x_0,$$
$$x_0 = \lambda_n z + (1-\lambda_n)x.$$

Indeed,

$$y - x_0 = \frac{1}{\lambda_n}(x - x_0) \in \frac{1}{\lambda_n}\lambda_n V = V,$$

and

$$z - x_0 = \frac{1-\lambda_n}{\lambda_n}(x_0 - x) \in \frac{1-\lambda_n}{\lambda_n}\lambda_n V = (1-\lambda_n)V \subseteq V.$$

According to (λ_n, h)-convexity of f,

$$f(x) \leqslant h(\lambda_n)f(y) + h(1-\lambda_n)f(x_0) \leqslant h(\lambda_n)K + h(1-\lambda_n)f(x_0),$$
$$f(x_0) \leqslant h(\lambda_n)f(z) + h(1-\lambda_n)f(x) \leqslant h(\lambda_n)K + h(1-\lambda_n)f(x).$$

We get

$$f(x) - f(x_0) \leqslant h(\lambda_n)K + [h(1-\lambda_n) - 1]f(x_0) < \varepsilon \tag{2}$$

and

$$f(x) \geqslant \frac{f(x_0) - h(\lambda_n)K}{h(1-\lambda_n)},$$

which implies

$$f(x) - f(x_0) \geqslant \left[\frac{1}{h(1-\lambda_n)} - 1\right] f(x_0) - \frac{h(\lambda_n)}{h(1-\lambda_n)}K > -\varepsilon. \tag{3}$$

The inequalities (2) and (3) show that $|f(x) - f(x_0)| < \varepsilon$, that is f is continuous at x_0, which was to be proved. □

REMARK 3. The previous limit conditions are not necessary, since in the case of Jensen-convexity are not fulfilled. However, the result of Bernstein and Doetsch is valid for Jensen-convex functions. In contrary, the nonnegative monotone functions - which are not necessary continuous - belongs to a special class of the h-convex functions, to the class of Godunova-Levin functions. Therefore, in this setting, the limit conditions in question cannot be ignored.

COROLLARY 3. *Let $D \subset X$ be a nonempty, convex, open set and let $h : [0,1] \to \mathbb{R}$ be a given nonnegative function satisfying the limit conditions*

$$\lim_{x \to 0} h(x) = 0 \qquad \textit{and} \qquad \lim_{x \to 1} h(x) = 1.$$

Let $H \subseteq [0,1]$ and assume that 0 or 1 is an accumulation point of H and $h(\lambda)$ and $h(1-\lambda)$ are not zero simultaneously for all $\lambda \in H$. If $f : D \to \mathbb{R}$ is (H,h)-convex and locally bounded at a point of D, then f is continuous at that point.

Proof. Since f is (H,h)-convex, it is also $(1-H,h)$-convex, so there exists a sequence in H or in $1-H$, which tends to zero. Now, we can apply the previous theorem. □

THEOREM 4. *Let $D \subset X$ be a nonempty, convex, open set and let $h : [0,1] \to \mathbb{R}$ be a given nonnegative function satisfying the limit conditions*

$$\lim_{x\to 0} h(x) = 0 \qquad \text{and} \qquad \lim_{x\to 1} h(x) = 1.$$

Let $H \subseteq [0,1]$ and assume that 0 or 1 is an accumulation point of H and $h(\lambda)$ and $h(1-\lambda)$ are not zero simultaneously for all $\lambda \in H$. If $f : D \to \mathbb{R}$ is (H,h)-convex and locally bounded at a point of D, then f continuous on D.

Proof. According to Theorem 1, f is locally bounded at every point of D. So, we can use the previous corollary to get the continuity of f at every point of D. □

4. Convexity property of $(\mathbb{Q},h)$-convex functions

The following result offers a generalization of the theorem of Bernstein-Doetsch [1], Breckner [4] and Burai-Házy-Juhász [2] for *(rationally, s)*-convex functions

THEOREM 5. *Let $D \subset X$ be a nonempty, convex, open set and let $h : [0,1] \to \mathbb{R}$ be a given continuous, nonnegative function satisfying the limit conditions*

$$\lim_{x\to 0} h(x) = 0 \qquad \text{and} \qquad \lim_{x\to 1} h(x) = 1.$$

Assume that $h(\lambda)$ and $h(1-\lambda)$ are not zero simultaneously for all $\lambda \in \mathbb{Q}\cap[0,1]$. If $f : D \to \mathbb{R}$ is $(\mathbb{Q},h)$-convex and locally bounded at a point of D, then f is continuous on D and is h-convex.

Proof. We prove that the function f is (λ,h)-convex for all $\lambda \in [0,1]$. Let $\lambda \in [0,1]$ arbitrary. Then there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \in \mathbb{Q}$ and $\lambda_n \to \lambda$ (when n tends to ∞). Applying $(\mathbb{Q},h)$-convexity of f, we get

$$f(\lambda_n x + (1-\lambda_n)y) \leqslant h(\lambda_n)f(x) + h(1-\lambda_n)f(y). \tag{4}$$

The local upper boundedness of f implies the continuity of f (according to Theorem 3). Therefore, taking the limit $n \to \infty$ in (4), we get

$$f(\lambda x + (1-\lambda)y) \leqslant h(\lambda)f(x) + h(1-\lambda)f(y),$$

which proves the h-convexity of f. □

COROLLARY 4. *Let $D \subset X$ be a nonempty, convex, open set and let $h : [0,1] \to \mathbb{R}$ be a given continuous, nonnegative function satisfying the limit conditions*

$$\lim_{x\to 0} h(x) = 0 \qquad \text{and} \qquad \lim_{x\to 1} h(x) = 1.$$

Let H is a dense subset of $[0,1]$ and assume that $h(\lambda)$ and $h(1-\lambda)$ are not zero simultaneously for all $\lambda \in H$. If $f : D \to \mathbb{R}$ is (H,h)-convex and locally bounded at a point of D, then f is continuous on D and is h-convex.

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