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ON SCHUR–CONVEXITY AND SCHUR–GEOMETRIC CONVEXITY OF FOUR–PARAMETER FAMILY OF MEANS

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Abstract. We prove that the four-parameter family of means

$$R(u,v;r,s;x,y) = \left[\frac{E(r,s;x^u,y^u)}{E(r,s;x^v,y^v)}\right]^{1/(u-v)}$$

is Schur-geometrically convex (concave) in x,y if $(u+v)(r+s) \geqslant (\leqslant) 0$, and Schur-concave (convex) in $u,v \geqslant 0$ if $r+s \geqslant (\leqslant) 0$.

1. Introduction

In [2] the authors investigated Schur-geometric convexity of extended mean values (called also Stolarsky means, as they were introduced by Kenneth B. Stolarsky in [7])

$$E(r,s;x,y) = \begin{cases} \left(\dfrac{r}{s}\dfrac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)} & \text{if} \quad sr(s-r)(x-y) \neq 0, \\ \left(\dfrac{1}{r}\dfrac{y^r-x^r}{\log y - \log x}\right)^{1/r} & \text{if} \quad r(x-y) \neq 0,\ s=0, \\ e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r-x^r)} & \text{if} \quad r=s,\ r(x-y)\neq 0, \\ \sqrt{xy} & \text{if} \quad r=s=0, \\ x & \text{if} \quad x=y \end{cases}.$$

Their main result is that $E(r,s;x,y)$ is Schur-geometrically convex in variables x,y if $r+s \geqslant 0$, and Schur-geometrically concave otherwise. Shi et al. in [6] were working on Schur-geometrical convexity of Gini means defined by

$$G(r,s;x,y) = \begin{cases} \left[\dfrac{x^r+y^r}{x^s+y^s}\right]^{1/(r-s)} & r \neq s \\ \exp\left(\dfrac{x^r\log x + y^r \log y}{x^r+y^r}\right) & r = s \end{cases}.$$

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They proved that $G(r,s;x,y)$ is Schur-geometrically convex in x,y if $r,s \geqslant 0$.

Qi ([4]) proved that for fixed $x \neq y$ Stolarsky means are Schur-geometrically concave in variables r,s if r,s are positive, and Schur-geometrically convex for negative r,s. In [5] Sándor gave another proof of this fact and obtained the same result for Gini means.

It is worth noting that both Gini and Stolarsky means emerge from the logarithmic and arithmetic means in the same way: if $M(x,y)$ stands for either logarithmic mean (L) or the arithmetic mean (A), then the corresponding two-parameter family is defined as

$$M(r,s;x,y) = \begin{cases} \left(\dfrac{M(x^s,y^s)}{M(x^r,y^r)}\right)^{1/(s-r)} & r \neq s \\ \exp\left(\frac{d}{dr}\log M(x^r,y^r)\right) & s = r. \end{cases} \tag{1.1}$$

Obviously we could build two-parameter means taking as a starting point other known means like for example generalized Heronian means

$$H_n(x,y) = \frac{\sum_{i=0}^n x^{(n-i)/n}y^{i/n}}{n+1} \quad \rightarrow \quad H_n(r,s;x,y) = \left(\frac{\sum_{i=0}^n x^{(n-i)r/n}y^{ir/n}}{\sum_{i=0}^n x^{(n-i)s/n}y^{is/n}}\right)^{1/(r-s)} \tag{1.2}$$

As $L(x,y) = E(1,0;x,y)$, $A(x,y) = E(2,1;x,y)$, $H_n(x,y) = E(1/n,1+1/n;x,y)$ are all Stolarsky means, it is quite natural to ask what happens if we consider two-parameter means generated from Stolarsky means $E(u,v;x,y)$. This leads us to the four-parameter family of means defined by

$$R(r,s;u,v;x,y) = \begin{cases} \left[\frac{E(u,v;x^r,y^r)}{E(u,v;x^s,y^s)}\right]^{1/(r-s)} & r \neq s \\ \exp\left(\frac{d}{dr}\log E(u,v;x^r,y^r)\right) & r = s. \end{cases}$$

Given (1.1), we can write R in general case as

$$R(r,s;u,v;x,y) = \left[\frac{L(x^{ru},y^{ru})}{L(x^{su},y^{su})}\frac{L(x^{sv},y^{sv})}{L(x^{rv},y^{rv})}\right]^{\frac{1}{r-s}\frac{1}{u-v}}, \tag{1.3}$$

which shows that R are symmetric in respective pairs of variables, positively homogeneous in x,y and $R(u,v;r,s) = R(r,s;u,v)$.

R-means were introduced in [8], where the comparison theorem was established, and their monotonicity and convexity is discussed in [9].

In this paper we answer the following questions:

- when is R Schur-convex (Schur-concave) in variables r,s?

- when is R Schur-geometrically convex (Schur-geometrically concave) in variables x,y?

2. Definitions and lemmas

Let us recall some definitions: we say that $\mathbf{x} = (x_1, x_2)$ is majorized by $\mathbf{y} = (y_1, y_2)$ (and write $\mathbf{x} \prec \mathbf{y}$) if $\max(x_1, x_2) \leqslant \max(y_1, y_2)$ and $x_1 + x_2 = y_1 + y_2$. For positive x_i we denote $\log \mathbf{x} = (\log x_1, \log x_2)$. A real, symmetric function of two variables f is said to be Schur-convex (concave) if $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \leqslant (\geqslant) f(\mathbf{y})$. A real, symmetric function of two variables f is said to be Schur-geometrically convex (concave) if $\log \mathbf{x} \prec \log \mathbf{y}$ implies $f(\mathbf{x}) \leqslant (\geqslant) f(\mathbf{y})$.

If U is a permutation matrix, then $\mathbf{x} \prec \mathbf{x}U \prec \mathbf{x}$ which implies, that any Schur-convex or Schur-geometrically convex function is symmetric (i.e $f(\mathbf{x}) = f(\mathbf{x}U)$.

The lemma below can be easily deduced from [3, p. 55, Lemma A.2.b] and provides a useful characterization of Schur-convex and Schur-geometrically convex functions:

LEMMA 2.1. *Let $I \subset \mathbf{R}$ be an interval (possibly unbounded).*

A symmetric function $f : I \times I \to \mathbf{R}$ is Schur convex (concave) if and only if for every $a \in I$ the function $f_a^+(x) = f(a+x, a-x)$ is increasing (decreasing) on $(0,\infty) \cap (I-a) \cap (a-I)$.

A symmetric function $f : I \times I \to \mathbf{R}$ is Schur-geometrically convex (concave) if and only if for every positive $a \in I$ the function $f_a^*(x) = f(ax, a/x)$ is increasing (decreasing) on $(1,\infty) \cap (I/a) \cap (a/I)$.

In what follows we shall use frequently the following characterisation of convex functions (see [1, p. 26])

LEMMA 2.2. *Function f is convex (concave) if and only if the function*

$$g(x,y) = \frac{f(x) - f(y)}{x - y}, \quad x \neq y$$

is increasing (decreasing) in both variables.

Two more tools will be useful:

LEMMA 2.3. *For $t, A, B > 0$ let*

$$h(t,A,B) = At \coth At - Bt \coth Bt.$$

If $s \neq t$ and $A \neq B$, then

$$\operatorname{sgn}(h(t,A,B) - h(s,A,B)) = \operatorname{sgn}(t-s)(A-B).$$

Proof. The function $k(x) = x \coth x$ is even, so $k'(0) = 0$ and $k''(x) = \frac{2\cosh x}{\sinh^3 x}(x - \tanh x) \geqslant 0$, hence k is increasing for positive x, which is equivalent to $(A-B)h(t,A,B) > 0$.

Applying Lemma 2.2 to the convex function k, we see that $\frac{h(t,A,B)}{t(A-B)}$ increases in t. The inequality

$$0 \leqslant \left(\frac{h(t,A,B)}{t(A-B)}\right)' = \frac{t(A-B)h'(t,A,B)-(A-B)h(t,A,B)}{t^2(A-B)^2}$$

implies $(A-B)h'(t,A,B) > 0$ for every t, so h' and $A-B$ are of the same sign. The Mean Value Theorem gives now

$$\operatorname{sgn}(h(t,A,B)-h(s,A,B)) = \operatorname{sgn}(t-s)h'(\xi,A,B) = \operatorname{sgn}(t-s)(A-B),$$

which completes the proof. □

LEMMA 2.4. *Let $f:\mathbf{R}\to\mathbf{R}$ be even and strictly increasing in $\mathbf{R}_+$. Then*

$$\operatorname{sgn}\frac{f(a)-f(b)}{a-b} = \operatorname{sgn}(a+b).$$

If f strictly decreases in $\mathbf{R}_+$, then

$$\operatorname{sgn}\frac{f(a)-f(b)}{a-b} = -\operatorname{sgn}(a+b).$$

Proof. If $a=-b$, then the result is obvious. Otherwise in case of increasing f we have

$$\begin{aligned}\operatorname{sgn}\frac{f(a)-f(b)}{a-b} &= \operatorname{sgn}\frac{f(|a|)-f(|b|)}{a-b} = \operatorname{sgn}\frac{f(|a|)-f(|b|)}{|a|-|b|}\operatorname{sgn}\frac{|a|-|b|}{a-b}\\ &= \operatorname{sgn}\frac{|a|^2-|b|^2}{(|a|+|b|)(a-b)} = \operatorname{sgn}(a+b). \qquad \square\end{aligned}$$

LEMMA 2.5. *The function*

$$h(t) = \frac{t^3\cosh t}{\sinh^3 t}$$

is even, increases from 0 to 1 on $(-\infty,0)$ and decreases on $(0,\infty)$.

Proof. It is clear that $h(0)=1$ and $h(\pm\infty)=0$, and since it is even it is sufficient to show that it decreases for positive t. Direct diffrentiation leads to quite complicated inequality, so let us make a little trick here: let

$$g(t) = \frac{\sinh t}{\cosh^{1/3} t}.$$

Then

$$\begin{aligned} g'(t) &= \frac{2}{3}\cosh^{2/3} t + \frac{1}{3}\cosh^{-4/3} t,\\ g''(t) &= \frac{4}{9}\sinh t\cosh^{-7/3} t\left(\cosh^2 t - 1\right), \end{aligned}$$

so g is convex for $t \geqslant 0$, therefore its divided difference $g(t)/t$ increases and h, its cubed reciprocal, decreases. □

LEMMA 2.6. *If f''' is positive (negative), then the divided difference*

$$g(x,y)=\begin{cases}\dfrac{f(x)-f(y)}{x-y} & x\neq y\\ f'(x) & x=y\end{cases}$$

is Schur-convex (concave).

Proof. By Taylor's theorem $f(a)=f(a-t)+f'(a-t)t+f''(\xi_1)t^2/2$ and $f(a)=f(a+t)-f'(a+t)t+f''(\xi_2)t^2/2$, where $a-t<\xi_1<a<\xi_2<a+t$. Therefore, if $f'''>0$, then f'' increases and

$$\begin{aligned}\frac{d}{dt}g(a+t,a-t)&=\frac{[f'(a+t)+f'(a-t)]t-[f(a+t)-f(a-t)]}{2t^2}\\&=\frac{f''(\xi_2)-f''(\xi_1)}{4}>0,\end{aligned}$$

and we can apply Lemma 2.1 to complete the proof. □

3. Schur convexity of four-parameter means

THEOREM 3.1. *R-means are Schur-geometrically convex in variables x,y if $(u+v)(r+s)\geqslant 0$ and Schur-geometrically concave otherwise.*

Proof. The R-means inherit their homogeneity in x,y the same way Stolarsky means inherit it from the logarithmic mean, therefore, in spite of Lemma 2.1, one can easily see that $R(u,v;r,s;x,y)$ is Schur-geometrically convex (Schur-geometrically concave) in x,y if and only if $R(u,x;r,s;x,1/x)$ increases (decreases) for $x>1$, or equivalently, that $T(t)=\log R(u,v;r,s;e^t,e^{-t})$ increases (decreases) for $t>0$. We have

$$\begin{aligned}T(t)&=\frac{\log|\sinh urt|-\log|\sinh ust|-\log|\sinh vrt|+\log|\sinh vst|}{(u-v)(r-s)}\\&=\frac{\log\sinh|ur|t-\log\sinh|us|t-\log\sinh|vr|t+\log\sinh|vs|t}{(u-v)(r-s)}\end{aligned}$$

and

$$\begin{aligned}T'(t)&=\frac{|ur|t\coth|ur|t-|us|t\coth|us|t-|vr|t\coth|vr|t+|vs|t\coth|vs|t}{t(u-v)(r-s)}\\&=\frac{h(|u|,|r|t,|s|t)-h(|v|,|r|t,|s|t)}{t(u-v)(r-s)}.\end{aligned}$$

Applying Lemma 2.3, we obtain

$$\operatorname{sgn}T'(t)=\operatorname{sgn}\frac{|u|-|v|}{u-v}\operatorname{sgn}\frac{|r|-|s|}{r-s}=\operatorname{sgn}(u+v)(r+s),$$

since $\frac{|u|-|v|}{u-v} = \frac{u^2-v^2}{(u-v)(|u|+|v|)} = \frac{u+v}{|u|+|v|}$.

The argument presented above is valid on a dense set of parameters satisfying $uv(|u|-|v|)rs(|r|-|s|) \neq 0$, therefore, by continuity, the theorem holds for other values of parameters as well. □

As an application of the above theorem, observe that for $0 \leqslant \mu < \nu \leqslant \frac{1}{2}$ we have $(\log x^\nu y^{1-\nu}, \log x^{1-\nu} y^\nu) \prec (\log x^\mu y^{1-\mu}, \log x^{1-\mu} y^\mu)$, therefore when $(u+v)(r+s) \geqslant 0$, we have

$$R(u,v;r,s;x^\nu y^{1-\nu}, x^{1-\nu} y^\nu) \leqslant R(u,v;r,s;x^\mu y^{1-\mu}, x^{1-\mu} y^\mu).$$

Observe that $R(u,v;r,s;x^\mu y^{1-\mu}, x^{1-\mu} y^\mu)$ is the four-parameter mean generated by

$$L(x^\mu y^{1-\mu}, x^{1-\mu} y^\mu)) = L_\mu(x,y) = \frac{1}{1-2\mu} \int_\mu^{1-\mu} x^t y^{1-t}\, dt.$$

Let us consider now Schur-convexity of R in variables u,v.

THEOREM 3.2. *For arbitrary positive x,y, the function $R(u,v;r,s;x,y)$ is Schur-convave in $\{(u,v) : u,v \geqslant 0\}$ if $r+s \geqslant 0$ and Schur-convex if $r+s \leqslant 0$. In $\{(u,v) : u,v \leqslant 0\}$ the Schur-convexity reverses.*

Proof. Setting $\omega = \log \sqrt{x/y}$ we can write

$$\log R(u,v;r,s;x,y) = \log \sqrt{xy} + \frac{\log E(r,s;e^{u\omega}, e^{-u\omega}) - \log E(r,s;e^{v\omega}, e^{-v\omega})}{u-v},$$

and by Lemma 2.6 Schur-convexity of $R(u,v)$ depends on third derivative of

$$K(t) = \log E(r,s;e^{t\omega}, e^{-t\omega}) = \frac{\log|\sinh r\omega t| - \log|\sinh s\omega t|}{r-s} - \frac{\log r - \log s}{r-s}.$$

However

$$K'''(t) = 2\frac{(r\omega)^3 \dfrac{\cosh r\omega t}{\sinh^3 r\omega t} - (s\omega)^3 \dfrac{\cosh s\omega t}{\sinh^3 s\omega t}}{r-s}$$
$$= \frac{2\omega}{t^2} \frac{(r\omega t)^3 \dfrac{\cosh r\omega t}{\sinh^3 r\omega t} - (s\omega t)^3 \dfrac{\cosh s\omega t}{\sinh^3 s\omega t}}{r\omega t - s\omega t}$$

and applying Lemmas 2.4 and 2.5, we obtain

$$\operatorname{sgn} K'''(t) = -\operatorname{sgn}(\omega)\operatorname{sgn}(r\omega t + s\omega t) = -\operatorname{sgn}(t)\operatorname{sgn}(r+s),$$

which completes the proof. □

REFERENCES

[1] P.S. BULLEN, *Handbook of Means and their Inequalities*, Kluwer Academic Press, Dordrecht, 2003.

[2] Y. CHU, X. ZHANG AND G. WANG, *The Schur Geometrical Convexity of the Extended Mean Values*, Journal of Convex Analysis **15** (2008), 707–718.

[3] A.W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

[4] F. QI, *A note on Schur-convexity of extended mean values*, Rocky Mountain J.Math. **35**, 5 (2005), 1787–1793.

[5] J. SÁNDOR, *The Schur-convexity of Stolarky and Gini means*, Banach J. Math. Anal. **1**, 2 (2007), 212–215.

[6] H.-N. SHI, Y.-M. JIANG AND W.-D. JIANG, *Schur-convexity and Schur-geometrically concavity of Gini means*, Comput. Math. Appl. **57**, 2 (2009), 266–274.

[7] K.B. STOLARSKY, *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92.

[8] A. WITKOWSKI, *Comparison theorem for two-parameter means*, Math. Inequal. Appl. **12**, 1 (2009), 11–20.

[9] A. WITKOWSKI, *Monotonicity and convexity of S-means*, Math. Inequal. Appl, **13**, 1 (2010), 33–42.

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