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POLARIZATION OF AN INEQUALITY

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Abstract. We generalize a previous inequality related to a sharp version of the Littlewood conjecture on the minimal L_1-norm of N-term exponential sums f on the unit circle. The new result concerns replacing the expression $\log(1+t|f|^2)$ with $\log\left(\sum_{k=1}^K t_k|f_k|^2\right)$. The proof occurs on the level of finite Toeplitz matrices, where it reduces to an inequality between their polarized determinants (or "mixed discriminants").

1. Introduction

We will prove a simple generalization of the main results of [10]. For each integer $N \geqslant 1$, let $\mathscr{L}(N)$ denote the collection of all complex polynomials f of the form $f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_{N-1} z^{N-1}$ where each $c_j \in \{1, -1\}$ ("Littlewood polynomials" ; see [3]). Let $\widetilde{\mathscr{L}}(N)$ denote the collection of all complex polynomials f of the form $f(z) = c_0 + c_1 z^{m_1} + c_2 z^{m_2} + \ldots + c_{N-1} z^{m_{N-1}}$ where each coefficient c_j is a complex number with $|c_j| \geqslant 1$, and $0 < m_1 < m_2 < \ldots < m_{N-1}$ are integers. Clearly $\mathscr{L}(N) \subset \widetilde{\mathscr{L}}(N)$. Define $D_N \in \mathscr{L}(N)$ by $D_N(z) = 1 + z + z^2 + \ldots + z^{N-1}$. Define the 1-norm $||f||_1$ on the unit circle by $||f||_1 := \int_0^{2\pi} |f(e^{i\theta})| d\theta / 2\pi$. The Littlewood conjecture concerning $\widetilde{\mathscr{L}}(N)$ was that there is an absolute constant $C > 0$ such that for all N and all $f \in \widetilde{\mathscr{L}}(N)$, $||f||_1 \geqslant C ||D_N||_1$, and was proved in [14] and [16], independently. The "sharp" Littlewood conjecture is that one can take $C = 1$, and this remains open. The main result of [10, Theorem 1.2] concerns only the smaller family $\mathscr{L}(N)$, and states that for all $N \in \mathbb{N}, f \in \mathscr{L}(N)$ and $t > 0$,

$$\int_0^{2\pi} \log\left(1 + t|D_N(e^{i\theta})|^2\right) d\theta \leqslant \int_0^{2\pi} \log\left(1 + t|f(e^{i\theta})|^2\right) d\theta. \tag{1}$$

As discussed in [10], this implies the sharp Littlewood conjecture for $f \in \mathscr{L}(N)$, as well as similar sharp p-norm inequalities (see below) for the range $0 < p \leqslant 4$, by means of some simple integrations over the $t > 0$. The result in the present paper still concerns only the smaller family $\mathscr{L}(N)$, but we generalize (1) in the sense of giving a "vectorized" or "polarized" version of it, as follows:

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THEOREM 1.1. *Let $K \geqslant 1$ and let $N_1, N_2, \ldots, N_K \geqslant 1$ be given integers. Then for any $f_k \in \mathscr{L}(N_k)$ and any real $t_k > 0$, $1 \leqslant k \leqslant K$, we have the inequality*

$$\int_0^{2\pi} \log\left(\sum_{k=1}^K t_k |D_{N_k}(e^{i\theta})|^2\right) d\theta \quad \leqslant \quad \int_0^{2\pi} \log\left(\sum_{k=1}^K t_k |f_k(e^{i\theta})|^2\right) d\theta\,. \tag{2}$$

The special case $K = 2$, $N_1 = 1$ is the old result (1). As in [10], the above theorem immediately implies some p-norm inequalities in $L_p(d\theta)$ in the range $0 < p \leqslant 4$:

$$\left\|\left(\sum_{k=1}^K t_k |D_{N_k}(e^{i\theta})|^2\right)^{\frac{1}{2}}\right\|_p \leqslant \left\|\left(\sum_{k=1}^K t_k |f_k(e^{i\theta})|^2\right)^{\frac{1}{2}}\right\|_p, \quad 0 < p \leqslant 2, \tag{3}$$

$$\left\|\left(\sum_{k=1}^K t_k |D_{N_k}(e^{i\theta})|^2\right)^{\frac{1}{2}}\right\|_p \geqslant \left\|\left(\sum_{k=1}^K t_k |f_k(e^{i\theta})|^2\right)^{\frac{1}{2}}\right\|_p, \quad 2 \leqslant p \leqslant 4. \tag{4}$$

(To deduce this, we replace K by $K+1$ in the theorem, put $N_{K+1} = 1$, and then use certain integral identities for the power functions x^p in terms of $\log(1+tx), 0 < t < \infty$, as in [6, p. 211–212, Lemma 11.1, Ch. 4], [15, Theorem 4], or [10, page 9].)

The proof of Theorem 1.1 is essentially the same as the proof of the special case (1) in [10], as will be seen in §2. The basic lemma is again the total unimodularity of $(0,1)$ "interval matrices" M (whose intervals of 1's occur in their columns, for instance). The only new step is to invoke this fact for the general case of a "polarized determinant" $D_n(A_{k_1}, A_{k_2}, \ldots, A_{k_n})$ of several $n \times n$ Gram matrices $A_k = M_k M_k{}^*$, instead of only for cases of the type $D_n(I, I, \ldots, I, A, A, \ldots, A)$ with only the two matrices I and $A = MM^*$, as was implicitly done in [10]. The "polarized determinant" $D_n(A_1, A_2, \ldots, A_n)$ of the $n \times n$ matrices A_i can be defined as $\frac{1}{n!}$ times the coefficient of $x_1 \ldots x_n$ in $\det(x_1 A_1 + \ldots + x_n A_n)$, where the x_i are scalars. It has traditionally been called the "mixed discriminant" and has been useful in work on the van der Waerden conjecture [1], [8]. It was also used in [12] in connection with certain matrices M having complex entries of modulus $\geqslant 1$, or 0. In this paper we implicitly use the idea of polarized determinant, but we omit explicit use of the notation $D_n(A_1, A_2, \ldots, A_n)$ in the formal lemmas and proofs.

2. Proof of Theorem 1.1.

LEMMA 2.1. *Let $K, n \in \mathbb{N}$, and for $k = 1, \ldots, K$ let M_k be any (rectangular) $n \times m_k$ matrices over $\mathbb{C}$. If x_k are scalars and $A_k := M_k(M_k{}^*)$, then*

$$\det(x_1 A_1 + \ldots + x_K A_K) = \sum_{n_1 + \ldots + n_K = n} \gamma(n_1, \ldots, n_K)\, x_1^{n_1} \ldots x_K^{n_K}, \tag{5}$$

where the coefficients $\gamma(n_1, \ldots, n_K)$ are given by

$$\gamma(n_1, \ldots, n_K) \;=\; \sum_{(S_1, \ldots, S_K)} \left|\det\left(\, S_1 \;\middle|\; \ldots \;\middle|\; S_K \,\right)\right|^2 \tag{6}$$

where each S_k denotes an $n \times n_k$ matrix obtained by choosing some n_k columns of M_k (that is, from n_k distinct column indices), $\left(S_1 \mid \ldots \mid S_K \right)$ denotes the $n \times n$ matrix consisting of the K blocks $S_1, \ldots, S_K$, and the sum is over all ordered K-tuples $(S_1, \ldots, S_K)$ of such choices (an empty sum being zero by convention).

Proof. This is a known result [1]. To prove it, consider the two "block" matrices B and C defined by $B := \left(x_1 M_1 \mid \ldots \mid x_K M_K \right)$ and $C := \left(M_1 \mid \ldots \mid M_K \right)$ Clearly, $BC^* = x_1 A_1 + \ldots + x_K A_K$. Now apply the Binet-Cauchy theorem to expand $\det(BC^*)$. □

REMARK 1. In the notation of polarized determinants D_n, the above coefficients are given by

$$\gamma(n_1, \ldots, n_K) = \frac{n!}{n_1! \ldots n_K!} D_n(A_1^{\langle n_1 \rangle}, \ldots, A_K^{\langle n_K \rangle}), \tag{7}$$

where $A_1^{\langle n_1 \rangle}$ means $A_1, \ldots, A_1$ repeated n_1 times, etc. [1].

We review the following facts already used in [10]:

LEMMA 2.2. [4] *Let S be a square matrix with entries in $\{0,1\}$ such that in each column the 1's occur in consecutive row positions (i.e. in an "interval"). Then:* **(i)** $\det S \in \{-1, 0, 1\}$. **(ii)** *If S' is a matrix with integer entries satisfying $S' = S \bmod 2$, then $|\det S'| \geqslant |\det S|$.*

Proof. For (i) see [4, p. 853] or [10, Lemma 2.3]. (ii) follows from (i) by noting that $\det S' = \det S \bmod 2$. □

COROLLARY 2.3. *Let $K, n \in \mathbb{N}$, and for $k = 1, \ldots, K$ let M_k be $n \times m_k$ matrices with entries in $\{0,1\}$ such that in each column the 1's occur in consecutive row positions. Let M_k' be $n \times m_k$ matrices with integer entries such that $M_k' = M_k \bmod 2$ for each k. If $t_k \geqslant 0$ are scalars and $A_k := M_k(M_k^*)$, $A_k' := M_k'(M_k'^*)$, then*

$$\det(t_1 A_1 + \ldots + t_K A_K) \leqslant \det(t_1 A_1' + \ldots + t_K A_K') . \tag{8}$$

Proof. Expand both sides of (8) using Lemma 2.1. Then we have

$$\left| \det \left(S_1 \mid \ldots \mid S_K \right) \right|^2 \leqslant \left| \det \left(S_1' \mid \ldots \mid S_K' \right) \right|^2 \tag{9}$$

for each of the corresponding terms in (6), by Lemma 2.2 applied to the matrices $S := \left(S_1 \mid \ldots \mid S_K \right)$ and $S' := \left(S_1' \mid \ldots \mid S_K' \right)$ (which correspond to the same K-tuple of choices of column indices). □

REMARK 2. For an $n \times n$ Hermitian matrix $A \geqslant 0$ with eigenvalues λ_i, let $||A||_p$ denote the l_p-norm $(\lambda_1^p + \ldots + \lambda_n^p)^{1/p}$. Corollary 2.3 implies l_p-norm inequalities for the two matrices on either side of (8), in the same manner as discussed after Theorem 1.1 (by replacing K with $K+1$ and taking $M_{K+1} = M_{K+1}' = I$, the $n \times n$ identity matrix):

$$\left|\left| t_1 A_1 + \ldots + t_K A_K \right|\right|_p \leqslant \left|\left| t_1 A_1' + \ldots + t_K A_K' \right|\right|_p , \quad 0 < p \leqslant 1, \tag{10}$$

and, if each M'_k has its entries specifically in $\{-1,0,1\}$, then also

$$\left|\left|t_1A_1+\ldots+t_KA_K\right|\right|_p \geqslant \left|\left|t_1A'_1+\ldots+t_KA'_K\right|\right|_p, \quad 1\leqslant p\leqslant 2. \tag{11}$$

The extra assumption is needed for (11) since that part of the implication relies upon the l_1-norms of both sides being the same.

Proof of Theorem 1.1. The proof is the same as the one in [10, Theorem 1.2], except that here we use Corollary 2.3 in one of the steps. We will repeat the details for completeness: Let $\psi(\theta)=\sum_{k=1}^{K} t_k|f_k(e^{i\theta})|^2$, and for each $n\in\mathbb{N}$ let $T(n,\psi)$ be the $n\times n$ Toeplitz matrix $T(n,\psi)_{ij}=\widehat{\psi}(j-i)$, $1\leqslant i,j\leqslant n$, where $\widehat{\psi}(m)$ is the usual Fourier coefficient, $\widehat{\psi}(m)=\int_0^{2\pi}\psi(\theta)e^{-im\theta}d\theta/2\pi$. By a theorem of Szegö [7, §5.1, pp. 64-65], $(\det T(n,\psi))^{1/n}\to\exp(\int_0^{2\pi}\log\psi(\theta)d\theta/2\pi)$ as $n\to\infty$. Fix n. It is easy to check that $T(n,\psi)=t_1A'_1+\ldots+t_KA'_K$ where each $A'_k=M'_k(M'^*_k)$ with M'_k being the $n\times(n+N_k-1)$ Toeplitz matrix $(M'_k)_{ij}=\widehat{f_k}(j-i)$, where $\widehat{f_k}(m)=\int_0^{2\pi}f_k(e^{i\theta})e^{-im\theta}d\theta/2\pi$, $m\in\mathbb{Z}$; the coefficient of z^m in the polynomial $f_k(z)$. For the special case when all $f_k=D_{N_k}$, denote the matrices M'_k by M_k, and denote ψ by ψ_0. It is clear that M_k has entries in $\{0,1\}$, M'_k has entries in $\{-1,0,1\}$, $M'_k=M_k$ mod 2, and that in each column of M_k the 1 entries occur in an interval. Hence $\det T(n,\psi_0)\leqslant\det T(n,\psi)$, by Corollary 2.3. Taking nth roots and letting $n\to\infty$ on both sides of the latter inequality completes the proof, by the above theorem of Szegö. □

3. Remarks and Questions

3.1. The proof of (8) in Corollary 2.3 actually proceeded by establishing the formally stronger coefficient-wise inequality

$$D_n(A_1,A_2,\ldots,A_n)\leqslant D_n(A'_1,A'_2,\ldots,A'_n). \tag{12}$$

(This may be called a "polarization" of its special case $\det A_1\leqslant\det A'_1$, whence the title of this paper.) We want to note now that, as in (2) of Theorem 1.1, the stronger inequality (12) may also be written in terms of integrals over circles, that is, in the case of Toeplitz matrices A_k and A'_k generated as before by the M_k of some $D_{N_k}(e^{i\theta})$ and the M'_k of the $f_k(e^{i\theta})$ respectively, for $k=1,\ldots n$. In such a case one can invoke the polarized version of the identity of Heine and Szegö [2, Theorem 1], by which the right hand side of (12) becomes the multiple integral

$$\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left(\prod_{k=1}^{n}|f_k(e^{i\theta_k})|^2\right)\Delta(\theta_1,\ldots,\theta_n)d\theta_1\ldots d\theta_n \tag{13}$$

where $\Delta(\theta_1,\ldots,\theta_n)=\prod_{1\leqslant p<q\leqslant n}|e^{i\theta_p}-e^{i\theta_q}|^2$. (The left hand side of (12) is of course the same integral with the f_k replaced by the D_{N_k}.)

We note that a similar kind of polarization occurred in the original motivating work of Hardy, Littlewood, and Gabriel ([9], [5]) on L_p-norm results concerning $\widetilde{\mathscr{L}}$.

There the authors proved a rearrangement theorem for all polynomials, with arbitrary coefficients, which can be specialized to our context as the following result: For even integers $p = 2s \geqslant 2$ one has $||f(e^{i\theta})||_p \leqslant ||D_N(e^{i\theta})||_p$, for all $f \in \widetilde{\mathscr{L}}(N)$ having complex coefficients of modulus 1 or 0. Their proof proceeded via a stronger, polarized version, essentially that

$$\int_0^{2\pi} |f_1(e^{i\theta})|^2 |f_2(e^{i\theta})|^2 \dots |f_s(e^{i\theta})|^2 d\theta \leqslant \int_0^{2\pi} |D_{N_1}(e^{i\theta})|^2 |D_{N_2}(e^{i\theta})|^2 \dots |D_{N_s}(e^{i\theta})|^2 d\theta \tag{14}$$

for any N_k and $f_k \in \widetilde{\mathscr{L}}(N_k)$ with coefficients of modulus 1 or 0. By a stretch of the imagination, one could view these integrals as being similar to those in (13) above. In fact, one simply replaces Δ by a singular measure concentrated on the "diagonal" $\theta_1 = \dots = \theta_n$ of the n-torus. It would thus be interesting to investigate what other "weights" in place of Δ would yield true inequalities (in one direction or the other).

3.2. To what extent can the hypotheses that $f_k \in \mathscr{L}(N_k)$ be relaxed, in the inequalities proved in this paper? For example, can we allow $f_k \in \widetilde{\mathscr{L}}(N_k)$? The strong termwise inequality (9) is trivially false in general for $f_k \in \widetilde{\mathscr{L}}(N_k)$, even when the polynomials f_k have coefficients in $\{0,1\}$ (take $K = n = 1, N_1 = 2$ and $f_1(z) = 1 + z^2$). Going up one level of summation, does the inequality (12) hold whenever $f_k \in \widetilde{\mathscr{L}}(N_k)$? The answer is again no; numerical work by the author has uncovered counterexamples to (12) with $n = 7$, $N_1 = 2$, $f_1(z) = 1 + cz$, $N_2 = 6 = N_3 = \dots = N_7$, $f_2(z) = 1 + \sum_{j=1}^{5} c_j z^j = f_3(z) = \dots = f_7(z)$, with some complex coefficients satisfying $|c_j| = 1 = |c|$. What if only ± 1 coefficients (and gaps) are allowed in the definition of $\widetilde{\mathscr{L}}(N)$, that is $f_k(z) = \sum_{j=1}^{N_k} \pm z^{n_j}$?

3.3. We finally mention questions on the subject of sharp L_p inequalities for real p: It seems natural to ask whether (4) holds for all $p \geqslant 2$, and for all $f_k \in \widetilde{\mathscr{L}}(N_k)$ with coefficients of modulus 1 or 0, or at least for all $f_k \in \mathscr{L}(N_k)$, and whether matrix versions hold, such as (11) for all $p \geqslant 1$. In connection with the integer p case, a natural matrix version of (14) would be to ask whether

$$\operatorname{trace}(A_1' A_2' \dots A_s') \leqslant \operatorname{trace}(A_1 A_2 \dots A_s) \tag{15}$$

for the matrices discussed above in 3.1, when $f_k \in \widetilde{\mathscr{L}}(N_k)$ have coefficients of modulus 1 or 0. By taking absolute values of everything inside the trace, one sees that (15) holds trivially for $f_k \in \mathscr{L}(N_k)$, and one could envision adapting the rearrangement arguments of [5] to prove it for $\widetilde{\mathscr{L}}$. However, this would not imply anything for non-integer p's. Instead, to prove (11) for non-integers $p > 2$, one may need to study some further homogeneous polynomials in the matrix entries, generalizing elementary symmetric polynomials (such as in [11] and [13]).

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