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HÁJEK–RÉNYI–TYPE INEQUALITIES AND LAWS OF LARGE NUMBERS FOR L_q–MIXINGALE ARRAY

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Abstract. In the paper, we get the precise results of maximal inequalities and Hájek-Rényi-type inequalities for the partial sums of L_q-mixingale array ($q > 1$), which generalize and improve the results of Theorem 1 and Theorem 2 in Meng and Lin [8]. We also get the strong law of large numbers, strong growth rate and the integrability of supremum for L_q-mixingale sequence ($q > 1$), which generalize and improve the results of Corollary 1 in the above cited reference and Corollary 2 in Hansen [4]. At last, a weak law of large numbers for L_q-mixingale array ($q \geqslant 2$) is given.

1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a fixed probability space. The following random variables that we deal with are all defined on $(\Omega, \mathscr{F}, P)$. Let $\{X_{nt} : 1 \leqslant t \leqslant k_n \uparrow \infty; n \geqslant 1\}$ be a triangular array of random variables and $\{\mathscr{F}_t^n : t = \cdots, -1, 0, 1, \cdots; n = 1, 2, \cdots\}$ be an array of σ-subfields of $\mathscr{F}$ such that $\{\mathscr{F}_t^n\}$ for each $n \geqslant 1$ is nondecreasing in t. Whenever an expression like $E(X_{nt}|\mathscr{F}_j^n)$ is used, it is assumed that $\| X_{nt} \|_1$ is finite, where $\| X \|_q$, $q \geqslant 1$, is defined as $(E|X|^q)^{1/q}$.

DEFINITION 1.1. The triangular array $\{X_{nt}, \mathscr{F}_t^n\}$ is a triangular L_q-mixingale array, $q \geqslant 1$, if there exist nonnegative constants $\{c_{nt} : 1 \leqslant t \leqslant k_n \uparrow \infty; n \geqslant 1\}$ and $\{\psi(m) : m \geqslant 0\}$ such that $\psi(m) \downarrow 0$ as $m \to \infty$ and for all $t = 1, 2, \cdots, k_n$, $n \geqslant 1$ and $m \geqslant 0$, we have

(a) $\| E(X_{nt}|\mathscr{F}_{t-m}^n) \|_q \leqslant c_{nt}\psi(m)$,

(b) $\| X_{nt} - E(X_{nt}|\mathscr{F}_{t+m}^n) \|_q \leqslant c_{nt}\psi(m+1)$.

Similarly we shall give the definition of L_q-mixingale sequence. Let $\{X_n : n \geqslant 1\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathscr{F}, P)$ and $\{\mathscr{F}_n : n = \cdots, -1, 0, 1, \cdots\}$ be a sequence of σ-subfields of $\mathscr{F}$ such that $\{\mathscr{F}_n\}$ is nondecreasing in n.

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DEFINITION 1.2. The sequence $\{X_n, \mathscr{F}_n\}$ is an L_q-mixingale sequence, $q \geqslant 1$, if there exist nonnegative constants $\{c_n : n \geqslant 1\}$ and $\{\psi(m) : m \geqslant 0\}$ such that $\psi(m) \downarrow 0$ as $m \to \infty$ and for all $n \geqslant 1$ and $m \geqslant 0$, we have

(a) $\| E(X_n|\mathscr{F}_{n-m}) \|_q \leqslant c_n \psi(m)$,

(b) $\| X_n - E(X_n|\mathscr{F}_{n+m}) \|_q \leqslant c_n \psi(m+1)$.

The concept of mixingale was first introduced by McLeish [7]. He proved a maximal inequality and then proved strong laws of large numbers for L_2-mixingales under the conditions of be square integrable and the mixingale numbers of size $-1/2$. Andrews [1] extended the mixingale concept and established weak laws of large numbers for L_1-mixingales. In Andrews' theorems the L_1-mixingale numbers need not decay to zero at any particular rate. But they need the mixingales are uniformly integrable. Andrews pointed out that his conclusions for L_1-mixingale sequence can be extended to L_q-mixingale ($1 < q \leqslant 2$) sequence $\{X_n\}$ under the condition that $\{|X_n|^q\}$ is uniformly integrable.

Recently, Meng and Lin [8] obtained two inequalities of the maximum for the partial sums of L_q-mixingale array ($1 < q \leqslant 2$), a strong law of large numbers for L_q-mixingale sequence and a weak law of large numbers for L_q-mixingale array without the condition of uniform integrability which is needed in Andrews [1].

The main purpose of this paper is to study the maximal inequalities and the Hájek-Rényi-type inequalities for L_q-mixingale array ($q > 1$). In Section 2, we get the precise results of maximal inequalities and Hájek-Rényi-type inequalities for the partial sums of L_q-mixingale array ($q > 1$), which generalize and improve the results of Theorem 1 and Theorem 2 in Meng and Lin [8]. We also get the strong law of large numbers, strong growth rate and the integrability of supremum for L_q-mixingale sequence ($q > 1$) which generalize and improve the results of Corollary 1 in Meng and Lin [8] and Corollary 2 in Hansen [4]. In Section 3, a weak law of large numbers for L_q-mixingale array ($q \geqslant 2$) is given.

We use the same notations as that in Meng and Lin [8]. Let $\{X_{nt}, \mathscr{F}_t^n\}$ be a triangular L_q-mixingale array ($q > 1$). Denote

$$W_{t;i,n} = E\left(X_{nt}|\mathscr{F}_{t+i}^n\right) - E\left(X_{nt}|\mathscr{F}_{t+i-1}^n\right),$$
$$T_{j;i,n} = \sum_{t=1}^{j} W_{t;i,n} = \sum_{t=1}^{j} \left[E\left(X_{nt}|\mathscr{F}_{t+i}^n\right) - E\left(X_{nt}|\mathscr{F}_{t+i-1}^n\right)\right].$$

The main results of this paper depend on the following lemmas.

LEMMA 1.1. *For $q > 1$, we have*

$$\|W_{t;i,n}\|_q \leqslant 2c_{nt}\psi(|i|). \tag{1.1}$$

Proof. For $i < 0$, by Minkowski's inequality and Definition 1.1, we can get

$$\begin{aligned} \| W_{t;i,n} \|_q &\leqslant \| E(X_{nt}|\mathscr{F}_{t+i}^n) \|_q + \| E(X_{nt}|\mathscr{F}_{t+i-1}^n) \|_q \\ &\leqslant c_{nt}\psi(-i) + c_{nt}\psi(-i+1) \\ &\leqslant 2c_{nt}\psi(-i) = 2c_{nt}\psi(|i|), \end{aligned}$$

while for $i \geqslant 0$,

$$\begin{aligned}\|W_{t;i,n}\|_q &= \|(X_{nt} - E(X_{nt}|\mathscr{F}^n_{t+i})) - (X_{nt} - E(X_{nt}|\mathscr{F}^n_{t+i-1}))\|_q \\ &\leqslant \|X_{nt} - E(X_{nt}|\mathscr{F}^n_{t+i})\|_q + \|X_{nt} - E(X_{nt}|\mathscr{F}^n_{t+i-1})\|_q \\ &\leqslant c_{nt}\psi(i+1) + c_{nt}\psi(i) \\ &\leqslant 2c_{nt}\psi(i) = 2c_{nt}\psi(|i|).\end{aligned}$$

Thus for all i, it follows that

$$\|W_{t;i,n}\|_q \leqslant 2c_{nt}\psi(|i|). \qquad \square$$

LEMMA 1.2. (c.f. [3, Doob's inequality]). *If $\{S_i, \mathscr{F}_i, 1 \leqslant i \leqslant n\}$ is a martingale, then for $p > 1$*

$$\left\| \max_{1\leqslant i\leqslant n} |S_i| \right\|_p \leqslant \frac{p}{p-1} \|S_i\|_p .$$

LEMMA 1.3. (c.f. [3, Burkholder's inequality]). *Let $\{X_i, \mathscr{F}_i, 1 \leqslant i \leqslant n\}$ be a martingale difference sequence and $p > 1$. Denote $S_n = \sum_{i=1}^n X_i$. Then there exist constants C_1 and C_2 depending only on p such that*

$$C_1 E\left|\sum_{i=1}^n X_i^2\right|^{p/2} \leqslant E|S_n|^p \leqslant C_2 E\left|\sum_{i=1}^n X_i^2\right|^{p/2},$$

where $C_1^{-1} = \left(18p^{1/2}q\right)^p$, $C_2 = \left(18pq^{1/2}\right)^p$ and $p^{-1} + q^{-1} = 1$.

LEMMA 1.4. (cf. [6, Lemma 1.5]). *Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables. Denote $S_n = \sum_{i=1}^n X_i$ for each $n \geqslant 1$. Let $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers and $\alpha_1, \alpha_2, \cdots$ be nonnegative numbers. Let r and C be fixed positive numbers. Assume that for each $n \geqslant 1$*

$$E\left(\max_{1\leqslant l\leqslant n} |S_l|\right)^r \leqslant C\sum_{l=1}^n \alpha_l, \tag{1.2}$$

$$\sum_{l=1}^{\infty} \frac{\alpha_l}{b_l^r} < \infty, \tag{1.3}$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \ \ a.s., \tag{1.4}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \ \ a.s., \tag{1.5}$$

where

$$\beta_n = \max_{1\leqslant k\leqslant n} b_k v_k^{\delta/r}, \ \forall\, 0 < \delta < 1, \ v_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{b_k^r}, \ \lim_{n\to\infty} \frac{\beta_n}{b_n} = 0. \tag{1.6}$$

And

$$E\left(\max_{1\leqslant l\leqslant n}\left|\frac{S_l}{b_l}\right|\right)^r \leqslant 4C\sum_{l=1}^{n}\frac{\alpha_l}{b_l^r} < \infty, \tag{1.7}$$

$$E\left(\sup_{l\geqslant 1}\left|\frac{S_l}{b_l}\right|\right)^r \leqslant 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty. \tag{1.8}$$

If further we assume that $\alpha_n > 0$ for infinitely many n, then

$$E\left(\sup_{l\geqslant 1}\left|\frac{S_l}{\beta_l}\right|\right)^r \leqslant 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^r} < \infty. \tag{1.9}$$

We point out that (1.4), (1.7) and (1.8) are due to Fazekas and Klesov [2].

LEMMA 1.5. (cf. [2, Corollary 2.1] and [5, Corollary 2.1.1]). *Let $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers and $\alpha_1, \alpha_2, \cdots$ be nonnegative numbers. Denote $\Lambda_k = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ for $k \geqslant 1$. Let r be a fixed positive number satisfying (1.2). If*

$$\sum_{l=1}^{\infty}\Lambda_l\left(\frac{1}{b_l^r} - \frac{1}{b_{l+1}^r}\right) < \infty, \tag{1.10}$$

$$\frac{\Lambda_n}{b_n^r} \text{ is bounded}, \tag{1.11}$$

then (1.4)–(1.9) hold.

2. Maximal inequalities and Hájek-Rényi-type inequalities for L_q-mixingale array ($q > 1$)

THEOREM 2.1. *Let $\{X_{nt}, \mathscr{F}_t^n\}$ be a triangular L_q-mixingale array ($q > 1$) such that $\sum_{m=1}^{\infty}\psi(m) < \infty$. Then for all $n \geqslant 1$,*

$$E\left(\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}X_{nt}\right|\right)^q \leqslant K^q\sum_{t=1}^{k_n}c_{nt}^q, \quad 1 < q \leqslant 2, \tag{2.1}$$

and

$$E\left(\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}X_{nt}\right|\right)^q \leqslant K^q\left(\sum_{t=1}^{k_n}c_{nt}^2\right)^{q/2}, \quad q > 2, \tag{2.2}$$

where $K = \frac{36q^2}{q-1}\left(\frac{q}{q-1}\right)^{1/2}\sum\limits_{i=-\infty}^{\infty}\psi(|i|)$.

Proof. Using the standard method, it is easily shown that for fixed $n \geqslant 1$,

$$X_{nt} = \sum_{i=-\infty}^{\infty}\left[E\left(X_{nt}|\mathscr{F}_{t+i}^n\right) - E\left(X_{nt}|\mathscr{F}_{t+i-1}^n\right)\right] \quad a.s., \ \forall\, t \geqslant 1, \tag{2.3}$$

and $\sum_{i=-\infty}^{\infty}\left[E\left(X_{nt}|\mathscr{F}_{t+i}^n\right)-E\left(X_{nt}|\mathscr{F}_{t+i-1}^n\right)\right]$ converges almost surely (or see Meng and Lin [8]).

Note that for fixed n and i, $\{W_{t;i,n},\mathscr{F}_{t+i}^n,t\geqslant 1\}$ is a martingale difference sequence. Indeed, by the property of conditional expectation, it follows that

$$\begin{aligned}E(W_{t;i,n}|\mathscr{F}_{t+i-1}^n)&=E\left[E(X_{nt}|\mathscr{F}_{t+i}^n)|\mathscr{F}_{t+i-1}^n\right]-E\left[E(X_{nt}|\mathscr{F}_{t+i-1}^n)|\mathscr{F}_{t+i-1}^n\right]\\&=E(X_{nt}|\mathscr{F}_{t+i-1}^n)-E(X_{nt}|\mathscr{F}_{t+i-1}^n)\\&=0\ \ a.s.\end{aligned}$$

Now for $1<q\leqslant 2$,

$$\begin{aligned}\left\|\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}X_{nt}\right|\right\|_q&=\left\|\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}\sum_{i=-\infty}^{\infty}W_{t;i,n}\right|\right\|_q\\&\leqslant\left\|\sum_{i=-\infty}^{\infty}\left(\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}W_{t;i,n}\right|\right)\right\|_q\\&\leqslant\sum_{i=-\infty}^{\infty}\left\|\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}W_{t;i,n}\right|\right\|_q\leqslant\sum_{i=-\infty}^{\infty}\frac{q}{q-1}\left\|\sum_{t=1}^{k_n}W_{t;i,n}\right\|_q\\&\leqslant\frac{q}{q-1}\sum_{i=-\infty}^{\infty}18q\left(\frac{q}{q-1}\right)^{1/2}\left[E\left(\sum_{t=1}^{k_n}W_{t;i,n}^2\right)^{q/2}\right]^{1/q}\\&\leqslant\frac{18q^2}{q-1}\left(\frac{q}{q-1}\right)^{1/2}\sum_{i=-\infty}^{\infty}\left(\sum_{t=1}^{k_n}E\left|W_{t;i,n}\right|^q\right)^{1/q}\\&\doteq\ \tilde{K}\sum_{i=-\infty}^{\infty}\left(\sum_{t=1}^{k_n}E\left|W_{t;i,n}\right|^q\right)^{1/q}.\end{aligned}\tag{2.4}$$

The equality in the first line follows from (2.3). The five inequalities due to the triangle inequality, Minkowski's inequality, Doob's inequality, Burkholder's inequality and C_r inequality, respectively. The final equality sets $\tilde{K}=\frac{18q^2}{q-1}\left(\frac{q}{q-1}\right)^{1/2}<\infty$.

Together with Lemma 1.1 and (2.4), it follows that

$$\begin{aligned}\left\|\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}X_{nt}\right|\right\|_q&\leqslant\tilde{K}\sum_{i=-\infty}^{\infty}\left(\sum_{t=1}^{k_n}E\left|W_{t;i,n}\right|^q\right)^{1/q}\\&\leqslant\tilde{K}\sum_{i=-\infty}^{\infty}\left(\sum_{t=1}^{k_n}\left[2c_{nt}\psi(|i|)\right]^q\right)^{1/q}\end{aligned}$$

$$= 2\tilde{K} \sum_{i=-\infty}^{\infty} \psi(|i|) \left(\sum_{t=1}^{k_n} c_{nt}^q \right)^{1/q}$$
$$\doteq K \left(\sum_{t=1}^{k_n} c_{nt}^q \right)^{1/q},$$

Which implies (2.1).

For $q > 2$, similar to the proof of (2.4), we have

$$\begin{aligned} \left\| \max_{1 \leqslant j \leqslant k_n} \left| \sum_{t=1}^{j} X_{nt} \right| \right\|_q &\leqslant \frac{18q^2}{q-1} \left(\frac{q}{q-1} \right)^{1/2} \sum_{i=-\infty}^{\infty} \left[E \left(\sum_{t=1}^{k_n} W_{t;i,n}^2 \right)^{q/2} \right]^{1/q} \\ &\leqslant \frac{18q^2}{q-1} \left(\frac{q}{q-1} \right)^{1/2} \sum_{i=-\infty}^{\infty} \left[\left(\sum_{t=1}^{k_n} \left\| W_{t;i,n}^2 \right\|_{q/2} \right)^{q/2} \right]^{1/q} \\ &= \frac{18q^2}{q-1} \left(\frac{q}{q-1} \right)^{1/2} \sum_{i=-\infty}^{\infty} \left(\sum_{t=1}^{k_n} \| W_{t;i,n} \|_q^2 \right)^{1/2}. \end{aligned}$$

The second inequality above follows from Minkowski's inequality. Together with Lemma 1.1 and the above inequality, it is easily seen that

$$\begin{aligned} \left\| \max_{1 \leqslant j \leqslant k_n} \left| \sum_{t=1}^{j} X_{nt} \right| \right\|_q &\leqslant \frac{18q^2}{q-1} \left(\frac{q}{q-1} \right)^{1/2} \sum_{i=-\infty}^{\infty} \left(\sum_{t=1}^{k_n} [2c_{nt} \psi(|i|)]^2 \right)^{1/2} \\ &\doteq K \left(\sum_{t=1}^{k_n} c_{nt}^2 \right)^{1/2}, \end{aligned}$$

which implies (2.2). The proof is complete. □

COROLLARY 2.1. *Let $\{X_n, \mathscr{F}_n\}$ be an L_q-mixingale sequence ($1 < q \leqslant 2$) with respect to nonnegative constants $\{c_n, n \geqslant 1\}$ and assume that $\sum_{m=1}^{\infty} \psi(m) < \infty$. Denote $S_n = \sum_{i=1}^{n} X_i$ for each $n \geqslant 1$. Let $\{b_n, n \geqslant 1\}$ be a nondecreasing unbounded sequence of positive numbers. Assume that*

$$\sum_{n=1}^{\infty} \frac{c_n^q}{b_n^q} < \infty, \tag{2.5}$$

then (1.4)–(1.9) hold, where $r = q$, $C = K^q$, $\alpha_k = c_k^q$ for $k \geqslant 1$ and K is defined in Theorem 2.1.

Proof. It is easily seen that Theorem 2.1 is also true for L_q-mixingale sequence ($1 < q \leqslant 2$). Therefore,

$$E \left(\max_{1 \leqslant k \leqslant n} |S_k| \right)^q \leqslant K^q \sum_{k=1}^{n} c_k^q = C \sum_{k=1}^{n} \alpha_k, \tag{2.6}$$

and

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{b_n^q} = \sum_{n=1}^{\infty} \frac{c_n^q}{b_n^q} < \infty. \tag{2.7}$$

One can get (1.4)–(1.9) immediately by (2.6), (2.7) and Lemma 1.4. The desired result is obtained. □

COROLLARY 2.2. (Marcinkiewicz strong law of large numbers). *Let* $\{X_n, \mathscr{F}_n\}$ *be an* L_q*-mixingale sequence* ($q > 2$) *with respect to nonnegative constants* $\{c_n, n \geqslant 1\}$ *and assume that* $\sum_{m=1}^{\infty} \psi(m) < \infty$. $1 \leqslant p < 2$. *Denote* $S_n = \sum_{i=1}^{n} X_i$ *and* $Q_n = \max\limits_{1 \leqslant k \leqslant n} c_k^q$ *for each* $n \geqslant 1$. $Q_0 = 0$. *Assume that*

$$\sum_{n=1}^{\infty} \frac{Q_n}{n^{1+q(\frac{1}{p}-\frac{1}{2})}} < \infty, \tag{2.8}$$

then

$$\lim_{n \to \infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} X_i = 0 \ \ a.s., \tag{2.9}$$

and with the growth rate

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} X_i = O\left(\frac{\beta_n}{n^{1/p}}\right) \ \ a.s., \tag{2.10}$$

where

$$\beta_n = \max_{1 \leqslant k \leqslant n} k^{1/p} v_k^{\delta/q}, \ \forall\, 0 < \delta < 1, \ v_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{k^{q/p}}, \ \lim_{n \to \infty} \frac{\beta_n}{n^{1/p}} = 0,$$

$$\alpha_k = K^q (k^{q/2} Q_k - (k-1)^{q/2} Q_{k-1}), \ k \geqslant 1, \ K = \frac{36q^2}{q-1}\left(\frac{q}{q-1}\right)^{1/2} \sum_{i=-\infty}^{\infty} \psi(|i|). \tag{2.11}$$

And (1.7)–(1.9) hold with $C = 1$.

Proof. Denote $b_n = n^{1/p}$ and $\Lambda_n = \sum_{k=1}^{n} \alpha_k$, $n \geqslant 1$. It is easily seen that Theorem 2.1 is also true for L_q-mixingale sequence ($q > 2$). Therefore,

$$E\left(\max_{1 \leqslant k \leqslant n} |S_k|\right)^q \leqslant K^q \left(\sum_{k=1}^{n} c_k^2\right)^{q/2} \leqslant K^q n^{q/2} Q_n = \sum_{k=1}^{n} \alpha_k = \Lambda_n. \tag{2.12}$$

By (2.8), it follows that

$$\sum_{l=1}^{\infty} \Lambda_l \left(\frac{1}{b_l^q} - \frac{1}{b_{l+1}^q}\right) = K^q \sum_{l=1}^{\infty} l^{q/2} Q_l \left(\frac{1}{l^{q/p}} - \frac{1}{(l+1)^{q/p}}\right)$$
$$\leqslant \frac{qK^q}{p} \sum_{l=1}^{\infty} \frac{Q_l}{l^{1+q(\frac{1}{p}-\frac{1}{2})}} < \infty. \tag{2.13}$$

That is to say (1.10) holds. By Remark 2.1 in Fazekas and Klesov [2], (1.10) implies (1.11). Therefore, the desired results follow from Lemma 1.5 immediately. □

THEOREM 2.2. *Let* $\{X_{nt}, \mathscr{F}_t^n\}$ *be a triangular* L_q*-mixingale array* ($q>1$) *such that* $\sum_{m=1}^{\infty} \psi(m) < \infty$*. Suppose that* $\{b_t, t \geqslant 1\}$ *is a nondecreasing sequence of positive numbers. Then for all* $n \geqslant 1$ *and for any* $\varepsilon > 0$*,*

$$P\left(\max_{1\leqslant j\leqslant k_n}\left|\frac{\sum_{t=1}^{j} X_{nt}}{b_j}\right| \geqslant \varepsilon\right) \leqslant \frac{2^q K^q}{\varepsilon^q}\sum_{t=1}^{k_n}\frac{c_{nt}^q}{b_t^q}, \quad 1<q\leqslant 2, \tag{2.14}$$

and

$$P\left(\max_{1\leqslant j\leqslant k_n}\left|\frac{\sum_{t=1}^{j} X_{nt}}{b_j}\right| \geqslant \varepsilon\right) \leqslant \frac{2^q K^q}{\varepsilon^q}\left(\sum_{t=1}^{k_n}\frac{c_{nt}^2}{b_t^2}\right)^{q/2}, \quad q>2, \tag{2.15}$$

where K *is defined in Theorem 2.1.*

Proof. The proof is similar to that of Theorem 2 in Meng and Lin [8]. Meng and Lin have shown that

$$\left\{\max_{1\leqslant j\leqslant k_n}\left|\frac{\sum_{t=1}^{j} X_{nt}}{b_j}\right| \geqslant \varepsilon\right\} \subseteq \left\{\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}\frac{X_{nt}}{b_t}\right| \geqslant \frac{\varepsilon}{2}\right\}. \tag{2.16}$$

Since $b_t > 0$ for all $t \geqslant 1$, $\{X_{nt}/b_t, \mathscr{F}_t^n\}$ is also a triangular L_q-mixingale array with nonnegative constant sequences $\{c_{nt}/b_t\}$ and $\{\psi(m) : m \geqslant 0\}$ by the definition of L_q-mixingale array. By (2.16) and Markov's inequality, we have

$$\begin{aligned}P\left(\max_{1\leqslant j\leqslant k_n}\left|\frac{\sum_{t=1}^{j} X_{nt}}{b_j}\right| \geqslant \varepsilon\right) &\leqslant P\left(\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}\frac{X_{nt}}{b_t}\right| \geqslant \frac{\varepsilon}{2}\right) \\ &\leqslant \frac{2^q}{\varepsilon^q} E\left(\max_{1\leqslant j\leqslant k_n}\left|\sum_{t=1}^{j}\frac{X_{nt}}{b_t}\right|\right)^q.\end{aligned} \tag{2.17}$$

Therefore, (2.14) and (2.15) follow from (2.1) and (2.2), respectively. □

THEOREM 2.3. *Let* $\{X_{nt}, \mathscr{F}_t^n\}$ *be a triangular* L_q*-mixingale array* ($q>1$) *such that* $\sum_{m=1}^{\infty} \psi(m) < \infty$*. Suppose that* $\{b_n, n \geqslant 1\}$ *is a nondecreasing sequence of positive numbers. Then for any* $\varepsilon > 0$ *and for any positive integers* $m < k_n$*,*

$$P\left(\max_{m\leqslant j\leqslant k_n}\left|\frac{\sum_{t=1}^{j} X_{nt}}{b_j}\right| \geqslant \varepsilon\right) \leqslant \frac{2^q K^q}{\varepsilon^q}\left(\sum_{t=1}^{m}\frac{c_{nt}^q}{b_m^q} + 2^q\sum_{t=m+1}^{k_n}\frac{c_{nt}^q}{b_t^q}\right), \quad 1<q\leqslant 2, \tag{2.18}$$

and

$$P\left(\max_{m\leqslant j\leqslant k_n}\left|\frac{\sum_{t=1}^{j} X_{nt}}{b_j}\right| \geqslant \varepsilon\right) \leqslant \frac{2^q K^q}{\varepsilon^q}\left[\left(\sum_{t=1}^{m}\frac{c_{nt}^2}{b_m^2}\right)^{q/2} + 2^q\left(\sum_{t=m+1}^{k_n}\frac{c_{nt}^2}{b_t^2}\right)^{q/2}\right], \quad q>2, \tag{2.19}$$

where K is defined in Theorem 2.1.

Proof. Observe that

$$\max_{m\leqslant j\leqslant k_n}\left|\frac{1}{b_j}\sum_{t=1}^{j}X_{nt}\right|\leqslant\left|\frac{1}{b_m}\sum_{t=1}^{m}X_{nt}\right|+\max_{m+1\leqslant j\leqslant k_n}\left|\frac{1}{b_j}\sum_{t=m+1}^{j}X_{nt}\right|,$$

thus

$$P\left(\max_{m\leqslant j\leqslant k_n}\left|\frac{1}{b_j}\sum_{t=1}^{j}X_{nt}\right|\geqslant\varepsilon\right)\leqslant P\left(\left|\frac{1}{b_m}\sum_{t=1}^{m}X_{nt}\right|\geqslant\frac{\varepsilon}{2}\right)+P\left(\max_{m+1\leqslant j\leqslant k_n}\left|\frac{1}{b_j}\sum_{t=m+1}^{j}X_{nt}\right|\geqslant\frac{\varepsilon}{2}\right)\doteq I+II. \tag{2.20}$$

(i) If $1<q\leqslant 2$, then by Markov's inequality and (2.1), we have

$$I\leqslant\frac{2^q}{\varepsilon^q b_m^q}E\left|\sum_{t=1}^{m}X_{nt}\right|^q\leqslant\frac{2^qK^q}{\varepsilon^q}\sum_{t=1}^{m}\frac{c_{nt}^q}{b_m^q}. \tag{2.21}$$

For II, we will apply Theorem 2.1 to $\{X_{m+i},1\leqslant i\leqslant k_n-m\}$ and $\{b_{m+i},1\leqslant i\leqslant k_n-m\}$. Noting that

$$\max_{m+1\leqslant j\leqslant k_n}\left|\frac{1}{b_j}\sum_{t=m+1}^{j}X_{nt}\right|=\max_{1\leqslant j\leqslant k_n-m}\left|\frac{1}{b_{m+j}}\sum_{t=1}^{j}X_{n,m+t}\right|. \tag{2.22}$$

Thus, by (2.22) and (2.14), we can get

$$II=P\left(\max_{1\leqslant j\leqslant k_n-m}\left|\frac{1}{b_{m+j}}\sum_{t=1}^{j}X_{n,m+t}\right|\geqslant\frac{\varepsilon}{2}\right)\leqslant\frac{2^qK^q}{(\varepsilon/2)^q}\sum_{t=1}^{k_n-m}\frac{c_{n,m+t}^q}{b_{m+t}^q}=\frac{4^qK^q}{\varepsilon^q}\sum_{t=m+1}^{k_n}\frac{c_{nt}^q}{b_t^q}. \tag{2.23}$$

Therefore, the desired result (2.18) follows from (2.20)–(2.23) immediately.

(ii) For $q>2$, the proof of (2.19) is similar to that of (2.18), so we omit it. We complete the proof of the theorem. □

REMARK 2.1. (2.1) and (2.14) have been studied by Meng and Lin [8]. But we get the precise results of maximal inequalities and Hájek-Rényi-type inequalities for the partial sums of L_q-mixingale array ($q>1$). Therefore, the results of maximal inequalities and Hájek-Rényi-type inequalities for the partial sums of L_q-mixingale array generalize and improve the results of Theorem 1 and Theorem 2 in Meng and Lin [8].

REMARK 2.2. In Corollary 2.1, we not only get the strong law of large numbers, but also give the strong growth rate and the integrability of supremum for L_q-mixingale sequence ($1<q\leqslant 2$). Therefore, our results improve the results of Corollary 1 in Meng and Lin [8] and Corollary 2 in Hansen [4].

3. Weak law of large numbers for L_q-mixingale array ($q \geqslant 2$)

Andrews [1] established a weak law of large numbers for L_1- mixingale array as follows:

THEOREM A. *Suppose the triangular array* $\{X_{nt}, \mathscr{F}_t^n\}$ *is a uniformly integrable* L_1*-mixingale. If* $\limsup_{n\to\infty} \frac{1}{k_n}\sum_{t=1}^{k_n} c_{nt} < \infty$*, then* $E|\frac{1}{k_n}\sum_{t=1}^{k_n} X_{nt}| \to 0$ *as* $n \to \infty$ *and in consequence* $\frac{1}{k_n}\sum_{t=1}^{k_n} X_{nt} \xrightarrow{P} 0$ *as* $n \to \infty$.

Meng and Lin [8] extended the weak law of large numbers for L_1-mixingale array to the case of L_q-mixingale array ($1 < q \leqslant 2$) by weakening the condition of uniform integrability, and obtained the following result:

THEOREM B. *Suppose* $\{X_{nt}, \mathscr{F}_t^n\}$ *is a triangular* L_q*-mixingale array* ($1 < q \leqslant 2$) *such that the following conditions are satisfied:*

(a) $\limsup_{n\to\infty} \frac{1}{k_n}\sum_{t=1}^{k_n} c_{nt} < \infty$,

(b) $\limsup_{n\to\infty} \frac{1}{k_n^2}\sum_{t=1}^{k_n} c_{nt}^2 = 0$.

Then for any $r \in (1,q)$, $\|\frac{1}{k_n}\sum_{t=1}^{k_n} X_{nt}\|_r \to 0$ *as* $n \to \infty$*, and in consequence* $\frac{1}{k_n}\sum_{t=1}^{k_n} X_{nt} \xrightarrow{P} 0$ *as* $n \to \infty$.

In this section, we will extend the weak law of large numbers for L_q-mixingale array ($1 < q \leqslant 2$) to the case of $q \geqslant 2$ without the condition of uniform integrability. We get the following result:

THEOREM 3.1. *Suppose* $\{X_{nt}, \mathscr{F}_t^n\}$ *is a triangular* L_q*-mixingale array* ($q \geqslant 2$) *such that the conditions* (a) *and* (b) *in Theorem B are satisfied. Then* $\|\frac{1}{k_n}\sum_{t=1}^{k_n} X_{nt}\|_q \to 0$ *as* $n \to \infty$*, and in consequence* $\frac{1}{k_n}\sum_{t=1}^{k_n} X_{nt} \xrightarrow{P} 0$ *as* $n \to \infty$.

Proof. Firstly, for each i, we will prove

$$\left\|\frac{1}{k_n} T_{k_n;i,n}\right\|_q \to 0, \quad as \ \ n \to \infty. \tag{3.1}$$

Indeed

$$\begin{aligned}
\left\|\frac{1}{k_n} T_{k_n;i,n}\right\|_q &= \frac{1}{k_n}\left\|\sum_{t=1}^{k_n} W_{t;i,n}\right\|_q \\
&\leqslant \frac{18q}{k_n}\left(\frac{q}{q-1}\right)^{1/2}\left[E\left(\sum_{t=1}^{k_n} W_{t;i,n}^2\right)^{q/2}\right]^{1/q} \\
&\leqslant \frac{18q}{k_n}\left(\frac{q}{q-1}\right)^{1/2}\left[\left(\sum_{t=1}^{k_n} \left\|W_{t;i,n}^2\right\|_{q/2}\right)^{q/2}\right]^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{18q}{k_n}\left(\frac{q}{q-1}\right)^{1/2}\left(\sum_{t=1}^{k_n}\|W_{t;i,n}\|_q^2\right)^{1/2} \\
&\leqslant \frac{18q}{k_n}\left(\frac{q}{q-1}\right)^{1/2}\left(\sum_{t=1}^{k_n}[2c_{nt}\psi(|i|)]^2\right)^{1/2} \\
&\leqslant 36q\left(\frac{q}{q-1}\right)^{1/2}\psi(0)\left(\frac{1}{k_n^2}\sum_{t=1}^{k_n}c_{nt}^2\right)^{1/2}. \qquad (3.2)
\end{aligned}
$$

The four inequalities above due to Burkholder's inequality, Minkowski's inequality, Lemma 1.1 and $\psi(m)\downarrow 0$ as $m\to\infty$, respectively. Therefore, (3.1) follows from (3.2) and the condition (b) immediately.

For fixed integer $M\geqslant 1$, it is easy to check that

$$
\sum_{i=1-M}^{M-1} T_{k_n;i,n} = \sum_{t=1}^{k_n} E\left(X_{nt}|\mathscr{F}_{t+M-1}^n\right) - \sum_{t=1}^{k_n} E\left(X_{nt}|\mathscr{F}_{t-M}^n\right). \qquad (3.3)
$$

Hence

$$
\frac{1}{k_n}\sum_{t=1}^{k_n}X_{nt} = \frac{1}{k_n}\sum_{i=1-M}^{M-1}T_{k_n;i,n} + \frac{1}{k_n}\sum_{t=1}^{k_n}\left(X_{nt}-E(X_{nt}|\mathscr{F}_{t+M-1}^n)\right) + \frac{1}{k_n}\sum_{t=1}^{k_n}E\left(X_{nt}|\mathscr{F}_{t-M}^n\right).
$$

By Minkowski's inequality and Definition 1.1, it follows that

$$
\begin{aligned}
\left\|\frac{1}{k_n}\sum_{t=1}^{k_n}X_{nt}\right\|_q &\leqslant \sum_{i=1-M}^{M-1}\left\|\frac{1}{k_n}T_{k_n;i,n}\right\|_q + \frac{1}{k_n}\sum_{t=1}^{k_n}\left\|X_{nt}-E(X_{nt}|\mathscr{F}_{t+M-1}^n)\right\|_q \\
&\qquad + \frac{1}{k_n}\sum_{t=1}^{k_n}\left\|E\left(X_{nt}|\mathscr{F}_{t-M}^n\right)\right\|_q \\
&\leqslant \sum_{i=1-M}^{M-1}\left\|\frac{1}{k_n}T_{k_n;i,n}\right\|_q + \frac{2}{k_n}\sum_{t=1}^{k_n}c_{nt}\psi(M).
\end{aligned}
$$

Let $n\to\infty$, by (3.1), we can see that

$$
\limsup_{n\to\infty}\left\|\frac{1}{k_n}\sum_{t=1}^{k_n}X_{nt}\right\|_q \leqslant 2\psi(M)\limsup_{n\to\infty}\frac{1}{k_n}\sum_{t=1}^{k_n}c_{nt}. \qquad (3.4)
$$

Since $\psi(m)\downarrow 0$ as $m\to\infty$, we can get

$$
\left\|\frac{1}{k_n}\sum_{t=1}^{k_n}X_{nt}\right\|_q \to 0, \ \ as \ \ n\to\infty
$$

by condition (a) and setting $M\to\infty$ in (3.4). The desired result is obtained. □

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REFERENCES

[1] D. W. K. ANDREWS, *Laws of large numbers for dependent non-identically distributed random variables*, Econometric Theory, **4** (1988), 458–467.
[2] I. FAZEKAS, AND O. KLESOV, *A general approach to the strong law of large numbers*, Theory Probab. Appl., **45** (2001), 436–449.
[3] P. HALL, AND C. C. HEYDE, *Martingale Limit Theory and Its Application*, Academic Press, New York, 1980.
[4] B. E. HANSEN, *Strong laws for dependent heterogeneous processes*, Econometric Theory, **7** (1991), 213–221.
[5] S. H. HU, *Some new results for the strong law of large numbers*, Acta Mathematica Sinica, **46** (2003), 1123–1134. (In Chinese.)
[6] S. H. HU, G. J. CHEN, AND X. J. WANG, *On extending the Brunk-Prokhorov strong law of large numbers for martingale differences*, Statist. Probab. Lett., **78** (2008), 3187–3194.
[7] D. L. MCLEISH, *A maximal inequality and dependent strong laws*, Ann. Probab., **3** (1975), 829–839.
[8] Y. J. MENG, AND Z. Y. LIN, *Maximal inequalities and laws of large numbers for L_q-mixingale array*, Statist. Probab. Lett., **79** (2009), 1539–1547.

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