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SOME COMPLETELY MONOTONIC FUNCTIONS RELATED TO THE PSI FUNCTION

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Abstract. Complete monotonicity properties of some functions involving the psi function are studied and some known results are extended and generalized. Moreover, a necessary and sufficient conditions for some functions to be completely monotonic are presented and proved.

1. Introduction

A function f is said to be completely monotonic on an interval I, if f has derivates of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geqslant 0 \tag{1}$$

for all $x \in I$ and $n \geqslant 0$. If the inequality (1) is strict, then f is said to be strictly completely monotonic on I. Let $\mathscr{C}[I]$ denote the set of completely monotonic functions on I.

The well-known Bernstein's theorem [28, p. 161.] states that f is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$, see [28]. In other words, $f \in \mathscr{C}[[0,\infty)]$ if and only if it is a Laplace transform of the measure μ. From this theorem it follows that completely monotonic functions on $(0,\infty)$ are always strictly completely monotonic unless they are constant (see [25]), so there is no need to discuss strict monotonicity in general.

A positive function f is said to be logarithmically completely monotonic on an interval I, if f has derivates of all orders on I and

$$(-1)^n [\log f(x)]^{(n)} \geqslant 0 \tag{2}$$

for all $x \in I$ and $n > 0$. Let $\mathscr{L}[I]$ denote the set of logarithmically completely monotonic functions on I. Among other things, it is proved in [11, 22, 27] that a logarithmically completely monotonic function is always completely monotonic, that is, $\mathscr{L}[I] \subset \mathscr{C}[I]$, but not conversely. For information on the history of the logarithmically completely monotonic functions, please refer to Remark 8 in [23, p. 2154].

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There have been a lot of literature about the completely monotonic functions. Main properties can be found in [28] and some classes of completely monotonic functions are given in [2].

Recall that the classical Euler gamma function is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

The logarithmic derivative of the gamma function

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is called the psi or digamma function. It can be expressed as (see [1, 4])

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt, \tag{3}$$

It is well-known that the gamma function is an important classical special function and has many applications to various areas of mathematics. (Logarithmically) completely monotonic functions also have application in many branches. For example, they play role in complex analysis, numerical and asymptotic analysis, integral transform, number theory and combinatorics. In recent years, many inequalities and (logarithmically) completely monotonic functions involving gamma, psi or polygamma functions are established, see [3, 5, 16, 17, 18, 21, 26]. and in the literature cited therein.

In this paper, using integral representations and analytic techniques, some functions related to the gamma and psi function are shown to be completely monotonic on $(0,\infty)$. Moreover, using fundamental lemma in § 3., we are able to present a necessary and sufficient conditions for some functions to be completely monotonic.

2. Monotonicity results

The quotient of two gamma functions has been explored by many mathematicians and various results are known. Some recent monotonicity results concerning this quotient can be found in [8, 10, 13, 14, 15, 19, 20, 24].

Since psi function is related to the quotient of gamma functions in a way

$$\left[\log \frac{\Gamma(x+a)}{\Gamma(x+b)}\right]' = \psi(x+a) - \psi(x+b),$$

it may be interesting to explore monotonicity properties of the difference of two psi functions.

THEOREM 1. *Let a, b, c, d be positive real numbers, $a \leqslant c$. Then the function*

$$f(x) = \psi(ax+b) - \psi(cx+d) + \log(c/a)$$

is completely monotonic on $(0,\infty)$ if and only if it holds

$$a(2d-1) \leqslant c(2b-1). \tag{4}$$

The following result is immediate consequence of Theorem 1.

COROLLARY 1. *Function*

$$\psi(ax+1)-\psi(cx+1)+\log(c/a)$$

is completely monotonic if and only if $a \leqslant c$.

Using duplication formula for psi function, it follows:

COROLLARY 2. *Function*

$$\psi(x+\alpha)-\tfrac{1}{2}\psi(x)-\tfrac{1}{2}\psi(x+\tfrac{1}{2})$$

is completely monotonic if and only if $\alpha \geqslant \frac{1}{4}$.

COROLLARY 3. *Let* a, b, c, d *be positive real numbers such that* (4) *is satisfied. Then the function*

$$f(x)=\frac{\Gamma(cx+d)^{1/c}}{\Gamma(ax+b)^{1/a}}\left(\frac{a}{c}\right)^{x}$$

is completely monotonic on $(0,\infty)$.

Therefore, f is decreasing and convex function. Hence, under the same conditions, $1/f$ is increasing. We conjecture that this function is also concave.

The function

$$W(x)=\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}}$$

called Wallis power function in [6], has been investigated in many papers. In [10] it is shown that W is convex for $|t-s|<1$ and concave for $|t-s|>1$.

As a consequence of the previous results, since

$$[\log W(x)]'=\frac{1}{t-s}[\psi(x+t)-\psi(x+s)]$$

we have the following:

COROLLARY 4. *Function* $1/W(x)$ *is completely monotonic on* $(0,\infty)$ *for all* $s,t>0$.

Function defined in Theorem 1 can be written as

$$f(x)=\psi(ax+b)-\log(ax)-\psi(cx+d)+\log(cx).$$

Therefore, we want to establish relation between the psi function and the logarithm. The first result is consequence of Theorem 1:

COROLLARY 5. *Let $a > 0$, $b \geqslant \frac{1}{2}$ be real numbers. Then*

$$x \mapsto \psi(ax+b) - \log(ax)$$

is decreasing on $(0,\infty)$.

The complete answer is given in the next result.

THEOREM 2. *Let a, b, c, d be positive real numbers. Then the function*

$$f(x) = \psi(ax+b) - \log(cx+d)$$

is completely monotonic on $(0,\infty)$ if and only if it holds

$$2ad \leqslant c(2b-1). \tag{5}$$

In [12] authors studied function

$$f(x) = x^{\alpha}(\log x - \psi(x))$$

and proved that f is completely monotonic on $(0,\infty)$ if and only if $\alpha \leqslant 1$. Similarly, in [9], author proved that the function

$$f(x) = (x+1)[\psi(x+1) - \log x]$$

is completely monotonic on $(0,\infty)$.

Motivated by this results, we generalized this function and proved the following theorem:

THEOREM 3. *Let $a \geqslant \frac{1}{2}$ and $b > 0$ be real numbers. Then the function*

$$f(x) = (x+b)(\psi(x+a) - \log x)$$

is completely monotonic on $(0,\infty)$ if it holds

$$12ab \geqslant 1 + 6a^2. \tag{6}$$

Necessary conditions for the function defined in previous theorem are still open problem. In the most important case, when $a = \frac{1}{2}$, from the theorem it follows:

COROLLARY 6. *Function*

$$(x+b)(\psi(x+\frac{1}{2}) - \log x)$$

is completely monotonic on $(0,\infty)$ if $b \geqslant 5/12$.

The lower bound of the constant b in this corollary is still open problem. Some calculation on Mathematica shows that b can be close to $2/29$.

3. Fundamental Lemma

In order to prove necessity part of our theorems, we begin with the following fundamental lemma on Laplace transform.

LEMMA 1. *Let φ be bounded and continuous at 0. Suppose that for all positive x we have*

$$\int_0^\infty e^{-xt}\varphi(t)dt > 0. \tag{7}$$

Then, it holds $\varphi(0) \geqslant 0$.

Proof. Denote $\varphi(0) = a_0$ and suppose $a_0 < 0$. Define function h by

$$\varphi(t) = a_0 + t\cdot h(t). \tag{8}$$

Let us suppose that h is also bounded, by constant M, say. It follows

$$\begin{aligned}\int_0^\infty e^{-xt}\varphi(t)dt &= \frac{a_0}{x} + \int_0^\infty e^{-xt}\, t\, h(t)\, dt\\ &= \frac{a_0}{x} + \frac{1}{x^2}\int_0^\infty e^{-u}\, u\, h\left(\frac{u}{x}\right) du\\ &< \frac{a_0}{x} + \frac{1}{x^2}\int_0^\infty e^{-u}\, u M\, du\\ &= \frac{a_0}{x} + \frac{M}{x^2}\end{aligned}$$

For x big enough, right side is less than 0, a contradiction.

In general, function h defined by (8) need not to be bounded. But, we can replace function φ by another function g, say, in a way that g satisfies (7) and function h, defined by the same formula with g instead of φ, be bounded.

By assumption, φ is continuous in 0. Since $\varphi(0) = a_0 < 0$, there exist a point $t_0 > 0$ such that $\varphi(t) \leqslant a_0/2$ for all $0 \leqslant t \leqslant t_0$ Let us define function g

$$g(t) = \begin{cases} a_0/2, & 0 \leqslant t \leqslant t_0,\\ \varphi(t), & t > t_0.\end{cases}$$

Then it holds $g(0) < 0$, $g(t) \geqslant \varphi(t)$ for all t and

$$\int_0^\infty e^{-xt} g(t)dt > \int_0^\infty e^{-xt}\varphi(t)dt \geqslant 0.$$

Therefore, g satisfies the same conditions of the lemma as function φ. But, for the function h defined by

$$g(t) = a_0/2 + t\cdot h(t)$$

it holds $h(t) = 0$ for $0 \leqslant t \leqslant t_0$, hence, h is bounded. This leads to a contradiction and the lemma is proved. □

4. Proof of the Theorems

Proof of Theorem 1.
Using integral representation of psi function (3) we can write:

$$\begin{aligned}(-1)^n f^{(n)}(x) &= (-1)^n[a^n\psi^{(n)}(ax+b) - c^n\psi^{(n)}(cx+d)]\\ &= \int_0^\infty \frac{c^n t^n e^{-(cx+d)t} - a^n t^n e^{-(ax+b)t}}{1-e^{-t}}\,dt\\ &= \int_0^\infty (act)^n\left[\frac{a\,e^{-(cx+d)at}}{1-e^{-at}} - \frac{c\,e^{-(ax+b)ct}}{1-e^{-ct}}\right]dt\\ &= \int_0^\infty (act)^n\left[\frac{a\,e^{-adt}}{1-e^{-at}} - \frac{c\,e^{-bct}}{1-e^{-ct}}\right]e^{-acxt}\,dt \end{aligned} \tag{9}$$

Sufficient condition for complete monotonicity is that the expression in square bracket is positive. It can be written in a way:

$$e^{(1-d)at}\left[\frac{a}{e^{at}-1} - \frac{c\,e^{\alpha t}}{e^{ct}-1}\right]$$

where α is defined by

$$\alpha = (1-b)c - (1-d)a. \tag{10}$$

For $\alpha < 0$, we have

$$a(e^{ct}-1) - c(e^{at}-1)e^{\alpha t} = ac\sum_{n=1}^{\infty}\frac{t^n}{n!}(c^{n-1} - a^{n-1}e^{\alpha t})$$

which is clearly positive since $c \geqslant a$.

Let $\alpha \geqslant 0$. We can write

$$\begin{aligned}&a(e^{ct}-1) - c(e^{at}-1)e^{\alpha t}\\ &\quad= a\sum_{n=1}^{\infty}\frac{c^n t^n}{n!} - c\left(\sum_{j=1}^{\infty}\frac{a^j t^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{\alpha^k t^k}{k!}\right)\\ &\quad= \sum_{n=1}^{\infty}\left(\frac{ac^n}{n!} - c\sum_{k=0}^{n-1}\frac{a^{n-k}}{(n-k)!}\cdot\frac{\alpha^k}{k!}\right)t^n\\ &\quad= \sum_{n=1}^{\infty}\left(\frac{ac^n}{n!} - \frac{c}{n!}\left[(a+\alpha)^n - \alpha^n\right]\right)t^n. \end{aligned}$$

For the positivity of this expression, it is sufficient that the following inequality is satisfied:

$$ac^{n-1} - (a+\alpha)^n + \alpha^n \geqslant 0 \tag{11}$$

for all natural n.

This inequality is satisfied for $n=1$. For $n=2$ it is equivalent to

$$a+2\alpha \leqslant c$$

i.e.

$$c(1-2b) \leqslant a(1-2d)$$

and this is exactly (4).

Let us suppose that (4) is satisfied. We shall prove (11) for $n>2$ by induction. Applying induction to (11), it is sufficient to prove

$$ac^{n-1}+\alpha^n+\alpha c^{n-1} \leqslant c^n.$$

By assumption, it holds $0<a\leqslant c$. Since it holds $a+2\alpha\leqslant c$, it follows also $\alpha\leqslant c$. Therefore,

$$ac^{n-1}+\alpha^n+\alpha c^{n-1} \leqslant c^{n-1}(a+2\alpha)+\alpha(\alpha^{n-1}-c^{n-1}) \leqslant c^n.$$

This proves sufficiency part of the theorem.

Proof of necessity part.

Suppose f is completely monotonic. Then it holds from (9), taking $n=0$,

$$\int_0^\infty e^{-acxt}\left[\frac{a\,e^{-adt}}{1-e^{-at}}-\frac{c\,e^{-bct}}{1-e^{-ct}}\right]dt \geqslant 0,$$

for all positive x.

From Lemma 1, the function

$$\varphi(t)=\frac{a\,e^{-adt}}{1-e^{-at}}-\frac{c\,e^{-bct}}{1-e^{-ct}}$$

must be nonnegative at 0.

Taking appropriate limit, it is easy to compute

$$\varphi(0)=\frac{ac}{2}(c-a-2\alpha).$$

Therefore, it follows $a+2\alpha\leqslant c$, which is equivalent to (4).

The proof is completed. $\square$

Proof of Theorem 2.

Using integral representation of psi function (3) and integral representation of $\log x$

$$\log x=\int_0^\infty \frac{e^{-t}-e^{-xt}}{t}dt \tag{12}$$

we have:

$$
\begin{aligned}
(-1)^n f^{(n)}(x) &= (-1)^n[a^n\psi^{(n)}(ax+b) - c^n\log^{(n)}(cx+d)] \\
&= \int_0^\infty \frac{c^n t^n e^{-(cx+d)t}}{t} - \frac{a^n t^n e^{-(ax+b)t}}{1-e^{-t}}\,dt \\
&= \int_0^\infty (act)^n \left[\frac{a\,e^{-(cx+d)at}}{at} - \frac{c\,e^{-(ax+b)ct}}{1-e^{-ct}}\right] dt \\
&= \int_0^\infty (act)^n \left[\frac{e^{-adt}}{t} - \frac{c\,e^{-bct}}{1-e^{-ct}}\right] e^{-acxt}\,dt
\end{aligned}
\tag{13}
$$

Sufficient condition for complete monotonicity is that the expression in square brackets is positive.

Let $\alpha = (1-b)c + ad$. Expression in brackets can be written as:

$$
e^{-adt}\left[\frac{1}{t} - \frac{c\,e^{\alpha t}}{e^{ct}-1}\right]
$$

Obviously,

$$
e^{ct} - 1 - ct\,e^{\alpha t}
$$

is positive if $\alpha < 0$.

For $\alpha \geqslant 0$, we have

$$
\begin{aligned}
e^{ct} - 1 - ct\,e^{\alpha t} &= \sum_{n=1}^{\infty} \frac{(ct)^n}{n!} - ct\sum_{n=0}^{\infty}(\alpha t)^n \\
&= c\sum_{n=1}^{\infty}\frac{t^n}{n!}[c^{n-1} - n\alpha^{n-1}]
\end{aligned}
$$

For the positivity of this expression, it is sufficient that the following inequality is satisfied:

$$
c^{n-1} - n\alpha^{n-1} \geqslant 0 \tag{14}
$$

for all $n \geqslant 1$.

It is satisfied for $n = 1$. For $n = 2$ it is equivalent to

$$
c \geqslant 2\alpha \tag{15}
$$

and this is exactly (5). For $n > 2$ we can prove by induction.

Suppose (5) is satisfied. Applying induction to (14), it is sufficient to prove

$$
\alpha^n + \alpha c^{n-1} \leqslant c^n.
$$

and this follows easily from (15).

To prove necessity, we again use Lemma 1. Suppose f is completely monotonic. Then it holds from (13), taking $n=0$,

$$\int_0^\infty e^{-acxt}\left[\frac{e^{-adt}}{t}-\frac{c\,e^{-bct}}{1-e^{-ct}}\right]dt \geqslant 0$$

From Lemma, the function

$$\varphi(t)=\frac{e^{-adt}}{t}-\frac{c\,e^{-bct}}{1-e^{-ct}}$$

must be nonnegative at 0.

Taking appropriate limit, it is easy to compute

$$\varphi(0)=\left(b-\frac{1}{2}\right)c-ad\geqslant 0$$

and this inequality is equivalent to (5).

The proof is completed. □

Proof of Theorem 3.

Using integral representations of psi function (3) and logarithm (12) we have:

$$f(x)=(x+b)\int_0^\infty \delta(t)e^{-(x+b)t}dt,$$

where

$$\delta(t)=\left(\frac{1}{t}-\frac{e^{-at}}{1-e^{-t}}\right)e^{bt}$$

is positive function and it holds

$$\lim_{t\to 0}\delta(t)=a-\frac{1}{2}.$$

Integration by parts gives:

$$f(x)=\delta(0)+\int_0^\infty \delta'(t)e^{-(x+b)t}dt \tag{16}$$

Now we have:

$$(-1)^n f^{(n)}(x)=\int_0^\infty \delta'(t)t^n e^{-(x+b)t}dt$$

for $n>1$.

Sufficient condition for complete monotonicity is that $\delta'(t)\geqslant 0$.

We can write:

$$\delta'(t)=\frac{e^{(b-a+1)t}}{t^2(e^t-1)^2}\Big[t^2(1-a+b+(a-b)e^t)+ +(bt-1)(e^{(a-1)t}+e^{(a+1)t}-2e^{at})\Big], \tag{17}$$

so for the positivity of $\delta'(t)$, it is sufficient that the expression in square brackets is positive.

We have:

$$\begin{aligned}
&t^2(1+(a-b)(e^t-1))+(bt-1)(e^{(a-1)t}+e^{(a+1)t}-2e^{at})\\
&\quad=t^2+(a-b)\sum_{n=1}^{\infty}\frac{t^{n+2}}{n!}+(bt-1)\sum_{n=2}^{\infty}\frac{t^n}{n!}[(a-1)^n+(a+1)^n-2a^n]\\
&\quad=(a-b)\sum_{n=1}^{\infty}\frac{t^{n+2}}{n!}+b\sum_{n=2}^{\infty}\frac{t^{n+1}}{n!}[(a-1)^n+(a+1)^n-2a^n]-\\
&\qquad-\sum_{n=3}^{\infty}\frac{t^n}{n!}[(a-1)^n+(a+1)^n-2a^n]\\
&\quad=\sum_{n=3}^{\infty}\frac{t^n}{n!}\Big[n(n-1)(a-b)+bn[(a-1)^{n-1}+(a+1)^{n-1}-2a^{n-1}]-\\
&\qquad-[(a-1)^n+(a+1)^n-2a^n]\Big]
\end{aligned}$$

Following inequality has to be satisfied:

$$n(n-1)(a-b)+bn[(a-1)^{n-1}+(a+1)^{n-1}-2a^{n-1}]-[(a-1)^n+(a+1)^n-2a^n]\geqslant 0$$

for $n\geqslant 3$.

It holds for $n=3$, and for $n=4$ we have

$$12ab-6b+6a-1-6a^2\geqslant 0. \tag{18}$$

If $b\geqslant a$, (18) holds true, since

$$b\geqslant a\geqslant \frac{6a^2-6a+1}{12a-6}$$

for $a>1/2$.

If $a\geqslant b$, it is sufficient that

$$12ab\geqslant 1+6a^2,$$

which is exactly (6). It can be written as

$$b\geqslant \frac{1}{12a}+\frac{a}{2}.$$

Now, for $n>4$ we have

$$\begin{aligned}
&n(n-1)(a-b)+bn[(a-1)^{n-1}+(a+1)^{n-1}-2a^{n-1}]-[(a-1)^n+(a+1)^n-2a^n]\\
&=n(n-1)a+bn[(a-1)^{n-1}+(a+1)^{n-1}-2a^{n-1}-(n-1)]\\
&\qquad-[(a-1)^n+(a+1)^n-2a^n]
\end{aligned}$$

$$\geqslant n(n-1)a+\left(\frac{1}{12a}+\frac{a}{2}\right)n[(a-1)^{n-1}+(a+1)^{n-1}-2a^{n-1}-(n-1)] \\ -[(a-1)^n+(a+1)^n-2a^n],$$

Previous inequality holds since

$$(a-1)^n+(a+1)^n-2a^n-n$$

is positive for all $n \geqslant 3$ (which can be proven easily by induction).

Expression

$$(a-1)^n+(a+1)^n-2a^n$$

can be expanded and simplified depending on parity of n. Therefore, we have two separate cases.

For n even:

$$n(n-1)a+n\left(\frac{1}{12a}+\frac{a}{2}\right)\left[2\sum_{i=1}^{n/2-1}\binom{n-1}{2i}a^{n-1-2i}-(n-1)\right]-2\sum_{i=1}^{n/2}\binom{n}{2i}a^{n-2i}$$

$$=n(n-1)\left(\frac{a}{2}-\frac{1}{12a}\right)-2+\frac{n}{6}\sum_{i=1}^{n/2-1}\binom{n-1}{2i}a^{n-2-2i}$$

$$+n\sum_{i=1}^{n/2-1}\binom{n-1}{2i}a^{n-2i}-2\sum_{i=1}^{n/2-1}\binom{n}{2i}a^{n-2i}$$

$$\geqslant \sum_{i=1}^{n/2-1}\left[n\binom{n-1}{2i}-2\binom{n}{2i}\right]a^{n-2i}\geqslant 0$$

For n odd:

$$n(n-1)a+n\left(\frac{1}{12a}+\frac{a}{2}\right)\left[2\sum_{i=1}^{(n-1)/2}\binom{n-1}{2i}a^{n-1-2i}-(n-1)\right]-2\sum_{i=1}^{(n-1)/2}\binom{n}{2i}a^{n-2i}$$

$$=n(n-2)\left(\frac{a}{2}-\frac{1}{12a}\right)+\frac{n}{6}\sum_{i=1}^{(n-1)/2}\binom{n-1}{2i}a^{n-2-2i}$$

$$+n\sum_{i=1}^{(n-3)/2}\binom{n-1}{2i}a^{n-2i}-2\sum_{i=1}^{(n-3)/2}\binom{n}{2i}a^{n-2i}$$

$$\geqslant \sum_{i=1}^{(n-3)/2}\left[n\binom{n-1}{2i}-2\binom{n}{2i}\right]a^{n-2i}\geqslant 0$$

In both cases, difference in the final sum is always positive, since the first term is exactly $(n-2i)/2$ times greater than the second.

The proof is completed. □

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