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WEYL TYPE THEOREMS AND CLASS $A(s,t)$ OPERATORS

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Abstract. In this note we show that if T belongs to the class $A(s,t)$ for some $0 < s,t \leqslant 1$ then Weyl's and generalized Weyl's theorems hold for $f(T)$ for every analytic function f on some open neighborhood of $\sigma(T)$. We also show that operators in $A(s,t)$ satisfy the Bishop's property (β).

1. Introduction

Let $\mathbf{B}(\mathscr{H})$ be the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space $\mathscr{H}$. We denote by $\sigma(T)$, $\sigma_s(T)$, $\sigma_a(T)$, $\sigma_p(T)$, $\sigma_{jp}(T)$, and $\sigma_{ja}(T)$, the spectrum, the surjectivity spectrum, the approximate point spectrum, the point spectrum, the joint point spectrum and the joint approximate point spectrum of T, respectively. If $T \in \mathbf{B}(\mathscr{H})$ we shall write $\ker(T)$ and $\mathscr{R}(T)$ for the null space and range of T, respectively.

An operator T is said to be p-hyponormal, for $p \in (0,1]$, if $(T^*T)^p \geqslant (TT^*)^p$ [4]. An 1-hyponormal operator is *hyponormal* and $\frac{1}{2}$-hyponormal is said to be *semi-hyponormal*. An invertible operator T is said to be log-hyponormal if $\log|T| \geqslant \log|T^*|$ [37]. p-hyponormality and log-hyponormality were defined as extension of hyponormality.

An operator T is said to be *paranormal* if $\|T^2x\| \geqslant \|Tx\|^2$ for every unit vector x. Paranormal operators have been studied by many authors, for examples [8, 17, 21, 23, 40]. Particularly, p-hyponormal operators and log-hyponormal operators are paranormal [40].

An operator T belongs to *class* $A(k)$ for $k > 0$ if

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geqslant |T|^2.$$

When $k = 1$ we say that T belongs to *class* A. Furuta *et al.* [23] showed that every p-hyponormal or log-hyponormal operator belongs to *class* A and that every class A is paranormal.

As a further generalization of class $A(k)$, Fujii *et al.* [24] introduced the *class* $A(s,t)$: T belongs to class $A(s,t)$ for $s > 0$ and $t > 0$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{t+s}} \geqslant |T^*|^{2t}.$$

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Class $AI(s,t)$ is the class of all invertible class $A(s,t)$ operators for $s>0$ and $t>0$. It was pointed out in [42] that *class* $A(k,1)$ equals *class* $A(k)$. They showed several properties of *class* $A(s,t)$ and *class* $AI(s,t)$ as extensions of the properties of class $A(k)$ shown in [23]. Particularly, they showed that T is log-hyponormal if and only if T belongs to *class* $AI(s,t)$ for all $s,t>0$.

Let T be an operator whose polar decomposition is $T=U|T|$, where $|T|=\sqrt{T^*T}$. For $s,t>0$, the *generalized Aluthge transformation* $\widetilde{T}_{s,t}$ of T is

$$\widetilde{T}_{s,t}=|T|^sU|T|^t.$$

When $s=t=\frac{1}{2}$, then $\widetilde{T}_{s,t}$ is called the *Aluthge transformation* of T and denoted by $\widetilde{T}$, [4]. The relations between T and its transformation $\widetilde{T}_{s,t}$ are

$$\widetilde{T}_{s,t}|T|^s=|T|^sU|T|^t|T|^s=|T|^sT, \tag{1.1}$$

and

$$U|T|^t\widetilde{T}_{s,t}=U|T|^t|T|^sU|T|^t=TU|T|^t, \tag{1.2}$$

for each $s,t>0$ such that $s+t=1$.

Aluthge and Wang [5] introduced w-hyponormal operators defined as follows: An operator T is said to be *w-hyponormal* if

$$|\widetilde{T}|\geqslant|T|\geqslant|\widetilde{T}^*|.$$

As a generalization of w-hyponormality, Ito [28] introduced the class $wA(s,t)$:

DEFINITION 1.1. ([28]) An operator T belongs to the *class* $wA(s,t)$ for $s>0$ and $t>0$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{t+s}}\geqslant|T^*|^{2t} \tag{1.3}$$

and

$$|T|^{2s}\geqslant(|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}. \tag{1.4}$$

Ito pointed out the following fact which states that $wA(s,t)$ can be expressed via generalized Aluthge transformation.

PROPOSITION 1.2. ([28]) *An operator T belongs to the class $wA(s,t)$ for $s>0$ and $t>0$ if and only if*

$$|\widetilde{T}_{s,t}|^{\frac{2t}{s+t}}\geqslant|T|^{2t} \text{ and } |T|^{2s}\geqslant|\widetilde{T}^*_{s,t}|^{\frac{2s}{s+t}}.$$

In this note we study Weyl's theorem for operators belong to $A(s,t)$. In Section 2 we prove some properties of operator T belongs to $A(s,t)$ which we need in the sequel. In Section 3 we study the Weyl's and Browder's theorems for operators in $A(s,t)$. Section 4 is devoted to generalized Weyl's theorem.

2. Properties of class $A(s,t)$ for some $0 < s,t \leqslant 1$

THEOREM 2.1. ([39]) *Let T belong to the class $A(s,t)$ for $0 < s,t \leqslant 1$ and let $\lambda \neq 0$. Then*

$$(T-\lambda)x = 0 \Longrightarrow (T-\lambda)^* x = 0, x \in \mathscr{H}.$$

LEMMA 2.2. ([29]) *Class $A(s,t)$ coincides with $wA(s,t)$ for each $s > 0$ and $t > 0$.*

LEMMA 2.3. ([24]) *For $s > 0$ and $t > 0$, T belongs to the class $A(s,t)$ if and only if T belongs to the class $AI(s,t)$ and T is invertible if and only if T is* log-*hyponormal.*

LEMMA 2.4. *If $\widetilde{T}_{s,t}$ is invertible, then $|T|^t$ is invertible for each $t > 0$.*

Proof. Suppose that $|T|^t$ is not invertible for each $t > 0$. Then $|T|^s$ is not invertible for each $s > 0$. Hence, either the range $\mathscr{R}(|T|^s)$ of $|T|^s$ is not dense or $|T|^s$ is not bounded below. Since $\widetilde{T}_{s,t}x = |T|^s(U|T|^t x)$, $x \in \mathscr{H}$, $\mathscr{R}(\widetilde{T}_{s,t}) \subset \mathscr{R}(|T|^s)$. If $\mathscr{R}(|T|^s)$ is not dense, then $\mathscr{R}(\widetilde{T}_{s,t})$ is not dense and hence $\widetilde{T}_{s,t}$ is not invertible. On the other hand, if $|T|^t$ is not bounded below, then there is a sequence $\{x_n\}$ of unit vectors such that $\||T|^t x_n\| \to 0$. Since $\left\|\widetilde{T}_{s,t}x_n\right\| \leqslant \||T|^s U\| \||T|^t x_n\|$, then $\left\|\widetilde{T}_{s,t}x_n\right\| \to 0$. Thus, $\widetilde{T}_{s,t}$ is not bounded below and is therefore not invertible. The proof is complete. □

LEMMA 2.5. *The operator T is invertible if and only if $\widetilde{T}_{s,t}$ is invertible.*

Proof. If T is invertible, then clearly $\widetilde{T}_{s,t}$ is invertible. If $\widetilde{T}_{s,t}$ is invertible, then Lemma 2.4 implies that $|T|^t$ is invertible for each $t > 0$. Since $T = |T|^{-s}\widetilde{T}_{s,t}|T|^{1-t}$, T is invertible. □

THEOREM 2.6. *Let T be an invertible operator in the class $A(s,t)$ for some $0 < s,t \leqslant 1$, then so is T^{-1}.*

Proof. If T is invertible belong to the class $A(s,t)$, then T is log-hyponormal, so T^{-1} is log-hyponormal. Therefore T^{-1} belongs to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. □

We write $W(T)$ for the numerical range. $W(T)$ is convex and $conv\,\sigma(T) \subseteq cl\,W(T)$. An operator is called *convexoid* if $conv\,\sigma(T) = cl\,W(T)$.

REMARK 2.7. The class $A(s,t)$ operators is a subclass of the class of the normaloid operators, see [41, 24].

LEMMA 2.8. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$, $\lambda \in \mathbb{C}$, and assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda$.*

Proof. We consider two cases:

Case I ($\lambda = 0$): Since T belongs class $A(s,t)$ for some $0 < s,t \leqslant 1$, T is normaloid. Therefore $T = 0$.

Case II ($\lambda \neq 0$): Here T is invertible, and since T belongs the class $A(s,t)$ for some $0 < s,t \leqslant 1$, we see that T^{-1} is also belongs class $A(s,t)$ for some $0 < s,t \leqslant 1$. Therefore T^{-1} is normaloid. On the other hand, $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows that T is convexoid, so W(T) $= \{\lambda\}$. Therefore $T = \lambda$. □

LEMMA 2.9. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$, $\lambda = |\lambda|e^{i\theta} \in \mathbb{C}$, and $\lambda_{s+t} = |\lambda|^{s+t}e^{i\theta}$, then* $\ker(T-\lambda) = \ker(\widetilde{T}_{s,t} - \lambda_{s+t})$.

Proof. The inclusion $\ker(T-\lambda) \subseteq \ker(\widetilde{T}_{s,t} - \lambda_{s+t})$ follows from [6, 39]. For the other inclusion $\ker(\widetilde{T}_{s,t} - \lambda_{s+t}) \subseteq \ker(T-\lambda)$, let $\lambda \neq 0$ and $0 \neq x \in \ker(\widetilde{T}_{s,t} - \lambda_{s+t})$. By [28], $\widetilde{T}_{s,t}$ is $\frac{min(s,t)}{s+t}$-hyponormal and we have

$$|\widetilde{T}_{s,t}|x = |\lambda|^{s+t}x = |\widetilde{T}^*_{s,t}|x,$$
$$|\widetilde{T}_{s,t}|^{2\beta_1} - |\widetilde{T}^*_{s,t}|^{2\beta_1} \geqslant |\widetilde{T}_{s,t}|^{2\beta_1} - |T|^{2\beta_1(s+t)} \geqslant 0,$$

where $\beta_1 = \frac{min(s,t)}{s+t}$. Hence,

$$(|\widetilde{T}_{s,t}|^{2\beta_1} - |T|^{2\beta_1(s+t)})x = 0,$$
$$\left\| |T|^{2\beta_1(s+t)}x - |\lambda|^{2\beta_1(s+t)}x \right\| \leqslant \left\| |T|^{2\beta_1(s+t)}x - |\widetilde{T}_{s,t}|^{2\beta_1} \right\| + \left\| |\widetilde{T}_{s,t}|^{2\beta_1}x - |\lambda|^{2\beta_1(s+t)}x \right\| = 0.$$

On the other hand, $\widetilde{T}^*_{s,t}x = |\lambda|^{s+t}e^{-i\theta}x$ implies that $|T|^sU^*x = |\lambda|^se^{-i\theta}x$, $T^* = |\lambda|e^{-i\theta}$. Therefore,

$$\|(T-\lambda)x\|^2 = \|Tx\|^2 - \lambda\langle x, Tx\rangle - \overline{\lambda}\langle Tx, x\rangle + |\lambda|^2\|x\|^2$$
$$= \||T|x\|^2 - \lambda\langle T^*x, x\rangle - \overline{\lambda}\langle x, T^*x\rangle + |\lambda|^2\|x\|^2 = 0.$$

If $\lambda = 0$, let $0 \neq x \in \widetilde{T}_{s,t}$, then $x \in \ker(T) = \ker(|T|)$. So that $\ker(\widetilde{T}_{s,t} - \lambda_{s+t}) \subseteq \ker(T-\lambda)$. □

For $T \in \mathbf{B}(\mathscr{H})$ let λ be an isolated point in $\sigma(T)$. We denote by E_λ the Riesz idempotent with respect to λ defined by $E_\lambda = \frac{1}{2\pi i}\int_{\partial D_\lambda}(z-T)^{-1}dz$ where D_λ is a closed disk centered at λ which satisfies $D_\lambda \cap \sigma(T) = \emptyset$. It is well known that $\ker(T-\lambda) \subseteq E_\lambda\mathscr{H}$ and that $E_\lambda\mathscr{H} = \{x \in \mathscr{H} : \|(T-\lambda)^n x\|^{\frac{1}{n}} \to 0\}$ (see [35, p. 424]).

THEOREM 2.10. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$, $\lambda = |\lambda|e^{i\theta} \in \mathbb{C}$ such that $\lambda \in iso\sigma(T)$, and $\lambda_{s+t} = |\lambda|^{s+t}e^{i\theta}$, then the following assertions hold.*

(1) *If* $\lambda \neq 0$*, then* $E_\lambda = E_\lambda(s,t)$ *and* $E_\lambda \mathscr{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$*, where* $E_\lambda(s,t)$ *is the Riesz idempotent of the* $\widetilde{T}_{s,t}$ *with respect to* λ_{s+t}.

(2) *If* $\lambda = 0$*, then* $\ker T = E_0\mathscr{H} = E_0(s,t)\mathscr{H} = \ker(\widetilde{T}_{s,t})$.

Proof. (1) Since $\sigma(\widetilde{T}_{s,t}) \setminus \{0\} = \sigma(U|T|^{s+t}) \setminus \{0\} = \left\{\rho^{s+t}e^{i\theta} : \rho e^{i\theta \in \sigma(T)}\right\} \setminus \{0\}$ (see [9]), then $\lambda_{s+t} \in iso\,\sigma(\widetilde{T}_{s,t})$. Hence

$$(E_\lambda(t,s)\mathscr{H})^\perp = \ker(E_\lambda(t,s)) = I - E_\lambda(t,s)\mathscr{H} \tag{2.1}$$

by [13]. So $\lambda_{s+t} \notin \sigma(\widetilde{T}_{s,t}|_{(E_\lambda(t,s)\mathscr{H})^\perp})$. By Theorem 2.1 and Lemma 2.9, we have $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_{22} \end{pmatrix}$ on $\mathscr{H} = (E_\lambda(t,s)\mathscr{H})^\perp \oplus (E_\lambda(t,s)\mathscr{H})$, where $T_{22} = T|_{\ker(T-\lambda)^\perp}$. Since $\ker(T-\lambda)$ is reduced by T, T_{22} belong also to the class $A(s,t)$ and $\widetilde{T}_{22}(s,t) = \widetilde{T}_{s,t}|_{(E_\lambda(t,s)\mathscr{H})^\perp}$, so that $\lambda \notin \sigma(T_{22})$ because $\lambda_{s+t} \notin \sigma(\widetilde{T}_{22}(s,t))$. Hence $T - \lambda = 0 \oplus (T_{22} - \lambda)$ and $\ker(T-\lambda)^* = \ker(T-\lambda) \oplus \ker(T_{22}-\lambda)^* = \ker(T-\lambda)$. Meanwhile $E_\lambda = \int_{\partial\mathscr{D}} (z-\lambda)^{-1} \oplus (z-T_{22})^{-1}\,dz = 1 \oplus 0 = E_\lambda(s,t)$.

(2) Since $\widetilde{T}_{s,t}$ is $\frac{min(s,t)}{s+t}$-hyponormal [28], we only need to prove that $E_0\mathscr{H} \subseteq E_0(s,t)\mathscr{H}$ since $E_0\mathscr{H} \supseteq E_0(s,t)\mathscr{H}$ holds by Theorem 2.9 and [13]. We have

$$|T|^s \ker(T) \subseteq |T|^s E_0\mathscr{H}. \tag{2.2}$$

Let $x \in E_0\mathscr{H}$. Then it follows from Equality (1.1) that

$$\|\widetilde{T}_{s,t}^n |T|^s x\|^{\frac{1}{n}} = \||T|^s T^n x\|^{\frac{1}{n}} \to 0.$$

Thus $E_0\mathscr{H} \subseteq E_0(s,t)\mathscr{H}$. Hence $E_0(s,t)\mathscr{H}$ is reduced by $|T|^s$.

Let $x \in E_0\mathscr{H}$ and $x = x_1 + x_2 \in E_0(s,t)\mathscr{H} \oplus (E_0(s,t)\mathscr{H})^\perp$. Then $|T|^s x \in |T|^s E_0\mathscr{H} \subseteq E_0(s,t)\mathscr{H}$, $|T|^s x_1 \in E_0(s,t)\mathscr{H}$ and $|T|^s x_2 \in (E_0(s,t)\mathscr{H})^\perp$ by Equality (2.2), and $E_0(s,t)\mathscr{H}$ is reduced by $|T|^s$. $\square$

A bounded linear operator T is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue.

THEOREM 2.11. *Let* T *belong to the class* $A(s,t)$ *for any* $0 < s,t \leqslant 1$*. Then* T *is isoloid.*

Proof. Let λ be an isolated point in $\sigma(T)$ and let E_λ be the associated Riesz idempotent.

If $\lambda = 0$, then $\sigma(T|_{ran(E_\lambda)}) = \{0\}$. Since $T|_{ran(E_\lambda)}$ is class $A(s,t)$ operator for each $s > 0$ and $t > 0$, so it follows by Lemma 2.8 that $T|_{ran(E_\lambda)} = 0$. Therefore $\lambda = 0$ is an eigenvalue of T.

If $\lambda \neq 0$, then $T|_{ran(E_\lambda)}$ is invertible belongs to the class $A(s,t)$ for each $s > 0$ and $t > 0$ operator and hence $(T|_{ran(E_\lambda)})^{-1}$ is also belong to the class $A(s,t)$ for each $s > 0$ and $t > 0$. By [39], we see $\left\|T|_{ran(E_\lambda)}\right\| = |\lambda|$ and $\left\|(T|_{ran(E_\lambda)})^{-1}\right\| = \frac{1}{|\lambda|}$. Let $x \in ran(E_\lambda)$ be an arbitrary vector. Then $\|x\| \leqslant \left\|(T|_{ran(E_\lambda)})^{-1}\right\| \left\|T|_{ran(E_\lambda)}x\right\| = \frac{1}{|\lambda|}\left\|T|_{ran(E_\lambda)}x\right\| \leqslant \frac{1}{|\lambda|}|\lambda|\,\|x\| = \|x\|$. This implies that $\frac{1}{\lambda}T|_{ran(E_\lambda)}$ is unitary with its spectrum $\sigma(\frac{1}{\lambda}T|_{ran(E_\lambda)}) = \{1\}$. Hence $T|_{ran(E_\lambda)} = \lambda$ and λ is an eigenvalue of T. This ends the proof. □

T is called *regular* if there exist a bounded linear operator T' such that $T = TT'T$. A bounded linear operator is said to be *reguloid* if $T-\lambda$ is regular of every $\lambda \in iso\,\sigma(T)$.

THEOREM 2.12. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then T is reguloid.*

Proof. If $\lambda \in iso\,\sigma(T)$, then $\mathscr{H} = E_\lambda\mathscr{H} + (I-E_\lambda)\mathscr{H}$, where E_λ and $(I-E_\lambda)\mathscr{H}$ are topologically complemented. By $T = T|_{E_\lambda\mathscr{H}} + T|_{(I-E_\lambda)\mathscr{H}}$ and Theorem 2.10, we have

$$(T-\lambda)\mathscr{H} = (T|_{(I-E_\lambda)\mathscr{H}} - \lambda)(I-E_\lambda)\mathscr{H}. \tag{2.3}$$

Therefore $(T-\lambda)\mathscr{H}$ is closed because $\sigma(T|_{(I-E_\lambda)\mathscr{H}}) = \sigma(T) - \{\lambda\}$. □

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\mathscr{R}(T^q) = \mathscr{R}(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$.

THEOREM 2.13. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then T is of finite ascent.*

Proof. Let $x \in \ker(T^2)$, then $\|Tx\|^2 \leqslant \|T^2x\| = 0$, and so $x \in \ker(T)$. Since the non-zero eigenvalues of class $A(s,t)$ operators for each $s > 0$ and $t > 0$ are normal eigenvalues of T by Theorem 2.1, if $0 \neq \lambda \in \sigma_p(T)$ and $(T-\lambda)^2x = 0$, then $(T-\lambda)(T-\lambda)x = 0 = (T-\lambda)^*(T-\lambda)x$ and $\|(T-\lambda)x\|^2 = \langle (T-\lambda)^*(T-\lambda)x, x\rangle = 0$. Hence, if T belongs to class $A(s,t)$ for every $s > 0$ and $t > 0$, then $a(T-\lambda) = 1$. □

Let $Hol(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [20], we say that $T \in \mathbf{B}(\mathscr{H})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathscr{H}$ which satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathbf{B}(\mathscr{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} - \sigma(T)$ and at every isolated point of $\sigma(T)$. If T has SVEP, then for each λ, $T-\lambda$ is invertible if and only if it is surjective (see [20, 32]).

In [31, Proposition 1.8], its proved that if T is of finite ascent, then T has SVEP. Hence it follows from last Theorem.

COROLLARY 2.14. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then T has the SVEP.*

Following [32], an operator T is said to have *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if for every open neighborhood G of λ, the function $f_n \in Hol(\sigma(T))$ with $(T-\lambda)f_n(\mu) \to 0$ uniformly on every compact subset of G implies that $f_n(\mu) \to 0$ uniformly on every compact subset of G. When T has Bishop's property (β) at each $\lambda \in \mathbb{C}$, simply say that T has property (β).

LEMMA 2.15. ([30]) *Let G be open subset of complex plane $\mathbb{C}$ and let $f_n \in Hol(G)$ be functions such that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G, then $f_n(\mu) \to 0$ uniformly on every compact subset of G.*

In [19] it is shown that every p-hyponormal operator has Bishop's property (β).

THEOREM 2.16. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then T has the property (β).*

Proof. Since $\widetilde{T}_{s,t}$ is $\frac{min(s,t)}{s+t}$-hyponormal ([28]) it is suffices to show that T has property (β) if and only if $\widetilde{T}_{s,t}$ has property (β).

Let G be an open neighborhood of λ and let $f_n \in Hol(\sigma(T))$ be functions such that $(\mu - \widetilde{T}_{s,t})f_n(\mu) \to 0$ uniformly on every compact subset of G. By Equations 1.2, $(\mu - T)(U|T|^s f_n(\mu)) = U|T|^s(\mu - \widetilde{T}_{s,t})f_n(\mu) \to 0$ uniformly on every compact subset of G. Hence $\widetilde{T}_{s,t} f_n(\mu) \to 0$ uniformly on every compact subset of G, and $\widetilde{T}_{s,t}$ having property β follows by Lemma 2.15.

Suppose that $\widetilde{T}_{s,t}$ has property (β). Let G be an open neighborhood of λ and let $f_n \in Hol(\sigma(T))$ be functions such that $(\mu - T)f_n(\mu) \to 0$ uniformly on every compact subset of G. Since $(\widetilde{T}_{s,t} - \mu)|T|^s f_n(\mu) = |T|^s(T-\mu)f_n(\mu) \to 0$ uniformly on every compact subset of G. Hence $Tf_n(\mu) = U|T|^s|T|^t f_n(\mu) \to 0$ uniformly on every compact subset of G for $\widetilde{T}_{s,t}$ has property $(\beta$, so that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G, and T has property (β) follows by Lemma 2.15. $\square$

3. Weyl's Theorem

Let $\alpha(T) := \dim\ker(T)$, $\beta(T) := \dim \mathscr{R}(T)$. An operator $T \in \mathbf{B}(\mathscr{H})$ is called *upper semi-Fredholm*, $T \in \Phi_+(\mathscr{H})$, if $\mathscr{R}(T)$ is closed and $\alpha(T) < \infty$. T is *lower semi-Fredholm*, $T \in \Phi_-(\mathscr{H})$, if $\beta(T) < \infty$. T is semi-Fredhom, $T \in \Phi_\pm(\mathscr{H})$, if $T \in \Phi_-(\mathscr{H}) \cup \Phi_+(\mathscr{H})$ and T is called Fredhom, $T \in \Phi(\mathscr{H})$, if $T \in \Phi_-(\mathscr{H}) \cap \Phi_+(\mathscr{H})$. The index of a semi-Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T).$$

T is called *Weyl* if it is Fredholm of index 0, and *Browder* if it is Fredholm "of finite ascent and descent".

The essential spectrum $\sigma_F(T)$, the Weyl spectrum $\sigma_W(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_F(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$$

respectively. Evidently

$$\sigma_F(T) \subseteq \sigma_W(T) \subseteq \sigma_b(T) \subseteq \sigma_F(T) \cup acc\sigma(T)$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$.

Let

$$\Phi_+^-(\mathscr{H}) := \{T \in \Phi_+(\mathscr{H}) : i(T) \leqslant 0\}.$$

Let $\mathbf{K}(\mathscr{H})$ denote the ideal of compact operators in $\mathbf{B}(\mathscr{H})$, and consider the following spectral subsets:

$$\sigma_{aW}(T) := \bigcap \{\sigma_a(T+S) : S \in \mathbf{K}(\mathscr{H})\} = \left\{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(\mathscr{H})\right\},$$
$$\sigma_{ab}(T) := \bigcap \{\sigma_a(T+S) : S \in \mathbf{K}(\mathscr{H}) \quad \text{and} \quad TS = ST\}.$$

Following [14], we say that *Weyl's theorem* holds for T if $\sigma(T) \setminus \sigma_W(T) = \pi_{00}(T)$, where $\pi_{00}(T)$ is the set of all isolated point in $\sigma(T)$ which are eigenvalues of finite multiplicity. And *Browder's theorem* holds for T if $\sigma(T) \setminus \sigma_W(T) = \pi_0(T)$, where π_0 is the set of all poles of T of finite rank, and that T satisfies *a-Browder's theorem* if $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \pi_0^a(T)$, where $\pi_0^a(T)$ is the set of left poles of finite rank.

According to [34], we say that T satisfies *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{aW}(T) = \pi_{a0}(T)$, where $\pi_{a0}(T)$ is the set of all isolated point in $\sigma_a(T)$ which are eigenvalues of finite multiplicity. It follows from [18, 34] that

$$a\text{-Weyl's theorem} \Longrightarrow \text{Weyl's theorem} \Longrightarrow \text{Browder's theorem}$$
$$a\text{-Weyl's theorem} \Longrightarrow a\text{-Browder's theorem} \Longrightarrow \text{Browder's theorem.}$$

The investigation of operators obeying Weyl's theorem, a-Weyl's theorem, Browder's theorem or a-Browder's theorem was studied by many mathematicians [1, 2, 14, 16, 18, 26, 33, 34] and the references cited therein.

THEOREM 3.1. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then the following assertions hold.*

(1) *Weyl's theorem holds for T.*

(2) *$\Sigma(f(T)) = f(\Sigma(T))$ for every $f \in Hol(\sigma(T))$, where $\Sigma(T)$ denotes either of $\sigma_W(T)$ or $\sigma_{aW}(T)$.*

(3) *Weyl's theorem holds for $f(T)$ when $f \in Hol(\sigma(T))$.*

Proof. (1) Recall that if T belongs to the class $A(s,t)$ for some $0 < s,t \leqslant 1$ and is Fredholm, then $i(T) \leqslant 0$. Indeed, since T is Fredholm, $|T|^s$ is also Fredholm and $i(|T|^s) = 0$. By Equations 1.1 and 1.2,

$$i(T) = i(|T|^s T) = i(\widetilde{T}_{s,t}|T|^s) = i(\widetilde{T}_{s,t}). \tag{3.1}$$

Hence, $i(T) \leqslant 0$ implies $i(\widetilde{T}_{s,t}) \leqslant 0$.

Let $\lambda \in \sigma(T) \setminus \sigma_W(T)$, then $T - \lambda$ is Fredholm, $i(T - \lambda) = 0$, and $\dim\ker(T - \lambda) > 0$.

If λ is an interior point of $\sigma(T)$, there would be an open subset $G \subseteq \sigma(T)$ including λ such that $i(T - \mu) = i(T - \lambda) = 0$ for all $\mu \in G$ (see [15]). So $\dim\ker(T - \mu) > 0$ for all $\mu \in G$, this is impossible for T has SVEP by Theorem 2.16 and [20, Theorem 10]. Thus $\lambda \in \partial\sigma(T) \setminus \sigma_W(T)$, $\lambda \in iso\,\sigma(T)$ (see [15, Theorem 6.8, page 366]), and $\lambda \in \pi_{00}(T)$ follows.

Let $\lambda \in \pi_{00}(T)$, then the Riesz idempotent P_λ has finite rank by Theorem 2.10, and $\lambda \in \sigma(T) \setminus \sigma_W(T)$ follows.

(2) Since T has the SVEP then it follows from [16, Corollary 2.6] and [16, Theorem 3.1] that $\sigma_W(f(T)) = f(\sigma_W(T))$ and $\sigma_{aW}(f(T)) = f(\sigma_{aW}(T))$ for every $f \in Hol(\sigma(T))$.

(3) Since T is isoloid by Theorem 2.11, has the SVEP (Corollary 2.14) and satisfies Weyl's theorem, then it follows from [16, Theorem 2.5] that $f(T)$ satisfies Weyl's theorem. □

THEOREM 3.2. ([39]) *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. If $\widetilde{T}_{s,t}$ is normal, then T is normal.*

THEOREM 3.3. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$, then the following assertions hold.*

(1) *if $m_2(\sigma(T)) = 0$ where m_2 means the planer Lebesgue measure, then T is normal.*

(2) *If $\sigma_W(T) = \{0\}$, then T is compact and normal.*

Proof. (1) It follows from $(\frac{min(s,t)}{s+t})$-hyponormality and Putnam's inequality for $(\frac{min(s,t)}{s+t})$-hyponormal operators [12], that $\widetilde{T}_{s,t}$ is normal. Hence, (1) follows by Theorem 3.2.

(2) Since $\sigma_W(T) = 0$, $\sigma(T) \setminus \{0\} = \pi_0(T) \subseteq iso\,\sigma(T)$. Hence $m_2(\sigma(T)) = 0$ and T is normal by (1).

Next to prove that T is compact, we may assume that $\sigma(T) \setminus \{0\}$ is a countable infinite set for $\sigma(T) \setminus \{0\} \subseteq iso\,\sigma(T)$. Let $\sigma(T) \setminus \{0\} = \{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ and $\mu = \lim_{n\to\infty} |\lambda_n|$, then $\mu = 0$. Since every E_{λ_n} has finite rank by Theorem 2.10, for every $\varepsilon > 0$, $\bigoplus_{|\lambda_n| > \varepsilon} E_{\lambda_n}$ also has finite rank. Therefore T is compact (See [15, page 271]). □

Recall that an operator T satisfies *a-Browder's theorem* if and only if $\sigma_{ab}(T) = \sigma_{aW}(T)$.

THEOREM 3.4. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then T and T^* satisfy a-Browder's theorem.*

Proof. Recall [2, Corollary 2.4] that if either T or T^* has SVEP, then both T and T^* satisfy a-Browder's theorem. So the result is an immediate consequence of Corollary 2.14. □

LEMMA 3.5. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. If $\overline{\lambda} \in \pi_{00}(T^*)$, then it is a pole of the resolvent of T^*.*

Proof. If $0 \neq \overline{\lambda} \in \pi_{00}(T^*)$, then $\lambda \in iso\,\sigma(T)$, then it follows from Theorem 2.1 that λ is a normal eigenvalue of T, and hence a simple pole of the resolvent of T by Theorem 2.10. If, instead, $\lambda = 0$, then $\dim\ker(T^*) < \infty$ implies $\mathscr{R}(T^*)$ is closed and hence $T^* \in \Phi_+(\mathscr{H})$ implies $T \in \Phi_-(\mathscr{H})$. Since both T and T^* has SVEP at 0, it follows that, $a(T) = d(T) < \infty$ by [1, Theorem 2.3]. Hence 0 is a pole of the resolvent of T implies 0 is a pole of the resolvent of T^*. □

THEOREM 3.6. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then $f(T^*)$ satisfies a-Weyl's theorem for every $f \in Hol(\sigma(T))$.*

Proof. Recall from [1, Theorem 3.6] that for a Banach space operator T with SVEP, T^* satisfies Weyl's theorem if and only if T^* satisfies a-Weyl's theorem. Since the SVEP is stable under functional calculus, T has SVEP implies $f(T)$ has SVEP for every $f \in Hol(\sigma(T))$. It will suffices to prove that $f(T^*)$ satisfies Weyl's theorem. Observe that if T belongs to the class $A(s,t)$ for each $s > 0$ and $t > 0$, then $a(T-\lambda) < \infty \Longrightarrow i(T-\lambda) \leqslant 0 \Longrightarrow i(T^* - \overline{\lambda}) \geqslant 0$ for every λ; hence $\sigma_W(T) = \overline{\sigma_W(T^*)}$ and $\sigma(f(T^*)) = f(\sigma(T^*)))$ it follows from Theorem 2.11 and proof of Theorem 3.1 that it will suffice to prove that T^* satisfies Weyl's theorem. Since T^* satisfies Browder's theorem by Theorem 3.4, $\sigma(T^*) \setminus \sigma_W(T^*) = \pi_0(T^*) \subseteq \pi_{00}(T^*)$. Let $\overline{\lambda} \in \pi_{00}(T^*)$, then $\overline{\lambda} \in \pi_0(T^*)$ by Lemma 3.5. Hence $\pi_0(T^*) = \pi_{00}(T^*)$. Therefore T^* satisfies Weyl's theorem. □

THEOREM 3.7. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. If F is an operator that commutes with T and for which there exits a positive integer n such that F^n is finite rank, then $T+F$ satisfies Weyl's theorem.*

Proof. Form Theorem 2.11 and Theorem 3.1, T is isoloid and satisfies Weyl's theorem. Now the result follows at once from [33, Theorem 2.4]. □

4. Applications to generalized Weyl's theorem

For each nonnegative integer n define T_n to be the restriction of T to $\mathscr{R}(T^n)$ viewed as a map from $\mathscr{R}(T^n)$ into $\mathscr{R}(T^n)$ (in particular $T_0 = T$). If for some n, $\mathscr{R}(T^n)$ is closed and T_n is an upper (resp. lower) semi-Fredholm operator then T is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or lower semi-B-Fredholm operators. If moreover, T_n is a Fredholm operator then T is called a *B-Fredholm* operator. The *index* of a semi-B-Fredholm is defined as the index of the semi-Fredholm operator T_n (see [10]). In [10] it is proved that an operator T is a B-Fredholm operator if and only if $T = F \oplus N$, where F is a Fredholm operator and N is a nilpotent operator. An operator $T \in \mathscr{B}(H)$ is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

Following [10] *generalized Weyl's theorem* holds for T if

$$\sigma(T) \setminus E(T) = \sigma_{BW}(T); \tag{4.1}$$

where $E(T)$ is the set of all isolated eigenvalues of T and *generalized Browder's theorem* holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T);$$

where $\pi(T)$ is the set of all poles of T.

Let $SBF_+(H)$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(H)$ the class of all $T \in SBF_+(H)$ such that $\operatorname{ind}(T) \leqslant 0$. Also let

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not in } SBF_+^-(H)\},$$

called the *semi-B-essential approximate point spectrum*. We say that T obeys *generalized a-Weyl's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus E^a(T),$$

where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_{ap}(T)$ ([10, Definition 2.13]). This also gives a generalization of a-Weyl's theorem. We say that T obeys *generalized a-Browder's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus \pi^a(T).$$

From [10, 34] we have the following implications

generalized a-Weyl's theorem $\Rightarrow$ generalized Weyl's theorem $\Rightarrow$ Weyl's theorem ,

generalized a-Weyl's theorem $\Rightarrow$ a-Weyl's theorem $\Rightarrow$ Weyl's theorem .

Recently, in [7] it is proved that

generalized Browder's theorem $\Leftrightarrow$ Browder's theorem,

generalized a-Browder's theorem $\Leftrightarrow$ a-Browder's theorem.

PROPOSITION 4.1. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then*

$$E(T) = \pi(T).$$

Proof. Let $\lambda \in E(T)$ then $\lambda \in iso\,\sigma(T)$. If $\lambda \neq 0$, then it follows from Theorem 2.1 that λ is a normal eigenvalue of T, and hence a simple pole of the resolvent of T by Theorem 2.10. Now if $\lambda = 0$, then T is a B-Weyl operator. For n large enough, $T - \frac{1}{n}$ is invertible then in particular a Fredholm operator. Since $T - \frac{1}{n}$ and $T^* - \frac{1}{n}$ have the SVEP at 0, then it follows from [3, Theorem 2.6 and Theorem 2.8] that $a(T - \frac{1}{n}) = d(T - \frac{1}{n})$. Hence by [25, Theorem 4.7] we have $a(T) = d(T) < \infty$. Therefore $0 \in \pi(T)$. □

THEOREM 4.2. *Let T belong to the class $A(s,t)$ for some $0 < s,t \leqslant 1$. Then the following assertions hold.*

(1) *generalized Weyl's theorem holds for T.*

(2) $\Sigma(f(T)) = f(\Sigma(T))$ *for every $f \in Hol(\sigma(T))$, where $\Sigma(T)$ denotes either of $\sigma_{BW}(T)$ or $\sigma_{SBF_+^-}(T)$.*

(3) *generalized Weyl's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.*

Proof. (1) Since $E(T) = \pi(T)$ by Proposition 4.1 and Weyl's theorem holds for T then it follows from [7, Corollary 2.1] that generalized Weyl's theorem holds for T.

(2) Since T satisfies the SVEP, then the result follows from [43, Theorem 2.1 and Theorem 2.3].

(3) T is isoloid, satisfies the SVEP and generalized Weyl's theorem holds for T, then it follows from [43, Theorem 2.2] that generalized Weyl's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$. □

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