AN OPERATOR EXTENSION OF THE PARALLELOGRAM
LAW AND RELATED NORM INEQUALITIES

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Dedicated to my dear teacher
Aziz Atai Langroudi
with respect and affection

Abstract. We establish a general operator parallelogram law concerning a characterization of inner product spaces, get an operator extension of Bohr's inequality and present several norm inequalities. More precisely, let $\mathfrak{A}$ be a C^*-algebra, T be a locally compact Hausdorff space equipped with a Radon measure μ and let $(A_t)_{t \in T}$ be a continuous field of operators in $\mathfrak{A}$ such that the function $t \mapsto A_t$ is norm continuous on T and the function $t \mapsto \|A_t\|$ is integrable. If $\alpha : T \times T \to \mathbb{C}$ is a measurable function such that $\overline{\alpha(t,s)}\alpha(s,t) = 1$ for all $t,s \in T$, then we show that

$$\int_T \int_T |\alpha(t,s)A_t - \alpha(s,t)A_s|^2 \, d\mu(t)d\mu(s) + \int_T \int_T |\alpha(t,s)B_t - \alpha(s,t)B_s|^2 \, d\mu(t)d\mu(s)$$
$$= 2\int_T \int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 \, d\mu(t)d\mu(s) - 2\left|\int_T (A_t - B_t)d\mu(t)\right|^2.$$

1. Introduction

Let $\mathfrak{A}$ be a C^*-algebra and let T be a locally compact Hausdorff space. A field $(A_t)_{t \in T}$ of operators in $\mathfrak{A}$ is called a continuous field of operators if the function $t \mapsto A_t$ is norm continuous on T. If $\mu(t)$ is a Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, one can form the Bochner integral $\int_T A_t \mathrm{d}\mu(t)$, which is the unique element in $\mathfrak{A}$ such that

$$\varphi\left(\int_T A_t \mathrm{d}\mu(t)\right) = \int_T \varphi(A_t)\mathrm{d}\mu(t)$$

for every linear functional φ in the norm dual $\mathfrak{A}^*$ of $\mathfrak{A}$; see [8, Section 4.1] and [7].

Let $\mathbb{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$ endowed with inner product $\langle \cdot, \cdot \rangle$. We denote the absolute value of $A \in \mathbb{B}(\mathscr{H})$ by $|A| = (A^*A)^{1/2}$. For $x,y \in \mathscr{H}$, the rank one operator $x \otimes y$ is defined on $\mathscr{H}$ by $(x \otimes y)(z) = \langle z, y \rangle x$.

Let $A \in \mathbb{B}(\mathscr{H})$ be a compact operator and let $0 < p < \infty$. The Schatten p-norm (p-quasi-norm) for $1 \leqslant p < \infty (0 < p < 1)$ is defined by $\|A\|_p = (\mathrm{tr}|A|^p)^{1/p}$, where tr is the usual trace functional. Clearly

$$\| |A|^p \|_1 = \|A\|_p^p \tag{1.1}$$

for $p > 0$. For $p > 0$, the Schatten p-class, denoted by $\mathscr{C}_p$, is defined to be the two-sided ideal in $\mathbb{B}(\mathscr{H})$ of those compact operators A for which $\|A\|_p$ is finite. In particular, $\mathscr{C}_1$ and $\mathscr{C}_2$ are the trace class and the Hilbert-Schmidt class, respectively. For $1 \leqslant p < \infty$, $\mathscr{C}_p$ is a Banach space; in particular the triangle inequality holds. However, for $0 < p < 1$, the quasi-norm $\|.\|_p$ does not satisfy the triangle inequality. In addition to Schatten p-norms and operator norm, there are other interesting norms defined on some ideals contained in the ideal of compact operators. A unitarily invariant norm $||| \cdot |||$ is defined only on a norm ideal $\mathscr{C}_{|||\cdot|||}$ associated with it and has the property $|||UAV||| = |||A|||$, where U and V are unitaries and $A \in \mathscr{C}_{|||\cdot|||}$. For more information on the theory of the unitarily invariant norms the reader is referred to [15].
One can show that

$$\sum_{i,j=1}^n \|x_i - x_j\|^2 + \sum_{i,j=1}^n \|y_i - y_j\|^2 = 2 \sum_{i,j=1}^n \|x_i - y_j\|^2 - 2 \left\| \sum_{i=1}^n (x_i - y_i) \right\|^2 \tag{1.2}$$

holds in an inner product space, which is indeed a generalization of the classical *parallelogram law*:

$$|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2 \qquad (z, w \in \mathbb{C}).$$

There are several extensions of parallelogram law among them we could refer the interested reader to [4, 5, 12, 17, 6]. Generalizations of the parallelogram law for the Schatten p-norms have been given in the form of the celebrated Clarkson inequalities (see [10] and references therein). Since $\mathscr{C}_2$ is a Hilbert space under the inner product $\langle A, B \rangle = \mathrm{tr}(B^*A)$, it follows from (1.2) that if $A_1, \cdots, A_n, B_1, \cdots, B_n \in \mathscr{C}_2$ with $\sum_{i=1}^n (A_i - B_i) = 0$, then

$$\sum_{i,j=1}^n \|A_i - A_j\|_2^2 + \sum_{i,j=1}^n \|B_i - B_j\|_2^2 = 2 \sum_{i,j=1}^n \|A_i - B_j\|_2^2. \tag{1.3}$$

The classical *Bohr's inequality* states that for any $z, w \in \mathbb{C}$ and any positive real numbers r, s with $\frac{1}{r} + \frac{1}{s} = 1$,

$$|z + w|^2 \leqslant r|z|^2 + s|w|^2.$$

Many interesting operator generalizations of this inequality have been obtained; cf. [1, 3, 9, 14, 16].

In this paper, we establish an extended operator parallelogram law and get a generalization of Bohr's inequality. We also present several unitarily invariant and Schatten p-norm inequalities. Our results can be regarded as extensions of main results of [11].

2. Joint extensions of the parallelogram law and Bohr's inequality

We start our work with following clear lemma. The first equality is called the operator parallelogram law.

LEMMA 2.1. *Let* $A, B \in \mathbb{B}(\mathscr{H})$. *Then*

$$|A + B|^2 + |A - B|^2 = 2|A|^2 + 2|B|^2 \qquad (a);$$

and

$$|A + B|^2 - |A - B|^2 = 4Re(A^*B) \qquad (b)$$

We now state our main result, which is an operator version of equality (1.2). As we will see later, it is indeed a joint operator extension of Bohr and parallelogram inequalities.

THEOREM 2.2. *Let* $\mathfrak{A}$ *be a* C^*-*algebra,* T *be a locally compact Hausdorff space equipped with a Radon measure* μ *and let* $(A_t)_{t \in T}$ *be a continuous field of operators in* $\mathfrak{A}$ *such that the function* $t \mapsto A_t$ *is norm continuous on* T *and the function* $t \mapsto \|A_t\|$ *is integrable. Let* $\alpha : T \times T \to \mathbb{C}$ *be a measurable function such that* $\overline{\alpha(t,s)}\alpha(s,t) = 1$ *for all* $t, s \in T$. *Then*

$$\int_T \int_T |\alpha(t,s)A_t - \alpha(s,t)A_s|^2 d\mu(t)d\mu(s) + \int_T \int_T |\alpha(t,s)B_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s)$$

$$= 2\int_T \int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s) - 2\left|\int_T (A_t - B_t)d\mu(t)\right|^2.$$

Proof.

$$\int_T \int_T |\alpha(t,s)A_t - \alpha(s,t)A_s|^2 d\mu(t)d\mu(s) + \int_T \int_T |\alpha(t,s)B_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s)$$

$$= \int_T \int_T \left(|\alpha(t,s)A_t - \alpha(s,t)A_s|^2 + |\alpha(t,s)B_t - \alpha(s,t)B_s|^2\right) d\mu(t)d\mu(s)$$

$$= \int_T \int_T \left(\frac{1}{2}|\alpha(t,s)A_t - \alpha(s,t)A_s + \alpha(t,s)B_t - \alpha(s,t)B_s|^2\right.$$

$$\left. + \frac{1}{2}|\alpha(t,s)A_t - \alpha(s,t)A_s - \alpha(t,s)B_t + \alpha(s,t)B_s|^2\right) d\mu(t)d\mu(s)$$

(by Lemma 2.1(a))

$$= \int_T \int_T \left(\frac{1}{2}|(\alpha(t,s)A_t - \alpha(s,t)B_s) - (\alpha(s,t)A_s - \alpha(t,s)B_t)|^2\right.$$

$$\left. + \frac{1}{2}|(\alpha(t,s)A_t - \alpha(t,s)B_t) - (\alpha(s,t)A_s - \alpha(s,t)B_s)|^2\right) d\mu(t)d\mu(s)$$

$$= \int_T \int_T \left[\left(|\alpha(t,s)A_t - \alpha(s,t)B_s|^2 + |\alpha(s,t)A_s - \alpha(t,s)B_t|^2\right.\right.$$

$$-\frac{1}{2}\big|(\alpha(t,s)A_t - \alpha(s,t)B_s) + (\alpha(s,t)A_s - \alpha(t,s)B_t)\big|^2\Big)$$
$$+\frac{1}{2}\big|(\alpha(t,s)A_t - \alpha(t,s)B_t) - (\alpha(s,t)A_s - \alpha(s,t)B_s)\big|^2\Big]d\mu(t)d\mu(s)$$

$$\text{(by Lemma 2.1(a))}$$

$$= \int_T\int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s) + \int_T\int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s)$$
$$-\frac{1}{2}\int_T\int_T \Big(\big|(\alpha(t,s)A_t - \alpha(t,s)B_t) + (\alpha(s,t)A_s - \alpha(s,t)B_s)\big|^2$$
$$-\big|(\alpha(t,s)A_t - \alpha(t,s)B_t) - (\alpha(s,t)A_s - \alpha(s,t)B_s)\big|^2\Big)d\mu(t)d\mu(s)$$
$$= 2\int_T\int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s)$$
$$-2\operatorname{Re}\int_T\int_T (\alpha(t,s)A_t - \alpha(t,s)B_t)^*(\alpha(s,t)A_s - \alpha(s,t)B_s)d\mu(t)d\mu(s)$$

$$\text{(by Lemma 2.1(b))}$$

$$= 2\int_T\int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s)$$
$$-2\operatorname{Re}\Big[\Big(\int_T (A_t - B_t)d\mu(t)\Big)^*\Big(\int_T (A_s - B_s)d\mu(s)\Big)\Big]$$
$$= 2\int_T\int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s) - 2\Big|\int_T (A_t - B_t)d\mu(t)\Big|^2. \qquad \square$$

If we let $T = \{1,\cdots,n\}$, μ be the counting measure on T and $\alpha(i,j) = \sqrt{\frac{r_i}{r_j}}$, where $r_i > 0$ $(1 \leqslant i \leqslant n)$, in Theorem 2.2, then we get

COROLLARY 2.3. (Generalized Parallelogram Law) *Let* $A_1,\cdots,A_n,B_1,\cdots,B_n \in \mathbb{B}(\mathcal{H})$ *and let* $r_1,\cdots,r_n$ *be positive numbers. Then*

$$\sum_{1\leqslant i<j\leqslant n}\Big|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j\Big|^2 + \sum_{1\leqslant i<j\leqslant n}\Big|\sqrt{\frac{r_i}{r_j}}B_i - \sqrt{\frac{r_j}{r_i}}B_j\Big|^2$$
$$= \sum_{i,j=1}^n\Big|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}B_j\Big|^2 - \Big|\sum_{i=1}^n(A_i - B_i)\Big|^2. \qquad (2.1)$$

If we set $B_1 = \cdots = B_n = 0$ in Corollary 2.3, then the following extension of parallelogram law is obtained

COROLLARY 2.4. [6, Theorem 4.2] *Suppose that* $A_1,\cdots,A_n \in \mathbb{B}(\mathcal{H})$ *and* $r_1,\cdots,r_n$ *are positive numbers with* $\sum_{i=1}^n \frac{1}{r_i} = 1$. *Then*

$$(0 \leqslant) \sum_{1\leqslant i<j\leqslant n}\Big|\sqrt{\frac{r_i}{r_j}}A_i - \sqrt{\frac{r_j}{r_i}}A_j\Big|^2 = \sum_{i=1}^n r_i|A_i|^2 - \Big|\sum_{i=1}^n A_i\Big|^2. \qquad (2.2)$$

REMARK 2.5. If $n = 2$ and $t := \frac{r_1}{r_2} > 0$, then operator equality (2.2) can be restated as the following form which is, as noted in [6], a generalization of [9, Theorem 1] and [3, Theorem 3].

$$|A_1 + A_2|^2 + \frac{1}{t}|tA_1 - A_2|^2 = (1+t)|A_1|^2 + \left(1 + \frac{1}{t}\right)|A_2|^2.$$

We also infer the following extension of Bohr's inequality [16, Theorem 7] from (2.2).

$$\left|\sum_{i=1}^{n} A_i\right|^2 \leqslant \sum_{i=1}^{n} r_i |A_i|^2.$$

The preceding inequality is, in turn, a special case of the following general form of the Bohr inequality being easily deduced from the fact that the left hand side of the operator equality in Theorem 2.2 is a positive element of the C^*-algebra $\mathfrak{A}$.

COROLLARY 2.6. (Generalized Operator Bohr's Inequality) *Let $\mathfrak{A}$ be a C^*-algebra, T be a locally compact Hausdorff space equipped with a Radon measure μ and let $(A_t)_{t \in T}$ be a continuous field of operators in $\mathfrak{A}$ such that the function $t \mapsto A_t$ is norm continuous on T and the function $t \mapsto \|A_t\|$ is integrable. Let $\alpha : T \times T \to \mathbb{C}$ be a measurable function such that $\overline{\alpha(t,s)}\alpha(s,t) = 1$ for all $t, s \in T$. Then*

$$\left|\int_T (A_t - B_t)d\mu(t)\right|^2 \leqslant \int_T \int_T |\alpha(t,s)A_t - \alpha(s,t)B_s|^2 d\mu(t)d\mu(s).$$

A weighted extension of norm equality (1.2) can be deduced from (2.1) as follows:

COROLLARY 2.7. *Let $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathscr{H}$ and let $r_1, \cdots, r_n$ be positive numbers. Then*

$$\sum_{1 \leqslant i < j \leqslant n} \left\|\sqrt{\frac{r_i}{r_j}}x_i - \sqrt{\frac{r_j}{r_i}}x_j\right\|^2 + \sum_{1 \leqslant i < j \leqslant n} \left\|\sqrt{\frac{r_i}{r_j}}y_i - \sqrt{\frac{r_j}{r_i}}y_j\right\|^2$$

$$= \sum_{i,j=1}^{n} \left\|\sqrt{\frac{r_i}{r_j}}x_i - \sqrt{\frac{r_j}{r_i}}y_j\right\|^2 - \left\|\sum_{i=1}^{n}(x_i - y_i)\right\|^2.$$

Proof. Let e be a non-zero vector of $\mathscr{H}$ and set $A_i = x_i \otimes e, B_i = y_i \otimes e$ for $i = 1, \cdots, n$. It follows from the elementary properties of rank one operators and equality (2.1) that

$$\left(\sum_{i,j=1}^{n} \left\|\sqrt{\frac{r_i}{r_j}}x_i - \sqrt{\frac{r_j}{r_i}}x_j\right\|^2 + \sum_{i,j=1}^{n} \left\|\sqrt{\frac{r_i}{r_j}}y_i - \sqrt{\frac{r_j}{r_i}}y_j\right\|^2\right)e \otimes e$$

$$= \sum_{i,j=1}^{n} \left|\left(\sqrt{\frac{r_i}{r_j}}x_i - \sqrt{\frac{r_j}{r_i}}x_j\right) \otimes e\right|^2 + \sum_{i,j=1}^{n} \left|\left(\sqrt{\frac{r_i}{r_j}}y_i - \sqrt{\frac{r_j}{r_i}}y_j\right) \otimes e\right|^2$$

$$= \sum_{i,j=1}^{n} \left| \sqrt{\frac{r_i}{r_j}} A_i - \sqrt{\frac{r_j}{r_i}} A_j \right|^2 + \sum_{i,j=1}^{n} \left| \sqrt{\frac{r_i}{r_j}} B_i - \sqrt{\frac{r_j}{r_i}} B_j \right|^2$$

$$= 2 \sum_{i,j=1}^{n} \left| \sqrt{\frac{r_i}{r_j}} A_i - \sqrt{\frac{r_j}{r_i}} B_j \right|^2 - 2 \left| \sum_{i=1}^{n} (A_i - B_i) \right|^2$$

$$= 2 \sum_{i,j=1}^{n} \left| \left(\sqrt{\frac{r_i}{r_j}} x_i - \sqrt{\frac{r_j}{r_i}} y_j \right) \otimes e \right|^2 - 2 \left| \sum_{i=1}^{n} (x_i - y_i) \otimes e \right|^2.$$

$$= \left(2 \sum_{i,j=1}^{n} \left\| \sqrt{\frac{r_i}{r_j}} x_i - \sqrt{\frac{r_j}{r_i}} y_j \right\|^2 - 2 \left\| \sum_{i=1}^{n} (x_i - y_i) \right\|^2 \right) e \otimes e,$$

from which we conclude the result. □

3. A general parallelogram law

An extension of a result of [2] for n-tuples may be stated as follows.

• If $f : [0, \infty) \to [0, \infty)$ is a convex function, then for any positive operators $A_1, \cdots, A_n$, any nonnegative numbers $\alpha_1, \cdots, \alpha_n$ with $\sum_{j=1}^{n} \alpha_j = 1$ and any unitarily invariant norm $||| \cdot |||$ on $\mathbb{B}(\mathscr{H})$

$$\left\| \left\| \sum_{j=1}^{n} \alpha_j f(A_j) \right\| \right\| \geqslant \left\| \left\| f \left(\sum_{j=1}^{n} \alpha_j A_j \right) \right\| \right\|. \tag{3.1}$$

The following result is also known [13]:

• If $f : [0, \infty) \to [0, \infty)$ is a convex function with $f(0) = 0$, then for any positive operators $A_1, \cdots, A_n$ and any unitarily invariant norm $||| \cdot |||$ on $\mathbb{B}(\mathscr{H})$

$$\left\| \left\| f \left(\sum_{j=1}^{n} A_j \right) \right\| \right\| \geqslant \left\| \left\| \sum_{j=1}^{n} f(A_j) \right\| \right\|. \tag{3.2}$$

The reverse inequalities hold for concave functions; see [10] for more details. We now prove another significant theorem.

THEOREM 3.1. *Let* $A_1, \cdots, A_n \in \mathscr{C}_{||| \cdot |||}$, $r_1, \cdots, r_n$ *be positive real numbers with* $\sum_{i=1}^{n} \frac{1}{r_i} = 1$, *let* g *be a nonnegative convex function on* $[0, \infty)$ *such that* $g(0) = 0$ *and let* $f(t) = g(t^2)$. *Then*

$$\left\| \left\| \sum_{i=1}^{n} \frac{1}{r_i} f(|r_i A_i|) \right\| \right\| \geqslant \left\| \left\| \sum_{1 \leqslant i < j \leqslant n} f \left(\left| \sqrt{\frac{r_i}{r_j}} A_i - \sqrt{\frac{r_j}{r_i}} A_j \right| \right) + f \left(\left| \sum_{i=1}^{n} A_i \right| \right) \right\| \right\| \tag{3.3}$$

for all unitarily invariant norm. If g *is concave, then the reverse of* (3.3) *holds.*

Proof.

$$\left\|\left|\sum_{i=1}^{n}\frac{1}{r_i}f\left(|r_iA_i|\right)\right|\right\| = \left\|\left|\sum_{i=1}^{n}\frac{1}{r_i}g\left(|r_iA_i|^2\right)\right|\right\|$$

$$\geqslant \left\|\left|g\left(\sum_{i=1}^{n}\frac{1}{r_i}|r_iA_i|^2\right)\right|\right\| \qquad\text{(by (3.1))}$$

$$= \left\|\left|g\left(\sum_{1\leqslant i<j\leqslant n}\left|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right|^2+\left|\sum_{i=1}^{n}A_i\right|^2\right)\right|\right\| \qquad\text{(by (2.2))}$$

$$\geqslant \left\|\left|\sum_{1\leqslant i<j\leqslant n}g\left(\left|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right|^2\right)+g\left(\left|\sum_{i=1}^{n}A_i\right|^2\right)\right|\right\| \text{(by (3.2))}$$

$$= \left\|\left|\sum_{1\leqslant i<j\leqslant n}f\left(\left|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right|\right)+f\left(\left|\sum_{i=1}^{n}A_i\right|\right)\right|\right\|. \qquad\square$$

The function $g(t)=t^p$ for $1\leqslant p<\infty$ ($g(t)=t^p$ for $0<p\leqslant 1$, resp.) is convex (concave, resp.) on $[0,\infty)$. Hence we get the following corollary.

COROLLARY 3.2. *Let* $A_1,\cdots,A_n\in\mathscr{C}_p$ *and* $r_1,\cdots,r_n$ *be positive real numbers with* $\sum_{i=1}^{n}\frac{1}{r_i}=1$. *Then*

$$\sum_{i=1}^{n}r_i^{p-1}\|A_i\|_p^p \geqslant \sum_{1\leqslant i<j\leqslant n}\left\|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right\|_p^p+\left\|\sum_{i=1}^{n}A_i\right\|_p^p$$

for any $2\leqslant p<\infty$. *The reverse inequality holds for any* $0<p\leqslant 2$.

Proof. Let $2\leqslant p<\infty$.

$$\sum_{i=1}^{n}r_i^{p-1}\|A_i\|_p^p = \sum_{i=1}^{n}r_i^{p-1}\|\,|A_i|^p\,\|_1 \qquad\text{(by (1.1))}$$

$$= \mathrm{tr}(\sum_{i=1}^{n}r_i^{p-1}|A_i|^p)$$

$$= \left\|\sum_{i=1}^{n}\frac{1}{r_i}|r_iA_i|^p\right\|_1$$

$$\geqslant \left\|\sum_{1\leqslant i<j\leqslant n}\left|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right|^p+\left|\sum_{i=1}^{n}A_i\right|^p\right\|_1$$

$$\text{(by Theorem 3.1 for } g(t)=t^{\frac{p}{2}}; 2\leqslant p<\infty)$$

$$= \text{tr}\left(\sum_{1\leqslant i<j\leqslant n}\left|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right|^p+\left|\sum_{i=1}^n A_i\right|^p\right)$$

$$= \sum_{1\leqslant i<j\leqslant n}\text{tr}\left(\left|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right|^p\right)+\text{tr}\left(\left|\sum_{i=1}^n A_i\right|^p\right)$$

$$= \sum_{1\leqslant i<j\leqslant n}\left\|\left|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right|^p\right\|_1+\left\|\left|\sum_{i=1}^n A_i\right|^p\right\|_1$$

$$= \sum_{1\leqslant i<j\leqslant n}\left\|\sqrt{\frac{r_i}{r_j}}A_i-\sqrt{\frac{r_j}{r_i}}A_j\right\|_p^p+\left\|\sum_{i=1}^n A_i\right\|_p^p. \qquad \text{(by (1.1))}$$

The proof for the reverse inequality is similar. □

REFERENCES

[1] S. ABRAMOVICH, J. BARIĆ AND J.E. PEČARIĆ, *A new proof of an inequality of Bohr for Hilbert space operators*, Linear Algebra Appl. **430**, 4 (2009), 1432–1435.

[2] J.S. AUJLA AND F.C. SILVA, *Weak majorization inequalities and convex functions*, Linear Algebra Appl. **369** (2003), 217–233.

[3] W.-S. CHEUNG AND J. PEČARIĆ, *Bohr's inequalities for Hilbert space operators*, J. Math. Anal. Appl. **323**, 1 (2006), 403–412.

[4] Y.J. CHO, V. ČULJAK, M. MATIĆ AND J.E. PEČARIĆ, *On parallelogram law and Bohr's inequality in n-inner product spaces*, Inequality theory and applications, Vol. 2 (Chinju/Masan, 2001), 79–90, Nova Sci. Publ., Hauppauge, NY, 2003.

[5] O. EĞECIOĞLU, *Parallelogram-law-type identities*, Linear Algebra Appl. **225** (1995), 1–12.

[6] M. FUJII AND H. ZHOU, *Matrix order in Bohr inequality for operators*, Banach J. Math. Anal., **4**, 1 (2010), 21–27.

[7] F. HANSEN, J.E. PEČARIĆ AND I. PERIĆ, *Jensen's operator inequality and its converses*, Math. Scand. **100**, 1 (2007), 61–73.

[8] F. HANSEN AND G.K. PEDERSEN, *Jensen's operator inequality*, Bull. London Math. Soc. **35** (2003), 553–564.

[9] O. HIRZALLAH, *Non-commutative operator Bohr inequality*, J. Math. Anal. Appl. **282** (2003), 578–583.

[10] O. HIRZALLAH AND F. KITTANEH, *Non-commutative Clarkson inequalities for n-tuples of operators*, Integral Equations Operator Theory **60**, 3 (2008), 369–379.

[11] O. HIRZALLAH, F. KITTANEH AND M.S. MOSLEHIAN, *Schatten p-norm inequalities related to a characterization of inner product spaces*, Math. Inequal. Appl. **13**, 2 (2010), 235–241.

[12] M. KATO, *A note on a generalized parallelogram law and the Littlewood matrices*, Bull. Kyushu Inst. Tech. Math. Natur. Sci. **33** (1986), 37–39.

[13] T. KOSEM, *Inequalities between* $\|f(A+B)\|$ *and* $\|f(A)+f(B)\|$, Linear Algebra Appl. **418**, 1 (2006), 153–160.

[14] M.S. MOSLEHIAN, J.E. PEČARIĆ AND I. PERIĆ, *An operator extension of Bohr's inequality*, Bull. Iranian Math. Soc. **35**, 2 (2009), 67–74.

[15] B. SIMON, *Trace Ideals and their Applications*, Cambridge University Press, Cambridge, 1979.

[16] F. ZHANG, *On the Bohr inequality of operators*, J. Math. Anal. Appl. **333** (2007), 1264–1271.

[17] W.L. ZENG, *An extension of the parallelogram characterization of inner product spaces*, J. Math. Res. Exposition **2**, 3 (1982), 23–25.

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