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MEANS INVOLVING LINEAR FUNCTIONALS AND n–CONVEX FUNCTIONS

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Abstract. In this article we study Cauchy type means for linear functionals. We examine their monotonicity properties, and even more we give new type of inequalities after we proved exponential convexity. We cover a number of well-known means such as generalized Stolarsky, Stolarsky-Tobey and Whiteley means.

1. Introduction and preliminary results

Mean-value theorems play a great role in mathematical analysis. In particular, Lagrange and Cauchy type mean-value theorems are most frequently used. The most common approach is to prove first the Lagrange type mean-value theorems and then deduce from them Cauchy type mean-value theorems. Those theorems enables us to define various classes of means that can be expressed in terms of linear functionals. Much of the paper is devoted to monotonicity properties of these means using log-convex properties. Positive semi-definite matrices represent a basic tool in our study and that approach enables use of log-convex and even exponentially convex functions. In Section 2 we define a class of linear functionals that we consider in the paper and after we prove mean-value theorems we cover functional power means, Gini means and means of Cauchy type introduced by Leach and Sholander in [5]. In Section 3 we examine applications of linear functional to n-convex functions, very fruitful class of functions that enables us to treat generalized Stolarsky means, generalized Pečarić-Šimić means, generalized Stolarsky-Tobey means and Whiteley means. Section 3 is devoted to further generalization i.e. bilinear functionals.

In the rest of this introduction part we give definitions and results that we need in the later sections. First, we give here definition of exponentially convex function as originally gave Bernstein in [4] (see also [2], [6], [7]).

In the sequel, let I stands for an open interval in $\mathbb{R}$.

DEFINITION 1. A function $h: I \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h(x_i + x_j) \geqslant 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$, $i = 1, \dots, n$ such that $x_i + x_j \in I$, $1 \leqslant i, j \leqslant n$.

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In the rest of the paper we will rely on the following proposition and its corollaries.

PROPOSITION 1.1. *Let $h: I \to \mathbb{R}$. The following propositions are equivalent.*

(i) *h is exponentially convex.*

(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geqslant 0,$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $1 \leqslant i \leqslant n$.

COROLLARY 1.2. *If h is exponentially convex then*

$$\det\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \geqslant 0,$$

for every $n \in \mathbb{N}$, and all $x_i \in I$, $i = 1, \ldots, n$.

COROLLARY 1.3. *If $h: I \to (0,\infty)$ is exponentially convex function then h is a log-convex function, i.e. $\log h$ is convex function on I.*

We will need the following result on log-convex functions, given in [11].

LEMMA 1.4. *Let f be log-convex function and if, $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2 - x_1)} \leqslant \left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2 - y_1)}. \tag{1.1}$$

Proof. This follows from [11], Remark 1.2. □

2. Means for positive linear functionals

Let $a, b \in \mathbb{R}$ and let $C[a,b]$ denotes the vector space of all real-valued continuous functions $f: [a,b] \to \mathbb{R}$. We consider positive linear functionals $A: C[a,b] \to \mathbb{R}$, that is, we assume that

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g) \text{ for all } f, g \in C[a,b],\ \alpha, \beta \in \mathbb{R} \tag{A_1}$$

$$f \in L,\ f(t) \geqslant 0 \text{ on } [a,b] \Rightarrow A(f) \geqslant 0\ (A \text{ is positive}). \tag{A_2}$$

If additionally A satisfies $A(1) = 1$ we call A positive normalized functional.

THEOREM 2.1. *Let $\Phi: [a,b] \to \mathbb{R}$ be a continuous function. Let A be a positive linear functional on $C[a,b]$ such that $A(1) \neq 0$. Then there exists $\xi \in [a,b]$ such that*

$$\Phi(\xi) = \frac{A(\Phi)}{A(1)}. \tag{2.1}$$

Proof. Let $m = \min_{x\in[a,b]} \Phi(x)$, $M = \max_{x\in[a,b]} \Phi(x)$. Then $A(M-\Phi) \geqslant 0$ and $A(\Phi - m) \geqslant 0$, implies

$$mA(1) \leqslant A(\Phi) \leqslant MA(1) \tag{2.2}$$

and we have desired $\xi \in [a,b]$ using continuity of Φ. □

COROLLARY 2.2. *Let Φ_1, Φ_2 be two continuous function on $[a,b]$ and let A be a positive linear functional on $C[a,b]$ such that $A(1) \neq 0$. Then there exists $\xi \in [a,b]$ such that*

$$\frac{\Phi_1(\xi)}{\Phi_2(\xi)} = \frac{A(\Phi_1)}{A(\Phi_2)}, \tag{2.3}$$

assuming both denominators not equal zero.

Proof. Let us define continuous function on $[a,b]$, $\phi(t) = \Phi_1(t)A(\Phi_2) - \Phi_2(t)A(\Phi_1)$. By Theorem 2.1 there exists $\xi \in [a,b]$ such that $A(\phi)/A(1) = \phi(\xi)$. Since $A(\phi) = 0$, it follows $\Phi_1(\xi)A(\Phi_2) - \Phi_2(\xi)A(\Phi_1) = 0$ and (2.3) is proved. □

In the following subsections we give applications of above results to some well-known means.

2.1. Functional power means

Theorem 2.1 enables us to define various types of means, because if Φ has inverse we have

$$\xi = \Phi^{-1}\left(\frac{A(\Phi)}{A(1)}\right) \in [a,b]. \tag{2.4}$$

Specially, let $g : [a,b] \to \mathbb{R}_+$ be a continuous function and let $B : C[a,b] \to \mathbb{R}$ normalized positive functional. After we apply (2.4) on normalized linear functional A, $A(\Psi) := B(\Psi(g))$ (now $A : C[c,d] \to \mathbb{R}$, where $[c,d] = g([a,b])$) and function $\Phi(x) = x^r$, $r \neq 0$, we get a functional power mean (see [11], pp. 117):

$$M^{[r]}(g,B) = \begin{cases} (B(g^r))^{1/r}, & r \neq 0; \\ \exp(B(\ln g)), & r = 0. \end{cases} \tag{2.5}$$

In the sequel we will generate inequalities for functional power means.

THEOREM 2.3. *Let J be some open interval in $\mathbb{R}$, $g:[a,b]\to\mathbb{R}_+$ a continuous function and let $A: C(g([a,b]))\to\mathbb{R}$ be a positive normalized linear functional.*

(i) If $t\mapsto A(g^t)$ is continuous function on J then it is exponentially convex function on J.

(ii) Let $n\in\mathbb{N}$ and let $t_1,\dots,t_n\in J$ be arbitrary. Then the matrix $\left[A\left(g^{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^n$ is a positive semi-definite matrix. In particular

$$\det\left[A\left(g^{\frac{t_i+t_j}{2}}\right)\right]_{i,j=1}^n \geqslant 0. \tag{2.6}$$

Proof. For fixed $n\in\mathbb{N}, u_1,\dots,u_n\in\mathbb{R}$ let us consider the following function

$$\Psi(x)=\sum_{i,j=1}^n u_iu_jx^{\frac{t_i+t_j}{2}}.$$

Since $\Psi(x)=\left(\sum_{i=1}^n u_ix^{\frac{t_i}{2}}\right)^2\geqslant 0$, we have $A(\Psi(g))\geqslant 0$ i.e.

$$\sum_{i,j=1}^n u_iu_jA\left(g^{\frac{t_i+t_j}{2}}\right)\geqslant 0. \tag{2.7}$$

Now (i) and (ii) are obvious. □

COROLLARY 2.4. *Let $g:[a,b]\to\mathbb{R}_+$ be continuous function and let A be a positive normalized functional on the vector space of all real, continuous functions on $g([a,b])$. If $t\mapsto A(g^t)$ is a continuous function on $\mathbb{R}$ then*

$$A(g^s)^{t-r}\leqslant A(g^r)^{t-s}A(g^t)^{s-r} \quad \textit{for} \quad r<s<t. \tag{2.8}$$

Proof. From Theorem 2.3 it follows that $t\mapsto A(g^t)$ is exponentially convex function and hence it is log-convex function. Assume first that $A(g^s)$ and $A(g^r)$ are strictly greater than zero. Using Lemma 1.4 we have

$$\left(\frac{A(g^s)}{A(g^r)}\right)^{\frac{1}{s-r}}\leqslant\left(\frac{A(g^t)}{A(g^s)}\right)^{\frac{1}{t-s}}$$

which validates (2.8). If $A(g^s)=0$ then (2.8) obviously holds. If $A(g^r)=0$ then using log-convexity of $t\mapsto A(g^t)$ we have

$$(A(g^s))^2\leqslant A(g^{2s-r})A(g^r)=0$$

concluding $A(g^s)=0$ and again (2.8) is valid. □

REMARK 2.5. Let us note that usual condition $0 < r < s < t$ in Lyapunov inequality is dropped (see [11] pp. 117).

Previous remark enables us to use Lyapunov inequality (2.8) to prove the following corollary.

COROLLARY 2.6. *Let g be a positive continuous function on $[a,b]$ and A be a normalized positive linear functional on $C[a,b]$. Let us assume that $t \mapsto A(g^t)$ is a continuous on $\mathbb{R}$.*

(a) Then for $p < q$, $p,q \neq 0$

$$M^{[p]}(g,A) \leqslant M^{[q]}(g,A). \tag{2.9}$$

(b) If $p \mapsto M^{[p]}(g,A)$ is continuous for $p = 0$, then (2.9) *is valid for all $p,q \in \mathbb{R}$, $p < q$.*

Proof. (a) The proof is deduced observing three different cases for p and q in (2.9).

(*Case I.*) $0 < p < q$. If we put $r = 0$, $s = p$, $t = q$ in (2.8) we get

$$A(g^p)^q \leqslant A(g^q)^p$$

and after we raise both sides of this inequality to the power $1/pq$ we get (2.9).

(*Case II.*) $p < 0 < q$. In this case we put $r = p$, $s = 0$, $t = q$ in (2.8) and we get

$$A(g^q)^p \leqslant A(g^p)^q.$$

Raising both sides of this inequality to the power $1/pq$ we get (2.9).

(*Case III.*) $p < q < 0$. In this case we put $r = p$, $s = q$, $t = 0$ in (2.8) and we get

$$A(g^q)^{-p} \leqslant A(g^p)^{-q}.$$

Raising both sides of this inequality to the power $-1/pq$ we get (2.9).

(b) part of Corollary is obvious. □

2.2. Hölder inequality

Starting from (2.9) we will here prove the well known Hölder inequality.

THEOREM 2.7. *Let $B : C[a,b] \to \mathbb{R}$ be a positive, normalized linear functional. Let $p > 1$ and $q = p/(p-1)$ so that $p^{-1} + q^{-1} = 1$. If $w, f, g \geqslant 0$ on $[a,b]$ and $wf^p, wg^q, wfg \in C[a,b]$, then we have*

$$B(wfg) \leqslant B^{1/p}(wf^p)B^{1/q}(wg^q). \tag{2.10}$$

In the case $0 < p < 1$ and $B(wg^q) > 0$ (or $p < 0$ and $B(wf^p) > 0$), the inequality (2.10) *is reversed.*

Proof. [Sketch of the proof] First we assume that $B(wg^q) > 0$, and $p > 1$. Then we apply inequality (2.9) for normalized linear functional A defined with $A(h) := B(wh)/B(w)$ and we get

$$\left(\frac{B(wh)}{B(w)}\right)^p \leqslant \frac{B(wh^p)}{B(w)}. \tag{2.11}$$

We now get (2.10) replacing $w \to wg^q, h \to fg^{-q+1}$ in (2.11). All others cases can be covered similarly as indicated in [11]. □

2.3. Gini means

Corollary 2.2 enables us to define new types of means, because if Φ_1/Φ_2 has an inverse, from (2.3) we conclude

$$\xi = \left(\frac{\Phi_1}{\Phi_2}\right)^{-1}\left(\frac{A(\Phi_1)}{A(\Phi_2)}\right) \tag{2.12}$$

If $\Phi_1(x) = x^p$, $\Phi_2(x) = x^q$, $p, q \neq 0$, we get Gini means(see [11], pp. 119)

$$E_{p,q}(g,B) = \begin{cases} \left(\frac{B(g^p)}{B(g^q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp(B(g^p \ln g)/B(g^p)), & p = q, \end{cases} \tag{2.13}$$

for $p, q \in \mathbb{R}$. Again we assume that $g : [a,b] \to \mathbb{R}_+$ is continuous function and $B : C[a,b] \to \mathbb{R}$ normalized positive functional.

THEOREM 2.8. *Let $p \leqslant u$, $q \leqslant v$, $p \neq q$, $u \neq v$ and let $t \mapsto B(g^t)$ be continuous positive function. Then*

$$E_{p,q}(g,B) \leqslant E_{u,v}(g,B). \tag{2.14}$$

Proof. From Theorem 2.3 we have log-convexity of $t \mapsto B(g^t)$. The proof now follows using Lemma 1.4. □

2.4. Means of Cauchy type

Let $f : [a,b] \to \mathbb{R}$ and denote by $[x_0, x_1, \dots, x_n; f]$ divided difference of the function f on the knots $x_0, x_1, \dots, x_n \in [a,b]$. The following mean-value result from [5] is known:

THEOREM 2.9. *Let $x_0, x_1, \dots, x_n$ be mutually different real numbers, $f, g \in C^n[\min x_i, \max x_i]$ with $f^{(n)}(t) \neq 0$ for all $t \in [\min x_i, \max x_i]$. Then there there exists $\xi \in [\min x_i, \max x_i]$, such that*

$$\frac{[x_0, x_1, \dots, x_n; g]}{[x_0, x_1, \dots, x_n; f]} = \frac{g^{(n)}(\xi)}{f^{(n)}(\xi)}. \tag{2.15}$$

REMARK 2.10. When the number ξ from Theorem 2.9 is uniquely determined, we call it the (f,g) extended mean of $\mathbf{x}=(x_0,\ldots,x_n)$ and denote

$$\xi = E(\mathbf{x};f,g). \tag{2.16}$$

In [5] one may also find

$$[x_0,x_1,\ldots,x_n;f] = \int_{S_n} f^{(n)}\left(\sum_{k=0}^{n} u_k x_k\right) du_1\cdots du_n \tag{2.17}$$

where $u_0 = 1-\sum_{k=1}^{n} u_k$ and S_n represents Euclidean simplex in $\mathbb{R}^n$:

$$S_n = \{(u_1\ldots,u_n)\in\mathbb{R} : \sum_{k=1}^{n} u_k \leqslant 1, \text{ and } u_k \geqslant 0,\ k=1,\ldots,n\}. \tag{2.18}$$

If $f^{(n)}/g^{(n)}$ has inverse, combining (2.15) and (2.17) we get

$$E(\mathbf{x};f,g) = \left(\frac{f^{(n)}}{g^{(n)}}\right)^{-1}\left(\frac{A(f^{(n)})}{A(g^{(n)})}\right). \tag{2.19}$$

where

$$A(h) = \frac{\int_{S_n} h(\sum_{k=0}^{n} u_k x_k) du_1\cdots du_n}{\mathrm{vol}(S_n)}. \tag{2.20}$$

2.5. Generalized Stolarsky-Tobey means

We will here replace linear functional (2.20) with the following one: for given $\mathbf{x}=(x_0,\ldots,x_n)\in\mathbb{R}_+^{n+1}$

$$B(h) = \int_{S_n} h\left(\left(\sum_{k=0}^{n} u_k x_k^r\right)^{1/r}\right) d\nu(u), \tag{2.21}$$

where ν represents a probability measure on S_n. Using equation (2.4) and (2.21) we define new means:

$$E_h(\mathbf{x};\nu) = h^{-1}\left(B(h)\right). \tag{2.22}$$

In particular, if we take $h(x)=x^{s-r}$ in (2.22) we will get Stolarsky-Tobey means introduced in [14]:

$$E_{r,s}(\mathbf{x};\nu) = \begin{cases} \left(\int_{S_n}\left(\sum\limits_{k=0}^{n} u_k x_k^r\right)^{\frac{s-r}{r}} d\nu(u)\right)^{1/(s-r)}, & r(s-r)\neq 0; \\ \exp\left(\int_{S_n}\left(\sum\limits_{k=0}^{n} u_k x_k^r\right)^{\frac{1}{r}} d\nu(u)\right), & s=r\neq 0; \\ \left(\int_{S_n}(\prod\limits_{k=0}^{n} x_k^{u_k})^s d\nu(u)\right)^{\frac{1}{s}}, & r=0,\ s\neq 0; \\ \exp\left(\int_{S_n}\ln(\prod\limits_{k=0}^{n} x_k^{u_k}) d\nu(u)\right), & r=s=0. \end{cases} \tag{2.23}$$

Further, using equation (2.12) and (2.21) we define new means:

$$E(\mathbf{x}; f, g; \nu) = \left(\frac{f}{g}\right)^{-1}\left(\frac{B(f)}{B(g)}\right). \tag{2.24}$$

Particulary, if we take $f(x) = x^{p-r}$ and $g(x) = x^{q-r}$ in (2.22) we will get quotient of Stolarsky-Tobey means.

$$E_{p,q;r}(\mathbf{x};\nu) = \begin{cases} \left(\frac{\int_{S_n} (\sum_{k=0}^n u_k x_k^r)^{\frac{p-r}{r}} d\nu(u)}{\int_{S_n} (\sum_{k=0}^n u_k x_k^r)^{\frac{q-r}{r}} d\nu(u)}\right)^{1/(p-q)}, & r(p-q) \neq 0; \\ \exp\left(\frac{\int_{S_n} (\sum_{k=0}^n u_k x_k^r)^{\frac{q-r}{r}} \ln(\sum_{k=0}^n u_k x_k^r)^{1/r} d\nu(u)}{\int_{S_n} (\sum_{k=0}^n u_k x_k^r)^{\frac{q-r}{r}} d\nu(u)}\right), & p = q \neq r \neq 0, \\ \left(\frac{\int_{S_n} \prod_{k=0}^n x_k^{pu_k} d\nu(u)}{\int_{S_n} \prod_{k=0}^n x_k^{qu_k} d\nu(u)}\right)^{1/(p-q)}, & r = 0,\ p \neq q, \\ \exp\left(\frac{\int_{S_n} (\prod_{k=0}^n x_k^{qu_k})(\sum_{k=0}^n u_k \ln x_k) d\nu(u)}{\int_{S_n} \prod_{k=0}^n x_k^{qu_k} d\nu(u)}\right), & r = 0,\ p = q \neq 0, \\ \prod_{k=0}^n x_k^{\int_{S_n} u_k d\nu(u)}, & r = p = q = 0. \end{cases} \tag{2.25}$$

Means $E_{p,q;r}(\mathbf{x};\nu)$ we call *generalized Stolarsky-Tobey means.*

Since, all conditions of Theorem 2.8 are satisfied the following theorem is valid.

THEOREM 2.11. *Let $p \leqslant u,\ q \leqslant v$. Then*

$$E_{p,q;r}(\mathbf{x};\nu) \leqslant E_{u,v;r}(\mathbf{x};\nu), \tag{2.26}$$

for all $r \in \mathbb{R}$ and $(x_0, x_1, \ldots, x_n) \in \mathbb{R}_+^{n+1}$.

Proof. From Theorem 2.8 we have

$$E_{p,q;r}(\mathbf{x};\nu) \leqslant E_{u,v;r}(\mathbf{x};\nu),$$

for $p \leqslant u,\ q \leqslant v,\ p \neq q,\ u \neq v$. Now, using continuity we have (2.26). □

2.6. Generalized Whiteley means

An important example of probability measure on simplex S_n is the Dirichlet measure $\mu_{\mathbf{b}}$ (see[1] Sec. 4.4).

Let $\mathbf{b} = (b_0, b_1, \ldots, b_n) \in \mathbb{R}_+^{n+1}$, $n \in \mathbb{N}$. The beta function of n variables is defined by

$$B(\mathbf{b}) = \frac{\Gamma(b_0)\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(b_0 + b_1 + \cdots + b_n)},$$

where Γ is the gamma function. The Dirichlet measure μ_b is defined on S_n by

$$d\mu_{\mathbf{b}}(u) := \frac{1}{B(\mathbf{b})} \prod_{i=0}^n u_i^{b_i - 1} du_0 du_1 \cdots du_{n-1}$$

where $u_n = 1 - \sum_{i=0}^{n-1} u_i$.

Generalized Whiteley means are (see [10])

$$w_n^{[r,\mathbf{b}]}(\mathbf{x}) = \int_{S_n} \left(\sum_{k=0}^{n} x_k u_k \right)^r d\mu_{\mathbf{b}}(u). \tag{2.27}$$

Now, if we define linear functional

$$A(h) = \int_{S_n} h\left(\sum_{k=0}^{n} u_k x_k \right) \mu_{\mathbf{b}}(u)$$

using (2.12) we can define new means

$$W_n^{[p,q;\mathbf{b}]}(\mathbf{x}) = \left(\frac{f}{g} \right)^{-1} \left(\frac{A(f)}{A(g)} \right)$$

where $f(x) = x^p$, $g(x) = x^q$, and we get

$$W_n^{[p,q;\mathbf{b}]}(\mathbf{x}) = \begin{cases} \left(\frac{w_n^{[p,\mathbf{b}]}(\mathbf{x})}{w_n^{[q,\mathbf{b}]}(\mathbf{x})} \right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\int_{S_n} (\sum_{k=0}^{n} u_k x_k)^q \ln(\sum_{k=0}^{n} u_k x_k) d\mu_{\mathbf{b}}(u)}{\int_{S_n} (\sum_{k=0}^{n} u_k x_k)^q d\mu_{\mathbf{b}}(u)} \right), & p = q. \end{cases} \tag{2.28}$$

In [10] it is shown that following inequality holds.

THEOREM 2.12. *Let p,q,u,v be real numbers such that $p \leqslant u$, $q \leqslant v$, $q < p$, $v < u$. Then the following inequality is valid*

$$\left(\frac{w_n^{[p,\mathbf{b}]}(\mathbf{x})}{w_n^{[q,\mathbf{b}]}(\mathbf{x})} \right)^{\frac{1}{p-q}} \leqslant \left(\frac{w_n^{[u,\mathbf{b}]}(\mathbf{x})}{w_n^{[v,\mathbf{b}]}(\mathbf{x})} \right)^{\frac{1}{u-v}}.$$

We can conclude even more:

THEOREM 2.13. *Let p,q,u,v be real numbers such that $p \leqslant u$, $q \leqslant v$. Then the following inequality is valid*

$$W_n^{[p,q;\mathbf{b}]}(\mathbf{x}) \leqslant W_n^{[u,v;\mathbf{b}]}(\mathbf{x}).$$

Proof. Using Theorem 2.8 we first conclude

$$W_n^{[p,q;\mathbf{b}]}(\mathbf{x}) \leqslant W_n^{[u,v;\mathbf{b}]}(\mathbf{x}), \quad \text{for} \quad p \leqslant u,\ q \leqslant v,\ q \neq p,\ v \neq u.$$

Using continuous extensions we now cover cases $q = p$ and $v = u$. □

We can generalize means $W_n^{[p,q;\mathbf{b}]}(\mathbf{x})$ adding one more parameter using linear functional similar to (2.21), and we get the new means:

$$W_n^{[p,q;r;\mathbf{b}]}(\mathbf{x}) = \begin{cases} \left(\frac{w_n^{[p/r,\mathbf{b}]}(\mathbf{x}^r)}{w_n^{[q/r,\mathbf{b}]}(\mathbf{x}^r)}\right)^{\frac{1}{p-q}}, & r(p-q) \neq 0; \\ \exp\left(\frac{\int_{S_n}(\sum_{k=0}^n u_k x_k^r)^{\frac{q}{r}} \ln(\sum_{k=0}^n u_k x_k^r)^{1/r} d\mu_{\mathbf{b}}(u)}{\int_{S_n}(\sum_{k=0}^n u_k x_k^r)^{\frac{q}{r}} d\mu_{\mathbf{b}}(u)}\right), & r \neq 0,\ p = q, \\ \left(\frac{\int_{S_n}\prod_{k=0}^n x_k^{pu_k} d\mu(u)}{\int_{S_n}\prod_{k=0}^n x_k^{qu_k} d\mu(u)}\right)^{1/(p-q)}, & r = 0,\ p \neq q, \\ \exp\left(\frac{\int_{S_n}(\prod_{k=0}^n x_k^{qu_k})(\sum_{k=0}^n u_k \ln x_k) d\mu_{\mathbf{b}}(u)}{\int_{S_n}\prod_{k=0}^n x_k^{qu_k} d\mu_{\mathbf{b}}(u)}\right), & r = 0,\ p = q \neq 0, \\ \prod_{k=0}^n x_k^{\int_{S_n} u_k d\mu_{\mathbf{b}}(u)}, & r = p = q = 0. \end{cases} \tag{2.29}$$

THEOREM 2.14. *Let* $p \leqslant u$, $q \leqslant v$. *Then*

$$W_n^{[p,q;r;\mathbf{b}]}(\mathbf{x}) \leqslant W_n^{[u,v;r;\mathbf{b}]}(\mathbf{x}), \tag{2.30}$$

for all $r \in \mathbb{R}$ *and* $(x_0, x_1, \ldots, x_n) \in \mathbb{R}_+^{n+1}$.

Proof. Again we use Theorem 2.8 to first conclude

$$W_n^{[p,q;r;\mathbf{b}]}(\mathbf{x}) \leqslant W_n^{[u,v;r;\mathbf{b}]}(\mathbf{x}), \quad \text{for} \quad r(p-q)(u-v) \neq 0.$$

Using continuous extensions we now cover all other possible cases. □

REMARK 2.15. Let us observe that taking $p \to p-r$, $q \to q-r$ in (2.29) we get a special type of generalized Stolarsky-Tobey means introduced in previous subsection.

3. Means involving linear functionals and n-convex functions

Let $D \subset \mathbb{R}$, and let $S(D)$ be one of normed subspaces of all real functions defined on D with respect to some norm $\|\cdot\|_1$. On $C[a,b]$ we take sup-norm and we will consider continuous linear operators $A : C[a,b] \to S(D)$. The set of all functions that are convex of order n and continuous on $[a,b]$ (continuous from the right at a and continuous from the left at b) will be denoted by $K_n[a,b]$. For $i \in \mathbb{N}_0$ we define

$$e_i(t) = t^i,\ t \in [a,b]. \tag{3.1}$$

Further, for $t, c \in [a,b]$ we define function $w_n(t,c)$ by

$$w_n(t,c) = \left(\frac{t-c+|t-c|}{2}\right)^{n-1} = (t-c)_+^{n-1}, \quad n \in \mathbb{N}. \tag{3.2}$$

In [11] pp. 263 the following characterization of positivity of continuous operators on the set $K_n[a,b]$ can be found:

THEOREM 3.1. *Assume that* $A : C[a,b] \to S(D)$ *is a linear and continuous operator. Then*

$$f \in K_n[a,b] \Rightarrow Af \geqslant 0 \tag{3.3}$$

if and only if

$$Ae_i = 0 \quad for \quad i = 0,1,\dots,n-1 \text{ and} \tag{3.4}$$

$$Aw_n(t,c) \geqslant 0 \quad for\ every \quad c \in [a,b]. \tag{3.5}$$

Throughout this section, in our applications, we will use one fixed continuous linear functional $A : C[a,b] \to \mathbb{R}$ that satisfies (3.3).

In the following Lemma we have well known example of n-convex functions that is connected with Stolarsky means (see [5],[15]).

LEMMA 3.2. *If* $f_\lambda : [a,b] \to \mathbb{R}$ $(0 < a < b < \infty)$

$$f_\lambda(t) = \begin{cases} \frac{t^\lambda}{\lambda(\lambda-1)\cdots(\lambda-n+1)}, & \lambda \notin \{0,1,\dots,n-1\}; \\ \frac{t^j \ln t}{(-1)^{n-1-j} j!(n-1-j)!}, & \lambda = j \in \{0,1,\dots,n-1\}, \end{cases} \tag{3.6}$$

then f_λ *is* n*-convex function.*

Proof. From $f_\lambda^{(n)}(t) = t^{\lambda-n} > 0, \quad t \in [a,b]$, we conclude $f_\lambda \in K_n[a,b]$. □

THEOREM 3.3. *Let* $A : C[a,b] \to \mathbb{R}$ *be a linear and continuous functional which satisfies* (3.3) *and let* J *be some open interval in* $\mathbb{R}$.

(a) If $\lambda \mapsto A(f_\lambda)$ *is continuous function on* J *then it is exponentially convex function on* J.

(b) Let $r_1,\dots r_m \in J$ *be arbitrary. The matrix* $\left[A(f_{\frac{r_i+r_j}{2}})\right]_{i,j=1}^m$ *is a positive semi-definite matrix. Particularly*

$$\det\left[A\left(f_{\frac{r_i+r_j}{2}}\right)\right]_{i,j=1}^m \geqslant 0.$$

Proof. Let $u_1,\dots,u_m$ be arbitrary real numbers. Consider the following function

$$\Phi(x) = \sum_{i,j=1}^m u_i u_j f_{\frac{r_i+r_j}{2}}(x).$$

Since $\Phi^{(n)}(x) = \sum_{i,j=1}^m u_i u_j x^{\frac{r_i+r_j}{2}-n} = \left(\sum_{i=1}^m u_i x^{\frac{r_i-n}{2}}\right)^2 \geqslant 0$, we have $A(\Phi) \geqslant 0$ i.e.

$$\sum_{i,j=1}^{m} u_i u_j A\left(f_{\frac{r_i+r_j}{2}}\right) \geqslant 0. \tag{3.7}$$

(i) and (ii) now follow easily. □

REMARK 3.4. The same conclusions of Theorem 3.3 are valid if we replace function f_λ with function

$$g_\lambda(t) = \begin{cases} \frac{e^{\lambda t}}{\lambda^n}, & \lambda \neq 0; \\ \frac{t^n}{n!}, & \lambda = 0. \end{cases} \tag{3.8}$$

It is easy to see that $g_\lambda^{(n)}(t) = e^{\lambda t} > 0$ which means $g_\lambda \in K_n[a,b]$, $[a,b] \subseteq \mathbb{R}$.

THEOREM 3.5. *Let $f \in C^n[a,b]$ and let $A : C[a,b] \to \mathbb{R}$ be a linear continuous functional which satisfies (3.3). Then there exists $\xi \in [a,b]$ such that*

$$f^{(n)}(\xi)A(g_0) = A(f). \tag{3.9}$$

Proof. Let $m = \min f^{(n)}$, $M = \max f^{(n)}$. Let us observe that function $\varphi(x) = M\frac{x^n}{n!} - f(x) = Mg_0(x) - f(x)$ is n-convex function since $\varphi^{(n)}(x) = M - f^{(n)}(x) \geqslant 0$. Hence, $A(\varphi) \geqslant 0$ and we conclude

$$A(f) \leqslant MA(g_0).$$

Similarly,

$$mA(g_0) \leqslant A(f) \leqslant MA(g_0).$$

Now we have (3.9). □

COROLLARY 3.6. *Let $f, g \in C^n[a,b]$, let $A : C[a,b] \to \mathbb{R}$ be a linear and continuous functional which satisfies (3.3) and let additionally $A(g_0) \neq 0$. Then there exists $\xi \in [a,b]$ such that*

$$\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)} = \frac{A(f)}{A(g)}, \tag{3.10}$$

assuming both denominators not equal zero.

Proof. Let us define the function: $\phi(x) = f(x)A(g) - g(x)A(f)$. By Theorem 3.5 there exists $\xi \in [a,b]$ such that $A(\phi) = \phi^{(n)}(\xi)A(g_0)$. From $A(\phi) = 0$ it follows $f^{(n)}(\xi)A(g) - g^{(n)}(\xi)A(f) = 0$ and (3.10) is proved. □

Corollary 3.6 enables us to define various types of means, because if $f^{(n)}/g^{(n)}$ has an inverse, from (3.9) we conclude

$$\xi = \left(\frac{f^{(n)}}{g^{(n)}}\right)^{-1}\left(\frac{A(f)}{A(g)}\right) \in [a,b]. \tag{3.11}$$

Specially, if we take $f(x) = f_p(x),\ g(x) = f_q(x),$ (see Lemma 3.2) we get

$$F(A;p,q) := \xi = \left(\frac{A(f_p)}{A(f_q)}\right)^{\frac{1}{p-q}} \tag{3.12}$$

COROLLARY 3.7. *Let $p \leqslant r,\ q \leqslant s,$ and let $t \mapsto A(f_t)$ be a continuous function. Then*

$$F(A;p,q) \leqslant F(A;r,s). \tag{3.13}$$

Proof. From (a) part of Theorem 3.3 we conclude that $t \mapsto A(f_t)$ is exponentially convex function and hence log-convex function. Now using Lemma 1.4 we have inequality (3.13). □

3.1. Popoviciu legacy

Theorem 3.1 represents generalization of the next theorem proved by Popoviciu (see [12], [13]).

THEOREM 3.8. *Let $f : I \to \mathbb{R}$ be $n-$convex function, let $x_i \in I\ (i = 1,\ldots,m)$ and let $\mathbf{p} = (p_1,\ldots,p_m)$ be a real m-tuple. The inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \geqslant 0 \tag{3.14}$$

holds for every $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m,\ p_i \neq 0\ (i = 1,\ldots,m)$ and every n-convex function f if and only if

$$\sum_{i=1}^{m} p_i x_i^k = 0 \quad \text{for} \quad k = 0,1,\ldots,n-1 \tag{3.15}$$

and

$$-\sum_{r=1}^{k} p_r (x_r - t)^{n-1} = \sum_{r=k+1}^{m} p_r (x_r - t)^{n-1} \geqslant 0 \tag{3.16}$$

holds for every $t \in (x_k, x_{k+1}]$ and $k = 1,\ldots,m-n$.

In case $n = 2$ (convex functions) we have a more compact form (see [9]).

THEOREM 3.9. *The inequality* (3.14) *holds for all m-tuples* $\mathbf{x}$ *and* $\mathbf{p}$ *and all convex functions f if and only if $\sum_{i=1}^{m} p_i = 0$ and*

$$\sum_{i=1}^{m} p_i |x_i - x_k| \geqslant 0, \quad \text{for} \quad k = 1,\ldots,m.$$

Very simple form of linear functional $A(f)=\sum_{i=1}^{m} p_i f(x_i)$ that naturally follows from Theorem 3.8 enables us to cover limit cases in an explicit way for means $F(A;p,q)$ defined in (3.12) where we assume that $\mathbf{p}=(p_1,\ldots,p_m)$ and $\mathbf{x}=(x_1,\ldots,x_m)$ satisfy (3.15) and (3.16).

$$F(\mathbf{x};p,q)=\begin{cases}\left(\frac{\prod_{k=0}^{n-1}(q-k)}{\prod_{k=0}^{n-1}(p-k)}\frac{\sum_{i=1}^{m}p_ix_i^p}{\sum_{i=1}^{m}p_ix_i^q}\right)^{\frac{1}{p-q}}, & (p-q)\prod_{k=0}^{n-1}[(q-k)(p-k)]\neq 0;\\ \left(\frac{\prod_{k=0}^{n-1}(q-k)}{(-1)^{n-1-j}j!(n-1-j)!}\frac{\sum_{i=1}^{m}p_ix_i^j\ln x_i}{\sum_{i=1}^{m}p_ix_i^q}\right)^{\frac{1}{j-q}}, & \begin{matrix}\prod_{k=0}^{n-1}(q-k)\neq 0,\\ p=j\in\{0,1,\ldots,n-1\}\end{matrix};\\ \exp\left(\frac{\sum_{i=1}^{m}p_ix_i^p\ln x_i}{\sum_{i=1}^{m}p_ix_i^p}-\sum_{k=0}^{n-1}\frac{1}{p-k}\right), & p=q\notin\{0,1,\ldots,n-1\};\\ \exp\left(\frac{\sum_{i=1}^{m}p_ix_i^j\ln x_i^2}{2\sum_{i=1}^{m}p_ix_i^j\ln x_i}-\sum_{\substack{k=0\\k\neq j}}^{n-1}\frac{1}{p-k}\right), & p=q=j\in\{0,1,\ldots,n-1\}.\end{cases}$$

3.2. Generalized Stolarsky means

Generalized Stolarsky means are introduced in [5] using functions from Lemma 3.2 and means from subsection 2.3:

If $\mathbf{x}=(x_0,x_1,\ldots,x_n)\in\mathbb{R}^{n+1}$ is $n+1$-tuple of mutually different numbers then generalized Stolarsky mean of order (p,q) $(p\neq q)$ is

$$E_{p,q}(\mathbf{x})=\left(\frac{[x_0,\ldots,x_n;f_p]}{[x_0,\ldots,x_n;f_q]}\right)^{\frac{1}{p-q}}, \tag{3.17}$$

where f_p and f_q are functions defined in Lemma 3.6. However, we will first cover limit cases and for that purpose we introduce the following notation.

With $V(\mathbf{x};f)$ we denote

$$V(\mathbf{x};f):=\begin{vmatrix}1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & f(x_0)\\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & f(x_1)\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & f(x_n)\end{vmatrix}$$

Particulary, for $f(t)=t^r\ln^k t$ we will denote

$$V(\mathbf{x};r,k):=\begin{vmatrix}1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^r\ln^k x_0\\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^r\ln^k x_1\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^r\ln^k x_n\end{vmatrix}.$$

Similarly, we denote

$$W(\mathbf{x};r,k) := \begin{vmatrix} 1 & \ln x_0 & \ln^2 x_0 & \cdots & \ln^{n-1} x_0 & x_0^r \ln^k x_0 \\ 1 & \ln x_1 & \ln^2 x_1 & \cdots & \ln^{n-1} x_1 & x_1^r \ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln x_n & \ln^2 x_n & \cdots & \ln^{n-1} x_n & x_n^r \ln^k x_n \end{vmatrix},$$

It is known that for mutually different numbers $x_0, x_1, \ldots, x_n$.

$$[x_0, x_1, \ldots, x_n; f] = \frac{V(\mathbf{x}; f)}{V(\mathbf{x}; n, 0)}. \tag{3.18}$$

Using above notation we can here cover all limit cases (see also [16]):

$$E_{p,q}(\mathbf{x}) = \begin{cases} \left(\frac{\prod_{k=0}^{n-1}(q-k)}{\prod_{k=0}^{n-1}(p-k)} \frac{V(\mathbf{x};p,0)}{V(\mathbf{x};q,0)} \right)^{\frac{1}{p-q}}, & (p-q)\prod_{k=0}^{n-1}[(q-k)(p-q)] \neq 0; \\ \left(\frac{\prod_{k=0}^{n-1}(q-k)}{(-1)^{n-1-j} j!(n-1-j)!} \frac{V(\mathbf{x};j,1)}{V(\mathbf{x};q,0)} \right)^{\frac{1}{j-q}}, & q \neq p = j \in \{0,1,\ldots,n-1\}; \\ \left((-1)^{k-j} \binom{n-1}{j} \binom{n-1}{k}^{-1} \frac{V(\mathbf{x};j,1)}{V(\mathbf{x};k,1)} \right)^{\frac{1}{j-k}}, & p = k \neq j = q,\ k,j \in \{0,1,\ldots,n-1\}; \\ \exp\left(\frac{V(\mathbf{x};q,1)}{V(\mathbf{x};q,0)} - \sum_{k=0}^{n-1} \frac{1}{q-k} \right), & p = q \notin \{0,1,\ldots,n-1\}; \\ \exp\left(\frac{V(\mathbf{x};q,2)}{2V(\mathbf{x};q,1)} - \sum_{\substack{k=0 \\ k \neq q}}^{n-1} \frac{1}{q-k} \right), & p = q \in \{0,1,\ldots,n-1\}. \end{cases} \tag{3.19}$$

If we take $g(t) = t$ in Gini means (2.13) we get generalized Stolarsky means. So we conclude that following corollary is valid.

COROLLARY 3.10. *Let $p \leqslant u$, $q \leqslant v$. Then*

$$E_{p,q}(\mathbf{x}) \leqslant E_{u,v}(\mathbf{x}), \tag{3.20}$$

for all $x_0, x_1, \ldots, x_n$ mutually different, positive real numbers.

3.3. Generalized Pečarić-Šimić means

We now make step further adding one more parameter in the generalized Stolarsky means $E_{p,q}(\mathbf{x})$ using the same technique from previous sections. Using abbreviations $\mathbf{x}^r = (x_0^r, x_1^r, \ldots, x_n^r)$ and $\ln \mathbf{x} = (\ln x_0, \ln x_1, \ldots, \ln x_n)$ we have new means $E_{p,q;r}(\mathbf{x})$ of

Cauchy type called generalized *Pečarić-Šimić* means (they are first considered in [16]):

$$E_{p,q;r}(\mathbf{x}) = \begin{cases} \left(\frac{\prod_{k=0}^{n-1}(q-rk)}{\prod_{k=0}^{n-1}(p-rk)} \frac{V(\mathbf{x}^r;\frac{p}{r},0)}{V(\mathbf{x}^r;\frac{q}{r},0)} \right)^{\frac{1}{p-q}}, & r(p-q)\prod_{k=0}^{n-1}[(q-rk)(p-rq)]\neq 0; \\ \left(\frac{\prod_{k=0}^{n-1}(q-rk)}{r^n(-1)^{n-1-j}j!(n-1-j)!} \frac{V(\mathbf{x}^r;j,1)}{V(\mathbf{x}^r;\frac{q}{r},0)} \right)^{\frac{1}{jr-q}}, & \begin{matrix} r(p-q)\prod_{k=0}^{n-1}[(q-rk)(p-rq)]\neq 0, \\ p=jr, \quad j\in\{0,1,\dots,n-1\} \end{matrix}; \\ \left((-1)^{k-j}\binom{n-1}{j}\binom{n-1}{k}^{-1} \frac{V(\mathbf{x}^r;j,1)}{V(\mathbf{x}^r;k,1)} \right)^{\frac{1}{(j-k)r}}, & \begin{matrix} r(p-q)\neq 0,\ p=jr,\ q=kr, \\ 0\leqslant j\neq k\leqslant n-1 \end{matrix}; \\ \exp\left(\frac{V(\mathbf{x}^r;\frac{q}{r},1)}{rV(\mathbf{x}^r;\frac{q}{r},0)} - \sum_{k=0}^{n-1}\frac{1}{q-kr} \right), & r\prod_{k=0}^{n-1}[(q-rk)(p-rq)]\neq 0,\ p=q; \\ \exp\left(\frac{V(\mathbf{x}^r;j,2)}{2V(\mathbf{x}^r;j,1)} - \sum_{\substack{k=0 \\ k\neq j}}^{n-1}\frac{1}{q-kr} \right), & r\neq 0,\ p=q=jr,\ j\in\{0,1,\dots,n-1\}; \\ \left(\left(\frac{q}{p}\right)^n \frac{W(\mathbf{x};p,0)}{W(\mathbf{x};q,0)} \right)^{\frac{1}{p-q}}, & pq(p-q)\neq 0,\ r=0; \\ \left(\frac{n!}{q^n}\frac{W(\mathbf{x};q,0)}{W(\mathbf{x};0,n)} \right)^{\frac{1}{q}}, & q\neq 0,\ p=r=0; \\ \exp\left(\frac{W(\mathbf{x};q,1)}{W(\mathbf{x};q,0)} - \frac{n}{q} \right), & p=q\neq 0,\ r=0; \\ \sqrt[n+1]{x_0\cdot x_1\cdots x_n}, & p=q=r=0. \end{cases}$$

Using Corollary 3.10 we have

COROLLARY 3.11. *Let* $p \leqslant u$, $q \leqslant v$. *Then*

$$E_{p,q;r}(\mathbf{x}) \leqslant E_{u,v;r}(\mathbf{x}), \tag{3.21}$$

for all $r \in \mathbb{R}$ *and for all* $x_0, x_1, \dots, x_n$ *mutually different, positive real numbers.*

4. Means involving bilinear functionals and n-convex functions

The following theorem is a version of Theorem 3.1 for bilinear operators. The proof of theorem can be found in [11] p. 265.

THEOREM 4.1. *Let the operator* $A : C[a,b] \times C[a,b] \to S(D)$ *be bilinear and continuous operator. Then for every pair of functions* (f,g)

$$(f,g) \in K_n[a,b] \times K_n[a,b] \Rightarrow A(f,g) \geqslant 0 \text{ for } n \geqslant 2 \tag{4.1}$$

is valid iff

(i) $A(e_i,e_j)=0$ *for* $0\leqslant i,\ j\leqslant n-1$,

(ii) $A(e_i,w_n(t,c))=A(w_n(t,c),e_j)=0$ *for every* $c\in[a,b]$ *and every* $i,j=0,1,\ldots,n-1$, *and*

(iii) $A(w_n(t,c_1),w_n(t,c_2))\geqslant 0$ *for every* $(c_1,c_2)\in[a,b]\times[a,b]$.

Throughout this section we will use one fixed continuous bilinear functional $A:C[a,b]\times C[a,b]\to\mathbb{R}$ that satisfies (4.1), and functions f_λ and g_λ are functions defined in Lemma 3.2 and Remark 3.4.

The proof of the next theorem and its corollaries follows the same line of reasoning as proofs in Sections 3 and 4 and therefore are omitted.

THEOREM 4.2. *Let* $A:C[a,b]\times C[a,b]\to\mathbb{R}$ *be a bilinear and continuous operator which satisfies* (4.1), $g\in K_n[a,b]$ *and let* J *be some open interval in* $\mathbb{R}$.

(a) If $\lambda\mapsto A(f_\lambda,g)$ *is continuous function on* J *then it is exponentially convex function on* J.

(b) Let $r_1,\ldots r_m\in J$ *be arbitrary. The matrix* $\left[A(f_{\frac{r_i+r_j}{2}},g)\right]_{i,j=1}^{m}$, *is a positive semi-definite matrix. Particulary*

$$\det\left[A\left(f_{\frac{r_i+r_j}{2}},g\right)\right]_{i,j=1}^{m}\geqslant 0.$$

COROLLARY 4.3. *Let* $g\in K_n[a,b]$, $t\mapsto A(f_t,g)$ *be a continuous function and let* $A(f_t,g)>0$, *for all* $t\in\mathbb{R}$.

(a) If $p\leqslant r,\ q\leqslant s,\ p\neq q,\ r\neq s$ *then*

$$\left\{\frac{A(f_p,g)}{A(f_q,g)}\right\}^{\frac{1}{p-q}}\leqslant\left\{\frac{A(f_r,g)}{A(f_s,g)}\right\}^{\frac{1}{r-s}}. \tag{4.2}$$

(b) If $r<s<t$

$$A(f_s,g)^{t-r}\leqslant A(f_r,g)^{t-s}A(f_t,g)^{s-r}. \tag{4.3}$$

COROLLARY 4.4. *Let* $f\in C^n[a,b]$ *and* $g\in K_n[a,b]$. *Then there exists* $\xi\in[a,b]$ *such that*

$$f^{(n)}(\xi)A(g_0,g)=A(f,g). \tag{4.4}$$

COROLLARY 4.5. *Let* $h_1,h_2\in C^n[a,b],g\in K_n[a,b]$ *and let bilinear functional* A *additionally satisfies conditions* $A(g_0,g)\neq 0$. *Then there exists* $\xi\in[a,b]$ *such that*

$$\frac{h_1^{(n)}(\xi)}{h_2^{(n)}(\xi)}=\frac{A(h_1,g)}{A(h_2,g)}, \tag{4.5}$$

assuming both denominators not equal zero.

If $h_1^{(n)}/h_2^{(n)}$ has an inverse, from (4.5) we then conclude

$$\xi = \left(\frac{f^{(n)}}{g^{(n)}}\right)^{-1}\left(\frac{A(h_1,g)}{A(h_2,g)}\right) \tag{4.6}$$

and we can define various means.

Specially, if we take $h_1(x) = f_p(x)$, $h_2(x) = f_q(x)$, we get

$$\xi = \left(\frac{A(f_p,g)}{A(f_q,g)}\right)^{\frac{1}{p-q}} \tag{4.7}$$

Mean defined in (4.7) we will denote by $T(A;g;p,q)$.

COROLLARY 4.6. *Let $p \leqslant r$, $q \leqslant s, g \in K_n[a,b]$ and let $t \mapsto A(f_t,g)$ be a continuous function. Then*

$$T(A;g;p,q) \leqslant T(A;g;r,s). \tag{4.8}$$

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