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WEIGHTED CRITERIA FOR ONE–SIDED POTENTIALS WITH PRODUCT KERNELS ON CONES OF DECREASING FUNCTIONS

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Abstract. Necessary and sufficient conditions governing two–weight estimates for one–sided potentials with multiple kernels and corresponding one–sided strong fractional maximal functions are established. The one–weight problem for the multiple Riemann–Liouville transform is also studied.

1. Introduction

In this paper necessary and sufficient conditions guaranteeing two–weight inequalities for one–sided potentials with product kernels and appropriate strong one–sided fractional maximal functions on the cone of decreasing functions are obtained, provided that the right-hand side weight is a product of one–dimensional weights. For the weighted Riemann–Liouville transform with product kernels we establish the one–weight inequality. To derive the main results of this paper first we show that the two–sided pointwise relation $R_{\alpha_1,\cdots,\alpha_n} f \approx H_n f$ holds on the class of functions $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ which is non-negative and decreasing in each variable separately, where

$$\left(R_{\alpha_1,\cdots,\alpha_n} f\right)(x_1,\cdots,x_n) = \frac{1}{\prod_{i=1}^n x_i^{\alpha_i}} \int_0^{x_1}\cdots\int_0^{x_n} \frac{f(t_1,\cdots,t_n)}{\prod_{i=1}^n (x_i-t_i)^{1-\alpha_i}} dt_1\cdots dt_n, \;\; 0<\alpha_i<1,$$

and

$$\left(H_n f\right)(x_1,\cdots,x_n) = \frac{1}{\prod_{i=1}^n x_i} \int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\cdots,t_n)dt_1\cdots dt_n.$$

The appropriate pointwise estimate for the multiple Weyl transform

$$\left(W_{\alpha_1,\cdots,\alpha_n} f\right)(x_1,\cdots,x_n) = \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} \frac{f(t_1,\cdots,t_n)\prod_{i=1}^n t_i^{-\alpha_i}}{\prod_{i=1}^n (t_i-x_i)^{1-\alpha_i}} dt_1\cdots dt_n, \;\; 0<\alpha_i<1,$$

is also discussed on the cone of increasing functions.

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A full characterization of the class of weights u for which the boundedness of the one-dimensional Hardy transform

$$\big(Hf\big)(x)=\frac{1}{x}\int_0^x f(t)dt$$

from $L^p_{dec}(u,\mathbb{R}_+)$ to $L^p(u,\mathbb{R}_+)$ holds, was given in [2]. Two-weight Hardy inequalities on cones of decreasing (resp. increasing) functions were established in the paper [18]. We mention also [5] for other relevant references. The multidimensional analogs of these results were studied in [3], [4], [1] (see also [17], [13] for related topics).

For the weight theory for Hardy–type and potential operators we refer to the monographs [15], [14], [8], [7], [6] and references cited therein. The monograph [12] is dedicated to the two–weight problem for multiple integral operators (see also the papers [9], [10], [11] for criteria guaranteeing trace inequalities for potential operators with multiple kernels).

Finally we mention that constants (often different constants in the same series of inequalities) will generally be denoted by c or C. Under the symbol $Tf\approx Kf$, where T and K are linear positive operators defined on appropriate classes of functions, we mean that there are positive constants c_1 and c_2 independent of f and x such that $(Tf)(x)\leqslant c_1(Kf)(x)\leqslant c_2(Tf)(x)$.

2. Preliminaries

We say that a function $f:\mathbb{R}^n_+\to\mathbb{R}_+$ is decreasing (resp. increasing) if f is decreasing (resp. increasing) in each variable separately. Further, a set $D\subset\mathbb{R}^n_+$ is decreasing (resp. increasing) if the function χ_D is decreasing (resp. increasing). We shall need the following notation:

$$D_{x_1,\cdots,x_n}:=D\cap([0,x_1]\times\cdots\times[0,x_n]),\quad D\subset\mathbb{R}^n_+.$$

Let $\mathscr{D}$ (resp. $\mathscr{I}$) be the class of all non–negative decreasing (resp. increasing) functions on $\mathbb{R}^n_+$. Suppose that u is measurable a.e. positive function (weight) on $\mathbb{R}^n_+$. We denote by $L^p(u,\mathbb{R}^n_+)$, $0<p<\infty$, the class of all non–negative functions on $\mathbb{R}^n_+$ for which

$$\|f\|_{L^p(u,\mathbb{R}^n_+)}:=\left(\int_{\mathbb{R}^n_+}f^p(x_1,\cdots,x_n)u(x_1,\cdots,x_n)dx_1\cdots dx_n\right)^{1/p}<\infty.$$

Under the symbol $L^p_{dec}(u,\mathbb{R}^n_+)$ we mean the class $L^p(u,\mathbb{R}^n_+)\cap\mathscr{D}$.

Now we list the well–known results regarding the one–weight problem for the operator H_n. For the following statement we refer to [2] (see also [5]):

THEOREM A. *Let $0<p<\infty$. Then the inequality*

$$\int_0^\infty\big(Hf(x)\big)^p u(x)dx\leqslant c\int_0^\infty (f(x))^p u(x)dx,\quad f\in L^p_{dec}(u,\mathbb{R}_+),$$

holds if and only if there is a positive constant C such that for all $r>0$,

$$\int_r^\infty \left(\frac{r}{x}\right)^p u(x)dx \leqslant C\int_0^r u(x)dx. \tag{1}$$

Condition (1) is called B_p condition and was introduced in [2].

THEOREM B. [1] *Let $0<p<\infty$. Then H_n is bounded from $L^p_{dec}(u,\mathbb{R}^n_+)$ to $L^p(u,\mathbb{R}^n_+)$ if and only if there is a positive constant c such that for all decreasing sets D, $D\subset\mathbb{R}^n_+$,*

$$\int_{\mathbb{R}^n\setminus D} \frac{|D_{x_1,\cdots,x_n}|^p}{(x_1\cdots x_n)^p} u(x_1,\cdots,x_n)dx_1\cdots dx_n \leqslant c\int_D u(x_1,\cdots,x_n)dx_1\cdots dx_n. \tag{2}$$

The next statement regarding two–weight criteria for the operator H on the cone of decreasing functions was proved in [18]:

THEOREM C. *Let v and w be weight functions on $\mathbb{R}_+$. Suppose that $1<p\leqslant q<\infty$. Then the inequality*

$$\left[\int_0^\infty \big(Hf(x)\big)^q v(x)dx\right]^{1/q} \leqslant C\left[\int_0^\infty (f(x))^p w(x)dx\right]^{1/p}, \quad f\in L^p_{dec}(u,\mathbb{R}_+),$$

holds if and only if the following two conditions are satisfied:

$$\sup_{a>0}\left(\int_0^a v(x)dx\right)^{1/q}\left(\int_0^a w(x)dx\right)^{-1/p}<\infty$$

and

$$\sup_{a>0}\left(\int_a^\infty \frac{v(x)}{x^q}dx\right)^{1/q}\left(\int_0^a W^{-p'}(x)x^{p'}w(x)dx\right)^{1/p'}<\infty,$$

where $W(x):=\int_0^x w(t)dt$.

3. Two–Sided Pointwise Estimates

In this section we establish auxiliary two–sided pointwise estimates for the weighted Riemann–Liouville transform with product kernels on the cones of decreasing functions which might have an independent interest. The appropriate estimate for the weighted multiple Weyl transform is also studied on the cone of increasing functions.

PROPOSITION 3.1. *Let $0<\alpha_i<1$. Then the following relations hold:*
(a)

$$R_{\alpha_1,\cdots,\alpha_n}f\approx H_nf, \quad f\in\mathscr{D};$$

(b)

$$W_{\alpha_1,\cdots,\alpha_n}f\approx H'_nf, \quad f\in\mathscr{I},$$

where

$$\left(H_n' f\right)(x_1,\cdots,x_n)=\int\limits_{x_1}^{\infty}\cdots\int\limits_{x_n}^{\infty}\frac{f(t_1,\cdots,t_n)}{\Pi_{i=1}^n t_i}dt_1\cdots dt_n.$$

Proof. (a) *Upper estimate.* For simplicity assume that $n=2$. Represent $R_{\alpha_1,\cdots,\alpha_n}$ as follows:

$$\begin{aligned}\left(R_{\alpha_1,\alpha_2}f\right)(x_1,x_2) &= \frac{1}{x_1^{\alpha_1}x_2^{\alpha_2}}\int_0^{x_1/2}\int_0^{x_2/2}\frac{f(t_1,t_2)}{(x_1-t_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}}dt_1dt_2\\ &\quad+\frac{1}{x_1^{\alpha_1}x_2^{\alpha_2}}\int_0^{x_1/2}\int_{x_2/2}^{x_2}\frac{f(t_1,t_2)}{(x_1-t_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}}dt_1dt_2\\ &\quad+\frac{1}{x_1^{\alpha_1}x_2^{\alpha_2}}\int_{x_1/2}^{x_1}\int_0^{x_2/2}\frac{f(t_1,t_2)}{(x_1-t_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}}dt_1dt_2\\ &\quad+\frac{1}{x_1^{\alpha_1}x_2^{\alpha_2}}\int_{x_1/2}^{x_1}\int_{x_2/2}^{x_2}\frac{f(t_1,t_2)}{(x_1-t_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}}dt_1dt_2\\ &:= \left(R^{(1)}_{\alpha_1,\alpha_2}f\right)(x_1,x_2)+\left(R^{(2)}_{\alpha_1,\alpha_2}f\right)(x_1,x_2)\\ &\quad+\left(R^{(3)}_{\alpha_1,\alpha_2}f\right)(x_1,x_2)+\left(R^{(4)}_{\alpha_1,\alpha_2}f\right)(x_1,x_2).\end{aligned}$$

Observe that if $0<t_1<x_1/2$ and $0<t_2<x_2/2$, then $x_1/2\leqslant x_1-t_1$ and $x_2/2\leqslant x_2-t_2$. Hence

$$\left(R^{(1)}_{\alpha_1,\alpha_2}f\right)(x_1,x_2)\leqslant c\left(H_2f\right)(x_1,x_2),$$

where the positive constant c does not depend on f, x_1 and x_2. Further, using the same argument for the first integral and the fact that f is decreasing in the second variable we find that

$$\begin{aligned}\left(R^{(2)}_{\alpha_1,\alpha_2}f\right)(x_1,x_2) &\leqslant c\frac{1}{x_1x_2^{\alpha_2}}\left(\int_0^{x_1}f(t_1,x_2/2)dt_1\right)\left(\int_{x_2/2}^{x_2}(x_2-t_2)^{\alpha_2-1}dt_2\right)\\ &=\frac{c}{x_1}\left(\int_0^{x_1}f(t_1,x_2/2)dt_1\right)=\frac{c}{x_1x_2}\int_0^{x_1}\int_0^{x_2/2}f(t_1,x_2/2)dt_1dt_2\\ &\leqslant c\left(H_2f\right)(x_1,x_2).\end{aligned}$$

The estimate for $R^{(3)}_{\alpha_1,\alpha_2}f$ is similar to that of $R^{(2)}_{\alpha_1,\alpha_2}f$.

It remains to estimate of $R^{(4)}_{\alpha_1,\alpha_2}f$. Since f is decreasing we have that

$$\begin{aligned}\left(R^{(4)}_{\alpha_1,\alpha_2}f\right)(x_1,x_2) &\leqslant c\frac{1}{x_2^{\alpha_1}}\frac{1}{x_2^{\alpha_1}}f\left(\frac{x_1}{2},\frac{x_2}{2}\right)\left(\int_{x_1/2}^{x_1}\frac{dt_1}{(x_1-t_1)^{1-\alpha_1}}\right)\left(\int_{x_2/2}^{x_2}\frac{dt_2}{(x_2-t_2)^{1-\alpha_2}}\right)\\ &\leqslant c\left(H_2f\right)(x_1,x_2).\end{aligned}$$

If $n\geqslant 2$, then we split the integral $\int_0^{x_1}\cdots\int_0^{x_n}$ as a sum of $n-$ dimensional multiple integrals consisting of one–dimensional integrals having one of the following forms $\int_0^{x_i/2}$, $\int_{x_j/2}^{x_j}$ and argue as in the case $n=2$.

Lower estimate follows immediately by using the fact that f is non–negative and the obvious estimates $x_i - t_i \leqslant x_i$, where $i = 1, \cdots, n$ and $0 < t_i < x_i$.

(b) *Upper estimate.* Let us assume for simplicity that $n = 2$. We have

$$\begin{aligned}\left(W_{\alpha_1,\alpha_2} f\right)(x_1,x_2) &= \int_{2x_1}^{\infty}\int_{2x_2}^{\infty}(\cdots) + \int_{2x_1}^{\infty}\int_{x_2}^{2x_2}(\cdots) + \int_{x_1}^{2x_1}\int_{2x_2}^{\infty}(\cdots) + \int_{x_1}^{2x_1}\int_{x_2}^{2x_2}(\cdots) \\ &=: \sum_{k=1}^{4}\left(W^{(k)}_{\alpha_1,\alpha_2} f\right)(x_1,x_2).\end{aligned}$$

The inequality

$$\left(W^{(1)}_{\alpha_1,\alpha_2} f\right)(x_1,x_2) \leqslant c\left(H_2' f\right)(x_1,x_2)$$

holds because $t_i/2 \leqslant t_i - x_i$ for $i = 1,2$ when $t_i > 2x_i$. By using the fact that f is increasing in each variable separately we find that

$$\begin{aligned}\left(W^{(4)}_{\alpha_1,\alpha_2} f\right)(x_1,x_2) &\leqslant c x_1^{-\alpha_1} x_2^{-\alpha_2} f(2x_1,2x_2) \int_{x_1}^{2x_1}\int_{x_1}^{2x_2}(t_1-x_1)^{\alpha_1-1}(t_2-x_2)^{\alpha_2-1}dt_1dt_2 \\ &\leqslant c f(2x_1,2x_2) \leqslant \frac{c}{x_1x_2}\int_{2x_1}^{4x_1}\int_{2x_2}^{4x_2} f(t_1,t_2)dt_1dt_2 \leqslant c\left(H_2' f\right)(x_1,x_2).\end{aligned}$$

These arguments with respect to each variable separately yield that

$$\left(W^{(k)}_{\alpha_1,\alpha_2} f\right)(x_1,x_2) \leqslant c\left(H_2' f\right)(x_1,x_2), \quad k = 2,3.$$

Finally we conclude that the desired upper estimate holds.

Lower estimate is a direct consequence of the inequalities $t_i - x_i \leqslant t_i$, $i \in \{1, \cdots, 4\}$, where $t_i > x_i$. □

4. One and two weight criteria for the multiple Riemann–Liouville transforms

This section is dedicated to one and two weighted criteria for the multiple Riemann-Liouville transforms and appropriate one–sided strong fractional maximal functions.

We begin with the one–weight result.

THEOREM 4.1. *Let $0 < p < \infty$. Then the following statements are equivalent:*
(i) $R_{\alpha_1,\cdots,\alpha_n}$ *is bounded from $L^p_{dec}(u,\mathbb{R}^n_+)$ to $L^p(u,\mathbb{R}^n_+)$;*
(ii) *condition* (2) *holds.*

To formulate our next result we need some notation:

$$\begin{aligned}W_j(x_j) &:= \int_0^{x_j} w_j(t)dt, \quad W(t_1,\cdots,t_n) := \Pi_{i=1}^n W_i(t_i); \\ V_j(x_j) &:= \int_{x_j}^{\infty} v_j(t)dt, \quad V(t_1,\cdots,t_n) := \Pi_{i=1}^n V_i(t_i); \\ p' &:= \frac{p}{p-1}, \quad 1 < p < \infty.\end{aligned}$$

Let

$$\left(\mathscr{R}_{\alpha_1,\alpha_2} f\right)(x_1,x_2) = \int_0^{x_1}\int_0^{x_2} \frac{f(t_1,t_2)dt_1dt_2}{(x_1-t_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}},$$

$$\left(\mathscr{M}^-_{\alpha_1,\alpha_2} f\right)(x_1,x_2) = \sup_{\substack{0<h_1\leqslant x_1\\ 0<h_2\leqslant x_2}} h_1^{\alpha_1-1}h_2^{\alpha_2-1}\int_{x_1-h_1}^{x_1}\int_{x_2-h_2}^{x_2} f(t_1,t_2)dt_1dt_2,$$

where $x_1,x_2\in\mathbb{R}_+$, $f\geqslant 0$ and $0<\alpha_i<1$, $i=1,2$.

The next statement gives two–weight criteria for $\mathscr{R}_{\alpha_1,\alpha_2}$ and $\mathscr{M}^-_{\alpha_1,\alpha_2}$.

THEOREM 4.2. *Let $1<p\leqslant q<\infty$ and let $0<\alpha_i<1$, $i=1,2$. Assume that v and w are weights on $\mathbb{R}^2_+$. Suppose also that $w(x_1,x_2)=w_1(x_1)w_2(x_2)$ for some one–dimensional weights w_1 and w_2, and that $W_i(\infty)=\infty$, $i=1,2$. Then the following conditions are equivalent:*

(a) $\mathscr{R}_{\alpha_1,\alpha_2}$ *is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$;*

(b) $\mathscr{M}^-_{\alpha_1,\alpha_2}$ *is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$;*

(c) *the following four conditions hold simultaneously:*

(i)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1}\int_0^{a_2} w(t_1,t_2)dt_1dt_2\right)^{-1/p} \times\left(\int_0^{a_1}\int_0^{a_2}\left(t_1^{\alpha_1}t_2^{\alpha_2}\right)^q v(t_1,t_2)dt_1dt_2\right)^{1/q}<\infty; \tag{3}$$

(ii)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1}\int_0^{a_2}(t_1t_2)^{p'}W^{-p'}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right)^{1/p'} \times\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty}\left(t_1^{\alpha_1-1}t_2^{\alpha_2-1}\right)^q v(t_1,t_2)dt_1dt_2\right)^{1/q}<\infty; \tag{4}$$

(iii)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1} w_1(t_1)dt_1\right)^{-1/p}\left(\int_0^{a_2} t_2^{p'}W_2^{-p'}(t_2)w_2(t_2)dt_2\right)^{1/p'} \times\left(\int_0^{a_1}\int_{a_2}^{\infty} t_1^{q\alpha_1}t_2^{q(\alpha_2-1)}v(t_1,t_2)dt_1dt_2\right)^{1/q}<\infty; \tag{5}$$

(iv)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1} t_1^{p'}W_1^{-p'}(t_1)w_1(t_1)dt_1\right)^{1/p'}\left(\int_0^{a_2} w_2(t_2)dt_2\right)^{-1/p} \times\left(\int_{a_1}^{\infty}\int_0^{a_2} t_1^{q(\alpha_1-1)}t_2^{q\alpha_2}v(t_1,t_2)dt_1dt_2\right)^{1/q}<\infty. \tag{6}$$

COROLLARY 4.1. *Let* $1 < p \leqslant q < \infty$ *and let* $0 < \alpha_i < 1,\ i = 1,2$. *Then the following conditions are equivalent:*

(a) *the boundedness of* $\mathscr{R}_{\alpha_1,\alpha_2}$ *from* $L^p_{dec}(w,\mathbb{R}^2_+)$ *to* $L^q(v,\mathbb{R}^2_+)$ *holds for* $w \equiv 1$;

(b) *the operator* $\mathscr{M}^-_{\alpha_1,\alpha_2}$ *is bounded from* $L^p_{dec}(w,\mathbb{R}^2_+)$ *to* $L^q(v,\mathbb{R}^2_+)$ *for* $w \equiv 1$;

(c)

$$B_1 := \sup_{a_1,a_2>0} B_1(a_1,a_2) := \sup_{a_1,a_2>0} (a_1a_2)^{-1/p}\left(\int_0^{a_1}\int_0^{a_2} x_1^{q\alpha_1}x_2^{q\alpha_2}v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty;$$

(d)

$$B_2 := \sup_{a_1,a_2>0} B_2(a_1,a_2) := \sup_{a_1,a_2>0} (a_1a_2)^{1/p'}\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty} x_1^{q(\alpha_1-1)}x_2^{q(\alpha_2-1)}v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty;$$

(e)

$$B_3 := \sup_{a_1,a_2>0} B_3(a_1,a_2) := \sup_{a_1,a_2>0} a_1^{-1/p}a_2^{1/p'}\left(\int_0^{a_1}\int_{a_2}^{\infty} x_1^{q\alpha_1}x_2^{q(\alpha_2-1)}v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty;$$

(f)

$$B_4 := \sup_{a_1,a_2>0} B_4(a_1,a_2) := \sup_{a_1,a_2>0} a_1^{1/p'}a_2^{-1/p}\left(\int_{a_1}^{\infty}\int_0^{a_2} x_1^{q(\alpha_1-1)}x_2^{q\alpha_2}v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty.$$

The following statement deals with the case $q < p$.

THEOREM 4.3. *Let* $1 < q < p < \infty$ *and let* $0 < \alpha_i < 1,\ i = 1,2$. *Assume that* v *and* w *are weights on* $\mathbb{R}^2_+$. *Suppose also that* $w(x_1,x_2) = w_1(x_1)w_2(x_2)$ *and that* $W_i(\infty) = \infty,\ i = 1,2$. *Then the following conditions are equivalent:*

(a) $\mathscr{R}_{\alpha_1,\alpha_2}$ *is bounded from* $L^p_{dec}(w,\mathbb{R}^2_+)$ *to* $L^q(v,\mathbb{R}^2_+)$;

(b) $\mathscr{M}^-_{\alpha_1,\alpha_2}$ *is bounded from* $L^p_{dec}(w,\mathbb{R}^2_+)$ *to* $L^q(v,\mathbb{R}^2_+)$;

(c) *the following four conditions hold:*

(i)

$$\left[\int_{\mathbb{R}^2_+}\left(\int_0^{t_1}\int_0^{t_2} v(x_1,x_2)\left(x_1^{\alpha_1}x_2^{\alpha_2}\right)^q dx_1dx_2\right)^{r/q} \times W^{-r/q}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right]^{1/r} < \infty;$$

(ii)

$$\Bigg[\int_{\mathbb{R}^2_+}\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} v(x_1,x_2)\left(x_1^{\alpha_1-1}x_2^{\alpha_2-1}\right)^q dx_1dx_2\right)^{r/q} \times\left(\int_0^{t_1}\int_0^{t_2}(x_1x_2)^{p'}W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2\right)^{r/q'} \times(t_1t_2)^{p'}W^{-p'}(t_1,t_2)w(t_1,t_2)dt_1dt_2\Bigg]^{1/r}<\infty;$$

(iii)

$$\Bigg[\int_{\mathbb{R}^2_+}\left(\int_0^{t_1}\int_{t_2}^{\infty} v(x_1,x_2)\left(x_1^{\alpha_1}x_2^{\alpha_2-1}\right)^q dx_1dx_2\right)^{r/q} W_1^{-r/q}(t_1) \times\left(\int_0^{t_2} x_2^{p'}W_2^{-p'}(x_2)w_2(x_2)dx_2\right)^{r/q'} t_2^{p'}W_2(t_2)w_2(t_2)dt_1dt_2\Bigg]^{1/r}<\infty;$$

(iv)

$$\Bigg[\int_{\mathbb{R}^2_+}\left(\int_{t_1}^{\infty}\int_0^{t_2} v(x_1,x_2)\left(x_1^{\alpha_1-1}x_2^{\alpha_2}\right)^q dx_1dx_2\right)^{r/q} W_2^{-r/q}(t_2) \times\left(\int_0^{t_1} x_1^{p'}W_1^{-p'}(x_1)w_1(x_1)dx_1\right)^{r/q'} t_1^{p'}W_1(t_1)w_1(t_1)dt_1dt_2\Bigg]^{1/r}<\infty,$$

where $1/r = 1/q - 1/p$.

REMARK 4.1. Analyzing the proof of Theorems 4.2 and 4.3 necessary and sufficient conditions governing the two–weight inequality

$$\left(\int_{\mathbb{R}^n_+}\left(\int_0^{x_1}\cdots\int_0^{x_n}\frac{f(t_1,\cdots,t_n)}{\Pi_{i=1}^n(x_i-t_i)^{1-\alpha_i}}dt_1\cdots dt_n\right)^q v(x_1,\cdots,x_n)dx_1\cdots dx_n\right)^{1/q} \leqslant c\left(\int_{\mathbb{R}^n_+} f^p(x_1,\cdots,x_n)w(x_1,\cdots,x_n)dx_1\cdots dx_n\right)^{1/p}$$

can also be obtained on $L^p_{dec}(w,\mathbb{R}^n_+)$, where w is a product weight.

Now we prove these statements.

Theorem 4.1 is a direct consequence of Theorem B and Proposition 3.1.

To prove Theorem 4.2 we need some auxiliary statements which are formulated for $n = 2$ but due to the induction they are also true for any $n > 2$ (see also [12], Chapter 1).

PROPOSITION 4.1. *Let* $1 < p \leqslant q < \infty$ *and let either* $v(x_1,x_2) = v_1(x_1)v_2(x_2)$ *or* $w(x_1,x_2) = w_1(x_1)w_2(x_2)$, *where* v_1, v_2, w_1 *and* w_2 *are one–dimensional weights. Then:*

(i) *the inequality*

$$\left(\int_{\mathbb{R}^2_+}\left(\int_0^{x_1}\int_0^{x_2} f\right)^q v(x_1,x_2)dx_1dx_2\right)^{1/q} \leqslant c\left(\int_{\mathbb{R}^2_+} f^p(x_1,x_2)w(x_1,x_2)dx_1dx_2\right)^{1/p} \quad (7)$$

holds for all functions $f \in L^p(w,\mathbb{R}^2_+)$ *if and only if*

$$\sup_{a_1,a_2>0}\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty} v(t_1,t_2)dt_1dt_2\right)^{1/q}\left(\int_0^{a_1}\int_0^{a_2} w^{1-p'}(t_1,t_2)dt_1dt_2\right)^{1/p'} < \infty;$$

(ii) *the inequality*

$$\left(\int_{\mathbb{R}^2_+}\left(\int_{x_1}^{\infty}\int_{x_2}^{\infty} f\right)^q v(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}} \leqslant c\left(\int_{\mathbb{R}^2_+} f^p(x_1,x_2)w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}} \quad (8)$$

holds for all $f \in L^p(w,\mathbb{R}^2_+)$ *if and only if*

$$D := \sup_{a_1,a_2>0}\left(\int_0^{a_1}\int_0^{a_2} v(t_1,t_2)dt_1dt_2\right)^{1/q}\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty} w^{1-p'}(t_1,t_2)dt_1dt_2\right)^{1/p'} < \infty;$$

(iii) *the inequality*

$$\left(\int_{\mathbb{R}^2_+}\left(\int_{x_1}^{\infty}\int_0^{x_2} f\right)^q v(x_1,x_2)dx_1dx_2\right)^{1/q} \leqslant c\left(\int_{\mathbb{R}^2_+} f^p(x_1,x_2)w(x_1,x_2)dx_1dx_2\right)^{1/p} \quad (9)$$

holds for all $f \in L^p(w,\mathbb{R}^2_+)$ *if and only if*

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1}\int_{a_2}^{\infty} v(t_1,t_2)dt_1dt_2\right)^{1/q}\left(\int_{a_1}^{\infty}\int_0^{a_2} w^{1-p'}(t_1,t_2)dt_1dt_2\right)^{1/p'} < \infty.$$

PROPOSITION 4.2. *Let* $1 < p \leqslant q < \infty$. *Assume that* v *and* w *are weights on* $\mathbb{R}^2_+$. *Suppose that* $w(x_1,x_2) = w_1(x_1)w_2(x_2)$ *and that* $W_i(\infty) = \infty$, $i = 1,2$. *Then inequality* (7) *holds for all functions* $f \in L^p_{dec}(w,\mathbb{R}^2_+)$ *if and only if the following four conditions are satisfied:*

(i)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1}\int_0^{a_2} w(t_1,t_2)dt_1dt_2\right)^{-1/p}\left(\int_0^{a_1}\int_0^{a_2}(t_1t_2)^q v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty;$$

(ii)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1}\int_0^{a_2}(t_1t_2)^{p'}W^{-p'}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right)^{1/p'} \times\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty} v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty;$$

(iii)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1} w_1(t_1)dt_1\right)^{-1/p}\left(\int_0^{a_2} t_2^{p'}W_2^{-p'}(t_2)w_2(t_2)dt_2\right)^{1/p'} \times\left(\int_0^{a_1}\int_{a_2}^{\infty} t_1^q v(t_1,t_2)dt_1dt_2\right)^{1/q}<\infty;$$

(iv)

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1} t_1^{p'}W_1^{-p'}(t_1)w_1(t_1)dt_1\right)^{1/p'}\left(\int_0^{a_2} w_2(t_2)dt_2\right)^{-1/p} \times\left(\int_{a_1}^{\infty}\int_0^{a_2} t_2^q v(t_1,t_2)dt_1dt_2\right)^{1/q}<\infty.$$

Proposition 4.2 was proved in [3] in the case when both v and w are product weights.

For the following statement we refer to [19] (see also [12], Section 1.4).

PROPOSITION 4.3. *Let $1<q<p<\infty$. Suppose that the weight v defined on $\mathbb{R}^2_+$ is of product type, i.e. $v(x_1,x_2)=v_1(x_1)v_2(x_2)$ for some one-dimensional weights v_1 and v_2. Suppose also that $V_1(0)=V_2(0)=\infty$. Then*

(i) *inequality (7) holds for all $f\in L^p(w,\mathbb{R}^2_+)$ if and only if*

$$\left[\int_{\mathbb{R}^2_+}\left(\int_0^{t_1}\int_0^{t_2} w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{r/p'} V^{r/p}(t_1,t_2)v(t_1,t_2)dt_1dt_2\right]^{1/r}<\infty;$$

(ii) *inequality (8) holds for all $f\in L^p(w,\mathbb{R}^2_+)$ if and only if*

$$\left[\int_{\mathbb{R}^2_+}\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{r/p'} \times\left(\int_0^{t_1}\int_0^{t_2} v(x_1,x_2)dx_1dx_2\right)^{r/p} v(t_1,t_2)dt_1dt_2\right]^{1/r}<\infty;$$

(iii) *inequality (9) holds for all $f\in L^p(w,\mathbb{R}^2_+)$ if and only if*

$$\left[\int_{\mathbb{R}^2_+}\left(\int_{t_1}^{\infty}\int_0^{t_2} w^{1-p'}(x_1,x_2)dx_1dx_2\right)^{r/p'} \times\left(\int_0^{t_1}\int_{t_2}^{\infty} v(x_1,x_2)dx_1dx_2\right)^{r/p} v(t_1,t_2)dt_1dt_2\right]^{1/r}<\infty.$$

PROPOSITION 4.4. *Let $1<p\leqslant q<\infty$. Assume that v and w are weights on $\mathbb{R}^n_+$. Suppose that $w(x_1,x_2)=w_1(x_1)w_2(x_2)$ and that $W_i(\infty)=\infty,\ i=1,2$. Then inequality*

(7) *holds for all functions* $f \in L^p_{dec}(w,\mathbb{R}^2_+)$ *if and only if the following conditions are satisfied:*

(i)

$$\left[\int_{\mathbb{R}^2_+}\left(\int_0^{t_1}\int_0^{t_2} v(x_1,x_2)\big(x_1x_2\big)^q dx_1dx_2\right)^{r/q} \times W^{-r/q}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right]^{1/r} < \infty;$$

(ii)

$$\left[\int_{\mathbb{R}^2_+} V^{r/q}(t_1,t_2)\left(\int_0^{t_1}\int_0^{t_2}\big(x_1x_2\big)^{p'}W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2\right)^{r/q'} \times\big(t_1t_2\big)^{p'}W^{-p'}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right]^{1/r} < \infty;$$

(iii)

$$\left[\int_{\mathbb{R}^2_+}\left(\int_0^{t_1}\int_{t_2}^{\infty} x_1^q v(x_1,x_2)dx_1dx_2\right)^{r/q} W_1^{-r/q}(t_1) \times\left(\int_0^{t_2} x_2^{p'}W_2^{-p'}(x_2)w_2(x_2)dx_2\right)^{r/q'} t_2^{p'}W_2^{-p'}(t_2)w(t_1,t_2)dt_1dt_2\right]^{1/r} < \infty;$$

(iv)

$$\left[\int_{\mathbb{R}^2_+}\left(\int_{t_1}^{\infty}\int_0^{t_2} x_2^q v(x_1,x_2)dx_1dx_2\right)^{r/q} W_2^{-r/q}(t_2) \times\left(\int_0^{t_1} x_1^{p'}W_1^{-p'}(x_1)w_1(x_1)dx_1\right)^{r/q'} t_1^{p'}W_1^{-p'}(t_2)w(t_1,t_2)dt_1dt_2\right]^{1/r} < \infty.$$

Proof of Proposition 4.1. First assume that $w(x_1,x_2) = w_1(x_1)w_2(x_2)$.

Part (i) was proved in [16] (see also [19], [12], Chapter 1); therefore we omit it.

Part (ii). Let us denote:

$$\begin{aligned}\overline{f}(x_1,x_2) &:= f\big(1/x_1,1/x_2\big)x_1^{-2}x_2^{-2},\\ \overline{v}(x_1,x_2) &:= v\big(1/x_1,1/x_2\big)x_1^{-2}x_2^{-2},\\ \overline{w}(x_1,x_2) &:= w\big(1/x_1,1/x_2\big)x_1^{2p-2}x_2^{2p-2}.\end{aligned}$$

Then it is easy to check that inequality (8) is equivalent to

$$\left[\int_{\mathbb{R}^2_+}\overline{v}(x_1,x_2)\left(\int_0^{x_1}\int_0^{x_2}\overline{f}\right)^q dx_1dx_2\right]^{1/q} \leqslant c\left(\int_{\mathbb{R}^2_+}\overline{f}^p\,\overline{w}\right)^{1/p}.$$

Observe now that

$$D = \sup_{a_1,a_2>0} \left(\int_{a_1}^{\infty} \int_{a_2}^{\infty} \overline{v} \right)^{1/q} \left(\int_0^{a_1} \int_0^{a_2} \overline{w}^{1-p'} \right)^{1/p}.$$

By using Part (i) of Proposition 4.1 and obvious change of variables we have the desired result.

Part (iii) can be proved in a similar manner; therefore we omit the details.

Let now $v(x_1,x_2) = v_1(x_1)v_2(x_2)$. Then the result follows using the duality arguments. For example, inequality (8) holds if and only if

$$\left(\int_{\mathbb{R}^2_+} \left(\int_0^{x_1} \int_0^{x_2} f \right)^{p'} w^{1-p'}(x_1,x_2)dx_1dx_2 \right)^{1/p'}$$
$$\leqslant c \left(\int_{\mathbb{R}^2_+} f^{q'}(x_1,x_2)v^{1-q'}(x_1,x_2)dx_1dx_2 \right)^{1/q'},$$

where the weight function $v^{1-q'}(x_1,x_2)$ is of product type. Consequently, using the result of the previous case we are done. □

Proof of Proposition 4.2. We follow the proof of Theorem 5.3 in [3]. In that paper it was shown that if w is a product weight, i.e., $w(x_1,x_2) = w_1(x_1)w_2(x_2)$, such that $W_i(\infty) = \infty$, $i = 1,2$, and v is any weight on $\mathbb{R}^2_+$, then inequality (7) holds for all $f \in L^p_{dec}(w,\mathbb{R}^2_+)$ if and only if

$$\left(\int_{\mathbb{R}^2_+} \left(\int_0^{x_1} \int_0^{x_2} \left[\int_{\tau_1}^{\infty} \int_{\tau_2}^{\infty} g(t_1,t_2)dt_1dt_2 \right] d\tau_1 d\tau_2 \right)^{p'} W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2 \right)^{1/p'}$$
$$\leqslant c \left(\int_{\mathbb{R}^2_+} g^{q'}(x_1,x_2)v^{1-q'}(x_1,x_2)dx_1dx_2 \right)^{1/q'}, \quad g \geqslant 0. \tag{10}$$

Further, we have that

$$\begin{aligned} &\int_0^{x_1} \int_0^{x_2} \left(\int_{\tau_1}^{\infty} \int_{\tau_2}^{\infty} g \right) d\tau_1 d\tau_2 \\ &= \int_0^{x_1} \int_0^{x_2} t_1t_2g(t_1,t_2)dt_1dt_2 + x_1 \int_{x_1}^{\infty} \int_0^{x_2} t_2g(t_1,t_2)dt_1dt_2 \\ &\quad + x_2 \int_0^{x_1} \int_{x_2}^{\infty} t_1g(t_1,t_2)dt_1dt_2 + x_1x_2 \int_{x_1}^{\infty} \int_{x_2}^{\infty} g(t_1,t_2)dt_1dt_2 \\ &:= I_1(x_1,x_2) + I_2(x_1,x_2) + I_3(x_1,x_2) + I_4(x_1,x_2). \end{aligned}$$

It is obvious that (10) is satisfied if and only if

$$\left(\int_{\mathbb{R}^2_+} I_j^{p'}(x_1,x_2)W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2 \right)^{1/p'} \leqslant c \left(\int_{\mathbb{R}^2_+} g^{q'}v^{1-q'} \right)^{1/q'} \tag{11}$$

for $j = 1,2,3,4$. By applying Proposition 4.1 (Part (i)) we find that

$$\left(\int_{\mathbb{R}^2_+} I_1^{p'}(x_1,x_2)W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2\right)^{1/p'} \leqslant c\left(\int_{\mathbb{R}^2_+} g^{q'}(x_1,x_2)v^{1-q'}(x_1,x_2)dx_1dx_2\right)^{1/q'}$$

if and only if

$$\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty} W^{-p'}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right)^{1/p'}\left(\int_0^{a_1}\int_0^{a_2}\left(\frac{v^{1-q'}(x_1,x_2)}{(x_1x_2)^{q'}}\right)^{1-q}dt_1dt_2\right)^{1/q}$$
$$= c_p\left(\int_0^{a_1}\int_0^{a_2} w(t_1,t_2)dt_1dt_2\right)^{-1/p}\left(\int_0^{a_1}\int_0^{a_2} v(x_1,x_2)(x_1x_2)^q dx_1dx_2\right)^{1/q} \leqslant C.$$

Taking now Propositions 4.1 (Part (ii)) into account we find that (11) holds for $j = 4$ if and only if condition (ii) is satisfied, while Proposition 4.1 (Part (iii)) and the following observation:

$$\sup_{a_1,a_2>0}\left(\int_{a_1}^{\infty} w_1(t_1)W_1^{-p'}(t_1)dt_1\right)^{1/p'}\left(\int_0^{a_2} t_2^{p'}W_2^{-p'}(t_2)w_2(t_2)dt_2\right)^{1/p'}$$
$$\times\left(\int_0^{a_1}\int_{a_2}^{\infty} t_1^q v(t_1,t_2)dt_1dt_2\right)^{1/q}$$
$$= c_p\sup_{a_1,a_2>0}\left(\int_0^{a_1} w_1(t_1)dt_1\right)^{-1/p}\left(\int_0^{a_2} t_2^{p'}W_2^{-p'}(t_2)w_2(t_2)dt_2\right)^{1/p'}$$
$$\times\left(\int_0^{a_1}\int_{a_2}^{\infty} t_1^q v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty;$$

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1} t_1^{p'}W_1^{-p'}(t_1)w_1(t_1)dt_1\right)^{1/p'}\left(\int_{a_2}^{\infty} w_2(t_2)W_2^{-p'}(t_2)dt_2\right)^{1/p'}$$
$$\times\left(\int_{a_1}^{\infty}\int_0^{a_2} t_2^q v(t_1,t_2)dt_1dt_2\right)^{1/q}$$
$$= c_p\sup_{a_1,a_2>0}\left(\int_0^{a_1} t_1^{p'}W_1^{-p'}(t_1)w_1(t_1)dt_1\right)^{1/p'}\left(\int_0^{a_2} w_2(t_2)dt_2\right)^{-1/p}$$
$$\times\left(\int_{a_1}^{\infty}\int_0^{a_2} t_2^q v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty$$

yield (11) for $j = 2,3$. □

Proof of Proposition 4.3. Part (i) was proved in [19] (see also [12], Section 1.4).

Part (ii) follows from Part (i) by using the arguments of the proof of Part (ii) of Proposition 4.1 and observing that

$$\int_0^{\infty}\overline{v}_i(t)dt = \infty$$

if and only if $V_i(\infty)=\infty$ $(i=1,2)$, where $\overline{v}_i(x)=v_i(1/x)x^{-2}$.

Part (iii) follows by the similar arguments. $\square$

Proof of Proposition 4.4. We keep the notation of the proof of Proposition 4.2. First observe that the condition $W_i(\infty)=\infty$ implies that $\int_0^\infty W_i^{-p'}(t)w(t)dt=\infty,\ i=1,2$.

Inequality (11) for $i=1$ holds if and only if

$$\Bigg[\bigg(\int_{\mathbb{R}_+}\int_0^{t_1}\int_0^{t_2} v(x_1,x_2)(x_1x_2)^q dx_1dx_2\bigg)^{r/q} \times\bigg(\int_{t_1}^\infty\int_{t_2}^\infty W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2\bigg)^{r/q'} W^{-p'}(t_1,t_2)w(t_1,t_2)dt_1dt_2\Bigg]^{1/r}$$
$$=\left[\int_{\mathbb{R}_+^2}\bigg(\int_0^{t_1}\int_0^{t_2} v(x_1,x_2)\big(x_1x_2\big)^q dx_1dx_2\bigg)^{r/q} W^{-r/q}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right]^{1/r}<\infty.$$

Analogously it follows that inequality (11) holds for $i=2,3,4$ if and only if conditions (iv), (iii) and (ii) are satisfied respectively. $\square$

Proof of Theorem 4.2. Let us denote

$$\big(\mathscr{H}_{\alpha_1,\alpha_2}f\big)(x_1,x_2):=x_1^{\alpha_1-1}x_2^{\alpha_2-1}\int_0^{x_1}\int_0^{x_2} f(t_1,t_2)dt_1dt_2.$$

It is easy to see that Proposition 3.1 yields the following relation:

$$\mathscr{R}_{\alpha_1,\alpha_2}f\approx\mathscr{H}_{\alpha_1,\alpha_2}f$$

for all decreasing functions $f:\mathbb{R}_+^2\to\mathbb{R}_+$. Now Proposition 4.2 completes the proof of the equivalence $(a)\Leftrightarrow(c)$.

The fact that $(b)\Leftrightarrow(c)$ can be obtained from the inequalities:

$$\big(\mathscr{H}_{\alpha_1,\alpha_2}f\big)(x_1,x_2)\leqslant\big(\mathscr{M}^-_{\alpha_1,\alpha_2}f\big)(x_1,x_2)\leqslant\big(\mathscr{R}_{\alpha_1,\alpha_2}f\big)(x_1,x_2)\leqslant c\big(\mathscr{H}_{\alpha_1,\alpha_2}f\big)(x_1,x_2)$$

and Proposition 4.2. $\square$

Proof of Corollary 4.1. Due to Theorem 4.2 it is enough to show that conditions (c), (d), (e) and (f) are equivalent. We prove, for example, that $(\mathrm{c})\Leftrightarrow(\mathrm{d})$. The facts that $(\mathrm{c})\Leftrightarrow(\mathrm{e})$, $(\mathrm{c})\Leftrightarrow(\mathrm{f})$ follow analogously.

Let $a_1,a_2>0$. Then there are integers m and n such that $a_1\in[2^m,2^{m+1})$, $a_2\in[2^n,2^{n+1})$. We have

$$\begin{aligned}
B_1^q(a_1,a_2) &\leqslant 2^{-mq/p}2^{-nq/p}\int_0^{2^{m+1}}\int_0^{2^{n+1}} t_1^{\alpha_1 q}t_2^{\alpha_2 q}v(x_1,x_2)dx_1dx_2\\
&= 2^{-mq/p}2^{-nq/p}\sum_{k=-\infty}^{m}\sum_{j=-\infty}^{n}\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}} t_1^{\alpha_1 q}t_2^{\alpha_2 q}v(x_1,x_2)dx_1dx_2\\
&\leqslant C_{\alpha_1,\alpha_2,q}2^{-\frac{mq}{p}}2^{-\frac{nq}{p}}\sum_{k=-\infty}^{m}\sum_{j=-\infty}^{n}2^{kq}2^{jq}\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}}\frac{v(x_1,x_2)}{x_1^{(1-\alpha_1)q}x_2^{(1-\alpha_2)q}}dx_1dx_2\\
&\leqslant C_{\alpha_1,\alpha_2,q}B_2^q 2^{-mq/p}2^{-nq/p}\sum_{k=-\infty}^{m}\sum_{j=-\infty}^{n}2^{kq/p}2^{jq/p}\leqslant CB_2^q.
\end{aligned}$$

Consequently, $B_1 \leqslant cB_2$. Conversely:

$$\begin{aligned}
B_2(a_1,a_2)^q &\leqslant C_{p,q}2^{mq/p'}2^{nq/p'}\sum_{k=m}^{\infty}\sum_{j=n}^{\infty}\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}}\frac{v(x_1,x_2)}{x_1^{(1-\alpha_1)q}x_2^{(1-\alpha_2)q}}dx_1dx_2\\
&\leqslant C_{p,q}2^{mq/p'}2^{nq/p'}\sum_{k=m}^{\infty}\sum_{j=n}^{\infty}2^{-kq}2^{-jq}\int_{2^k}^{2^{k+1}}\int_{2^j}^{2^{j+1}}v(x_1,x_2)x_1^{\alpha_1 q}x_2^{\alpha_2 q}dx_1dx_2\\
&\leqslant C_{p,q}B_1^q 2^{mq/p'}2^{nq/p'}\sum_{k=m}^{\infty}\sum_{j=n}^{\infty}2^{-kq/p'}2^{-jq/p'}\leqslant CB_1^q. \qquad \square
\end{aligned}$$

Proof of Theorem 4.3. is similar to that of Theorem 4.2. The difference is that in this case we apply Proposition 4.4 instead of Proposition 4.2. □

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REFERENCES

[1] N. ARCOZZI, S. BARZA, J. L. GARCIA-DOMINGO AND J. SORIA, *Hardy's inequalities for monotone functions on partially ordered measure spaces*, Proc. Roy. Soc. Edinburgh Sect. A **136**, 5 (2006), 909–919.

[2] M. A. ARINO AND B. MUCKENHOUPT, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320**, 2 (1990), 727–735.

[3] S. BARZA, P. H. HEINIG AND L.–E. PERSSON, *Duality theorem over the cone of monotone functions and sequences in higher dimensions*, J. Inequal. Appl. **7**, 1 (2002), 79–108.

[4] S. BARZA, L.-E PERSSON AND J. SORIA, *Sharp weighted multidimensional integral inequalities for monotone functions*, Math. Nachr. **210** (2000), 43–58.
[5] M. J. CARRO, J. A. RAPOSO AND J. SORIA, *Recent developments in the theory of Lorentz spaces and weighted inequalities*, Mem. Amer. Math. Soc. **187**, 887 (2007), 128 pp.
[6] D. E. EDMUNDS, V. KOKILASHVILI AND A. MESKHI, *Bounded and compact integral operators*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
[7] I.GENEBASHVILI, A. GOGATISHVILI,V.KOKILASHVILI AND M. KRBEC, *Weight theory for integral transforms on spaces of homogeneous type*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 92, Longman, Harlow, 1998.
[8] V.KOKILASHVILI AND M. KRBEC, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific, Singapore, New Jersey, London , Hong Kong, 1991.
[9] V. KOKILASHVILI AND A. MESKHI, *On one–sided potentials with multiple kernels*, Integr. Transf. Spec. Funct. **16**, 8 (2005), 669–683.
[10] V. KOKILASHVILI AND A. MESKHI, *On a trace inequality for one-sided potentials with multiple kernels*, Frac. Calc. Appl. Anal. **6**, 4 (2003), 461–472.
[11] V. KOKILASHVILI AND A. MESKHI, *Two-weight estimates for strong fractional maximal functions and potentials with multiple kernels*, J. Korean Math. Soc. **46**, 3 (2009), 523–550.
[12] V. KOKILASHVILI, A. MESKHI AND L.-E. PERSSON, *Weighted norm inequalities for integral transforms with product kernels*, Nova Science Publishers, New York, 2009.
[13] K. KRULIĆ, J. PEČARIĆ AND L.–E. PERSSON, *Some new Hardy type inequalities with general kernels*, Math. Inequal. Appl. **12**, 3 (2009), 473–485.
[14] A. KUFNER, L. MALIGRANDA AND L.–E. PERSSON, *The Hardy inequality–about its history and some related results*, Vydavatelsky Servis Publishing House, Pilsen, 2007.
[15] A. KUFNER AND L.–E. PERSSON, *Weighted inequalities of Hardy type,* World Scientific Publishing Co, Singapore, New Jersey, London, Hong Kong, 2003.
[16] A. MESKHI, *A note on two-weight inequalities for multiple Hardy-type operators*, J. Funct. Spaces Appl. **3** (2005), 223–237.
[17] J. OGUNTUASE, C. OKPOTI, L. -E. PERSSON AND F. ALLOTEY, *Weighted multidimensional Hardy type inequalities via Jensen's inequality*, Proc. A. Razmadze Inst. **144** (2007), 91–105.
[18] E. SAWYER, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145–158.
[19] E. USHAKOVA, *Norm inequalities of Hardy and Pólya–Knopp types*, PhD Thesis, Department of Mathematics, Luleå University of Technology, 2006.

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