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CHARACTERIZATIONS OF THE OPERATOR INEQUALITY $A \geqslant B \geqslant C$

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Abstract. Let A, B and C be positive invertible operators on a Hilbert space. Motivated by the characterization of the operator inequality $A \geqslant B$ due to Fujii, Kamei and Nakamoto in [3], in this paper, we prove the following: $A \geqslant B \geqslant C$ if and only if the two operator inequalities

$$A^{r-t} \geqslant [A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}},$$
$$[C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}]^{\frac{r-t}{(p-t)s+r}} \geqslant C^{r-t}$$

hold for all $p,s \geqslant 1$, $r \geqslant t$ and $t \in [0,1]$.

Finally, characterizations of operator inequalities in terms of the operator equalities are given.

1. Introduction

In this paper, we use capital letters as bounded linear operators on a Hilbert space and I denotes the identity operator. the operators $T \geqslant O$ and $T > O$, respectively, mean that T is positive and T is positive and invertible. We recall first the well-known classical Löwner-Heinz inequality in Hilbert spaces as it is used frequently in this paper. It should be mentioned that the Löwner-Heinz inequality does not hold in general if $\alpha > 1$.

THEOREM LH. ([Löwner-Heinz's Inequality]) *If $S \geqslant T \geqslant O$, then $S^\alpha \geqslant T^\alpha$ for any $\alpha \in [0,1]$.*

Next, we recall the grand Furuta inequality and the extended grand Furuta inequality as we need them later on. Notice that, in general, if $A \geqslant B \geqslant O$, then we may assume, without loss of generality, that $A \geqslant B > O$. It follows that $A \geqslant B$ if and only if $B^{-1} \geqslant A^{-1}$.

In what follows we assume that A, B, C, S, $T > O$.

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THEOREM G. ([5, Grand Furuta's Inequality]) *If* $A \geqslant B$, *then*

$(A1)$ $A^{1-t+r} \geqslant [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1-t+r}{(p-t)s+r}}$ *for all* $p,s \geqslant 1$, $r \geqslant t$ *and* $t \in [0,1]$.

The conditions on p,r,s, and t in Theorem G are proved to be the best possibility for the inequality $(A1)$ as proved in [10]. Correspondingly,

$(A2)$ $[B^{r/2}(B^{-t/2}A^pB^{-t/2})^sB^{r/2}]^{\frac{1-t+r}{(p-t)s+r}} \geqslant B^{1-t+r}$ *for all* $p,s \geqslant 1$, $r \geqslant t$ *and* $t \in [0,1]$.

Then $(A1)$ *is equivalent to* $(A2)$.

Theorem G was extended in [11] as follows:

THEOREM U. ([11, Extended grand Furuta's Inequality]) *If* $A \geqslant B \geqslant C$, *then*

$(B1)$ $A^{1-t+r} \geqslant [A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1-t+r}{(p-t)s+r}}$ *for all* $p,s \geqslant 1$, $r \geqslant t$ *and* $t \in [0,1]$.

By using Theorem G, a one-page simplified proof of Theorem U appeared in [8]. Correspondingly,

$(B2)$ $[C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}]^{\frac{1-t+r}{(p-t)s+r}} \geqslant C^{1-t+r}$ *for all* $p,s \geqslant 1$, $r \geqslant t$ *and* $t \in [0,1]$.

Then $(B1)$ *is equivalent to* $(B2)$.

The purpose of the present paper is precisely indicated in the abstract. In particular, some characterizations of the operator inequalities in terms of the operator equalities are given in the last section.

2. Characterizations of the operator inequality $\mathbf{A \geqslant B \geqslant C}$

Furuta posed two interesting questions in [7, Section 3] about characterizing chaotic operator order $A \gg B$ in terms of operator inequalities. The answer to the second question is negative and a nontrivial counterexample was given by Furuta himself in the same paper. Later on Fujii et al. [3, Theorem 1] gave a complete solution to the question. The proof of their result is trivial for the necessary condition, and nontrivial for the sufficient condition as it requires the Kantorovich type operator inequality. In fact, they use Theorem G to claim and prove the following:

THEOREM F. ([3, Fujii's Inequality]) $A \geqslant B$ $($$A \gg B$ *in the Furuta's second question*$)$ *if and only if*

(C) $A^{r-t} \geqslant [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}}$ *holds for all* $p \geqslant 1$, $r \geqslant t$, $s \geqslant 1$ *and* $t \in [0,1]$.

In this section, we use Theorem U instead to generalize the result (Theorem F) above to the operator inequality $A \geqslant B \geqslant C$ and here is the main result of the paper.

THEOREM 2.1. *Let A, B and C be positive invertible operators on a Hilbert space. Then $A \geqslant B \geqslant C$ if and only if two operator inequalities*

$$A^{r-t} \geqslant [A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}} \tag{2.1}$$

$$[C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}]^{\frac{r-t}{(p-t)s+r}} \geqslant C^{r-t} \tag{2.2}$$

hold for all $p,s \geqslant 1$, $r \geqslant t$ and $t \in [0,1]$.

Proof. $(\Rightarrow)$ The inequality (2.1) follows by $(B1)$ in Theorem U and Theorem LH for $\frac{r-t}{1-t+r} \in [0,1)$. The inequality (2.2) is due to $(B2)$ in Theorem U and Theorem LH for $\frac{r-t}{1-t+r} \in [0,1)$.

$(\Leftarrow)$ We adapt the same method as in the proof of [3, Theorem 1]. But, first, by Theorem 6′ in [2], note that, if $S \geqslant T$ and $0 < m \leqslant S \leqslant n$ for some m and n, then we have

$$\frac{(n+m)^2}{4nm}S^2 \geqslant T^2.$$

Now, let $p = t = 1$ and $r = 2$, in particular, in the inequality (2.1). Then we have

$$A \geqslant [A(B^{-1/2}CB^{-1/2})^sA]^{1/2}. \tag{2.3}$$

Let $0 < m \leqslant A \leqslant n$. Then from the result above we have

$$\frac{(n+m)^2}{4nm}A^2 \geqslant A(B^{-1/2}CB^{-1/2})^sA.$$

It follows that

$$\frac{(M+m)^2}{4nm}I \geqslant (B^{-1/2}CB^{-1/2})^s$$

and so

$$\left(\frac{(M+m)^2}{4nm}\right)^{1/s}I \geqslant B^{-1/2}CB^{-1/2}.$$

Passing to the limit as $s \to \infty$ yields $B \geqslant C$.

A similar setting as above for the inequality (2.2) and it follows that

$$[C(B^{-1/2}AB^{-1/2})^sC]^{1/2} \geqslant C. \tag{2.4}$$

Therefore, we have

$$\left(\frac{(n+m)^2}{4nm}\right)^{1/s}(B^{-1/2}AB^{-1/2}) \geqslant I,$$

which in turn implies that $A \geqslant B$. This completes the proof. □

A special case of Theorem 2.1 includes [3, Theorem 1] as follows:

COROLLARY 2.2. *For all $p,s \geqslant 1$, $r \geqslant t$ and $t \in [0,1]$, $A \geqslant B$ if and only if*

$$A^{r-t} \geqslant [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{r-t}{(p-t)s+r}}.$$

Proof. Let $B = A$ and $C = B$ in (2.1) of Theorem 2.1. □

3. Characterizations of operator inequalities in Theorem 2.1 and Corollary 2.2 in terms of operator equalities

Under certain conditions, more precisely, under $(r-t)(n+1)=(p-t)s+r,$ we could characterize the operator inequalities in Theorem 2.1 and Corollary 2.2 in Section 2 in terms of operator equalities. The tool to be used is due to the Douglas's majorization and factorization theorem.

Let us state the theorem first.

THEOREM D. ([Douglas's Theorem]) *For any operators* A *and* B *(not necessarily* $A,B>O$ *), the following statements are equivalent:*

(1) *range* $B\subseteq$ *range* A*;*

(2) A^* *majorizes* B^**, i.e.,* $BB^*\leqslant\lambda^2AA^*$*, i.e.,* $\|B^*x\|\leqslant\lambda\|A^*x\|$ *for some* $\lambda\geqslant 0$ *and all* $x\in\mathbf{H}$ *(: majorization);*

(3) *There exists an operator* C *such that* $B=AC$ *(: factorization).*

Moreover, $\|C\|^2=\inf\{\mu : BB^*\leqslant\mu AA^*\}$ *(by the equivalence of* (2) *and* (3) *).*

The next result is due to Theorem 2.1 and Theorem D.

THEOREM 3.1. *For all* $p,s\geqslant 1$, $r\geqslant t$ *and* $t\in(0,1]$, *let a nonnegative integer* $n\geqslant 1$ *be such that* $(r-t)(n+1)=(p-t)s+r$. *Then the following three statements are equivalent to one another:*

(3.1) $A\geqslant B\geqslant C$;

(3.2) *The following two operator inequalities hold:*

(a) $A^{r-t}\geqslant[A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}}$,

(b) $[C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}]^{\frac{1}{n+1}}\geqslant C^{r-t}$:

(3.3) *The following two operator equalities hold:*

(a') *there exists a unique operator* $S_1>O$ *with* $\|S_1\|\leqslant 1$ *such that*

$$A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}=(A^{\frac{r-t}{2}}S_1A^{\frac{r-t}{2}})^{n+1},$$

(b') *there exists a unique operator* $S_2>O$ *with* $\|S_2\|\leqslant 1$ *such that*

$$C^{r/2}(B^{-t/2}A^pB^{-t/2})^sC^{r/2}=(C^{\frac{r-t}{2}}S_2^{-1}C^{\frac{r-t}{2}})^{n+1}.$$

Proof. Because of the condition that $(r-t)(n+1)=(p-t)s+r,$ the equivalence of (3.1) and (3.2) is a special case of Theorem 2.1.

Next, we prove that $(3.2)\Longleftrightarrow(3.3)$. We firstly show that $(a)\Longrightarrow(a')$. By (a) and Theorem D, there exists an operator E with $\|E\|\leqslant 1$ such that

$$[A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1}{2(n+1)}}=A^{\frac{r-t}{2}}E=E^*A^{\frac{r-t}{2}}.$$

Let $S_1 = EE^*$ and so $\| S_1 \| = \| E \|^2 \leqslant 1$. Then we have

$$[A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}} = A^{\frac{r-t}{2}}S_1A^{\frac{r-t}{2}}$$

and S_1 is clearly unique, here we may assume $S_1 > O$, without loss of generality. Now, (a') in (3.3) follows immediately.

Secondly, we show that $(a') \Longrightarrow (a)$. By (a') we have

$$[A^{r/2}(B^{-t/2}C^pB^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}} = A^{\frac{r-t}{2}}S_1A^{\frac{r-t}{2}} \leqslant A^{r-t}.$$

The inequality above is due to the fact that $S_1 \leqslant \| S_1 \| I \leqslant I$ as S_1 is Hermitian and so we have (a) in (3.2).

Finally, we have to prove $(b) \Longleftrightarrow (b')$. To show that $(b) \Longrightarrow (b')$, we may rewrite (b) as follows:

$$C^{-(r-t)} \geqslant [C^{-r/2}(B^{t/2}A^{-p}B^{t/2})^sC^{-r/2}]^{\frac{1}{n+1}}.$$

The inequality implies, as in the proof of $(a) \Longrightarrow (a')$, that

$$C^{-r/2}(B^{t/2}A^{-p}B^{t/2})^sC^{-r/2} = (C^{-\frac{r-t}{2}}S_2C^{-\frac{r-t}{2}})^{n+1}$$

for some unique $S_2 > O$ with $\| S_2 \| \leqslant 1$. Hence we get (b').

Similarly, we have $(b') \Longrightarrow (b)$ and the proof should be omitted. This completes the proof. □

Our final result is a special case of Theorem 3.1, and the proof should be omitted.

COROLLARY 3.2. *For all* $p,s \geqslant 1$, $r \geqslant t$, $t \in (0,1]$, *let a nonnegative integer* $n \geqslant 1$ *be such that* $(r-t)(n+1) = (p-t)s+r$. *Then the following three statements are equivalent to one another:*

(3.4) $A \geqslant B$;

(3.5) $A^{r-t} \geqslant [A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}]^{\frac{1}{n+1}}$;

(3.6) *There exists a unique operator* $S_1 > O$ *with* $\| S_1 \| \leqslant 1$ *such that*

$$A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2} = (A^{\frac{r-t}{2}}S_1A^{\frac{r-t}{2}})^{n+1}.$$

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REFERENCES

[1] R. G. DOUGLAS, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc., **17** (1966), 413–415.
[2] M. FUJII, S. IZUMINO, R. NAKAMOTO AND Y. SEO, *Operator inequalities related to Cauchy-Schwarz and HÏder-McCarthy inequalities*, Nihonkai Math. J., **8** (1997), 117–122.
[3] M. FUJII, E. KAMEI AND R. NAKAMOTO, *On a question of Furuta on chaotic order*, Linear Algebra Appl., **341** (2002), 119–127.
[4] T. FURUTA, *$A \geqslant B \geqslant O$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.
[5] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl., **219** (1995), 139–155.
[6] T. FURUTA, *Invitation To Linear Operators,* ISBN: 0-415-26799-4, Taylor & Francis, 2001.
[7] T. FURUTA, *Results under $\log A \geqslant \log B$ can be derived from ones under $A \geqslant B \geqslant O$ by Uchiyama's method–associated with Furuta and Kantorovich type operator inequalities*, Math. Inequl. Appl., **3** (2000), 423–436.
[8] T. FURUTA, *A proof of an order preserving inequality*, Proc. Japan Acad., **78** (2002), 26.
[9] C.-S. LIN, *On operator order and chaotic operator order for two operators*, Linear Algebra Appl., **425** (2007), 1–6.
[10] K. TANAHASHI, *The best possibility of the grand Furuta inequality*, Proc. Amer. Math. Soc., **128** (2000), 511–519.
[11] M. UCHIYAMA, *Criteria for operator means*, J. Math. Soc. Japan, **55** (2003), 197–207.

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