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REVERSES OF ANDO'S INEQUALITY FOR POSITIVE LINEAR MAPS

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Abstract. Ando's inequality says that if A and B are positive operators on a Hilbert space H and Φ is a positive linear map, then for each $\alpha \in [0,1]$

$$\Phi(A \sharp_\alpha B) \leqslant \Phi(A) \sharp_\alpha \Phi(B)$$

where the α-geometric mean is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}.$$

In this paper, we give simple proofs of reverse Ando's inequalities: If A and B are positive operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$, then for each $\alpha \in [0,1]$

$$\Phi(A) \sharp_\alpha \Phi(B) \leqslant \Phi(A \sharp_\alpha B) - C(m,M,\alpha)\Phi(A)$$

where the Kantorovich constant for the difference $C(m,M,\alpha)$ is defined by

$$C(m,M,\alpha) = (\alpha - 1)\left(\frac{M^\alpha - m^\alpha}{\alpha(M-m)}\right)^{\frac{\alpha}{\alpha-1}} + \frac{Mm^\alpha - mM^\alpha}{M-m}$$

for any real number $\alpha \in \mathbb{R}$.

1. Introduction

Let Φ be a positive linear map from the space of $B(H)$ to $B(K)$, where $B(H)$ is the C^*-algebra of all bounded linear operators on a Hilbert space H. Ando [1] showed the following property of a positive linear map in connection with the α-geometric mean in the sense of Kubo-Ando theory [6]: Let A and B be positive operators on a Hilbert space H. Then for each $\alpha \in [0,1]$

$$\Phi(A \sharp_\alpha B) \leqslant \Phi(A) \sharp_\alpha \Phi(B) \tag{1}$$

where the α-geometric mean is defined by

$$A \sharp_\alpha B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\alpha A^{\frac{1}{2}}$$

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and the geometric mean $\sharp$ is defined by $\sharp = \sharp_{1/2}$, namely

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

In [8], by using the Mond-Pečarić method for convex functions, we obtained the comprementary inequality to Ando's one under a more general setting.

Recently, Fujii, Lee and Seo [4] showed a difference type reverse of the matrix Cauchy-Schwarz inequality: Let A_i and B_i be positive definite matrices satisfying the sandwich condition

$$mA_i \leqslant B_i \leqslant MA_i$$

for some scalars $0 < m \leqslant M$ and $i = 1, 2, \cdots, n$. Then

$$\left(\sum_{i=1}^{n} A_i\right) \sharp \left(\sum_{i=1}^{n} B_i\right) - \sum_{i=1}^{n} A_i \sharp B_i \leqslant \frac{(\sqrt{M}-\sqrt{m})^2}{4(\sqrt{M}+\sqrt{m})} \sum_{i=1}^{n} A_i.$$

The essential part of the proof is the following difference type reverse Ando's inequality: Let A and B be positive operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$ and $\Phi : B(H) \mapsto B(K)$ a positive linear map. Then

$$\Phi(A) \sharp \Phi(B) - \Phi(A \sharp B) \leqslant \frac{(\sqrt{M}-\sqrt{m})^2}{4(\sqrt{M}+\sqrt{m})} \Phi(A). \tag{2}$$

In [2], Ando gave an elementary proof of (2). We shall review it: Put $C = A^{-1/2}BA^{-1/2}$ and

$$\gamma = \frac{(\sqrt{M}-\sqrt{m})^2}{4(\sqrt{M}+\sqrt{m})} \qquad \text{and} \qquad \lambda = \frac{\sqrt{M}+\sqrt{m}}{2}.$$

Since $\sqrt{m}I \leqslant \sqrt{C} \leqslant \sqrt{M}I$, it follows that

$$\left(C^{1/2} - \lambda I\right)^2 \leqslant (\sqrt{m}-\lambda)^2 I = (\sqrt{M}-\lambda)^2 I = \left(\frac{\sqrt{M}-\sqrt{m}}{2}\right)^2 I.$$

So, we get

$$\frac{1}{2}\left(\frac{1}{\lambda}C + \lambda I\right) \leqslant C^{1/2} + \gamma I$$

and hence

$$\frac{1}{2}\left(\frac{1}{\lambda}B + \lambda A\right) \leqslant A \sharp B + \gamma A.$$

This implies

$$\frac{1}{2}\left(\frac{1}{\lambda}\Phi(B) + \lambda \Phi(A)\right) \leqslant \Phi(A \sharp B) + \gamma \Phi(A).$$

By the arithmetic-geometric mean inequality,

$$\frac{1}{2}\left(\frac{1}{\lambda}\Phi(B) + \lambda \Phi(A)\right) \geqslant (\lambda \Phi(A)) \sharp (\frac{1}{\lambda}\Phi(B)) = \Phi(A) \sharp \Phi(B)$$

and we have the desired inequality (2).

In this paper, inspired by Ando's proof above [2], we give simple proofs of reverses of Ando's inequality (1).

2. Differece type

First of all, we begin with a simple proof of the α-geometric mean version of (2), also see [4], [5, Corollary 5.33].

THEOREM 1. *Let A and B be positive operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$ and let $\Phi : B(H) \mapsto B(K)$ be a positive linear map. Then for each $\alpha \in [0,1]$*

$$\Phi(A) \sharp_\alpha \Phi(B) - \Phi(A \sharp_\alpha B) \leqslant -C(m,M,\alpha)\Phi(A) \tag{3}$$

where the Kantorovich constnat for the difference $C(m,M,\alpha)$ ([5, Theorem 2.58]) is defined by

$$C(m,M,\alpha) = (\alpha-1)\left(\frac{M^\alpha - m^\alpha}{\alpha(M-m)}\right)^{\frac{\alpha}{\alpha-1}} + \frac{Mm^\alpha - mM^\alpha}{M-m}$$

for any real number $\alpha \in \mathbb{R}$.

Proof. Put $C = A^{-1/2}BA^{-1/2}$ and for each $\alpha \in [0,1]$

$$\lambda = \left(\frac{M^\alpha - m^\alpha}{\alpha(M-m)}\right)^{\frac{1}{\alpha-1}}.$$

Since $mI \leqslant C \leqslant MI$, it follows that

$$(1-\alpha)\lambda^\alpha I + \alpha\lambda^{\alpha-1}C \leqslant C^\alpha - C(m,M,\alpha)I$$

and hence

$$(1-\alpha)\lambda^\alpha A + \alpha\lambda^{\alpha-1}B \leqslant A \sharp_\alpha B - C(m,M,\alpha)A.$$

This implies

$$(1-\alpha)\lambda^\alpha\Phi(A) + \alpha\lambda^{\alpha-1}\Phi(B) \leqslant \Phi(A \sharp_\alpha B) - C(m,M,\alpha)\Phi(A).$$

On the other hand, by the weighted arithmetic-geometric mean inequality

$$(1-\alpha)\lambda^\alpha\Phi(A) + \alpha\lambda^{\alpha-1}\Phi(B) \geqslant \Phi(A) \sharp_\alpha \Phi(B)$$

and hence we have the desired inequlity (3). $\square$

REMARK 2. Theorem 1 is sharp in the sense that for each $\alpha \in [0,1]$ there exist two positive operators A, B such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$ and a positive linear map Φ such that

$$\Phi(A) \sharp_\alpha \Phi(B) - \Phi(A \sharp_\alpha B) = -C(m,M,\alpha)\Phi(A).$$

As a matter of fact, let $\Phi : M_2(\mathbb{C}) \mapsto \mathbb{C}$ be a positive linear map defined by

$$\Phi(X) = rx_{11} + (1-r)x_{22} \quad \text{for} \quad X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{with } 0 < r < 1$$

and A and B positive definite matrices such as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix}.$$

Then it is clear that the sandwich condition $mA \leqslant B \leqslant MA$ holds. Moreover, it follows that

$$\Phi(A) \sharp_\alpha \Phi(B) = (m + r(M-m))^\alpha \qquad \text{and} \qquad \Phi(A \sharp_\alpha B) = m^\alpha + r(M^\alpha - m^\alpha).$$

If we put

$$r = \frac{1}{M-m}\left(\frac{M^\alpha - m^\alpha}{\alpha(M-m)}\right)^{\frac{1}{\alpha-1}} - \frac{m}{M-m},$$

then we have $0 < r < 1$ and

$$\begin{aligned} \Phi(A) \sharp_\alpha \Phi(B) - \Phi(A \sharp_\alpha B) &= (m + r(M-m))^\alpha - m^\alpha - r(M^\alpha - m^\alpha) \\ &= -C(m,M,\alpha) \\ &= -C(m,M,\alpha)\Phi(A) \end{aligned}$$

as desired.

3. Ratio type

In this section, we give a simple proof of a ratio type reverse Ando's inequality by a similar method as in §2, also see [3].

THEOREM 3. *Let A and B be positive operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$ and let $\Phi : B(H) \mapsto B(K)$ be a positive linear map. Then for each $\alpha \in [0,1]$*

$$\Phi(A) \sharp_\alpha \Phi(B) \leqslant K(m,M,\alpha)^{-1}\Phi(A \sharp_\alpha B) \tag{4}$$

where the generalized Kantorovich constant $K(m,M,\alpha)$ ([5, Definition 2.2]) is defined by

$$K(m,M,\alpha) = \frac{mM^\alpha - Mm^\alpha}{(\alpha-1)(M-m)}\left(\frac{\alpha-1}{\alpha}\frac{M^\alpha - m^\alpha}{mM^\alpha - Mm^\alpha}\right)^\alpha$$

for any real number $\alpha \in \mathbb{R}$.

Proof. Put $C = A^{-1/2}BA^{-1/2}$. If we put

$$\lambda_0 = \frac{\alpha}{1-\alpha}\frac{M^{1-\alpha}-m^{1-\alpha}}{m^{-\alpha}-M^{-\alpha}} \qquad \text{and} \qquad \mu_0 = \frac{\alpha(M-m)}{M^\alpha - m^\alpha},$$

then

$$\alpha t^{1-\alpha} + (1-\alpha)\lambda_0 t^{-\alpha} \leqslant \mu_0 \qquad \text{for all } t \in [m,M].$$

Since $mI \leqslant C \leqslant MI$, we get

$$\alpha C + (1-\alpha)\lambda_0 I \leqslant \mu_0 C^\alpha$$

and hence

$$\alpha B + (1-\alpha)\lambda_0 A \leqslant \mu_0 (A \sharp_\alpha B).$$

This implies

$$(1-\alpha)\lambda_0 \Phi(A) + \alpha\Phi(B) \leqslant \mu_0 \Phi(A \sharp_\alpha B).$$

By the weighted arithmetic-geometric mean inequality, it follows that

$$(1-\alpha)\lambda_0\Phi(A) + \alpha\Phi(B) \geqslant \lambda_0^{1-\alpha}\, \Phi(A) \sharp_\alpha \Phi(B).$$

On the other hand, since $\frac{\mu_0}{\lambda_0^{1-\alpha}} = K(m,M,\alpha)^{-1}$ by an easy calculation, we have the desired inequality (4). $\square$

REMARK 4. Theorem 3 is sharp in the sense that for each $\alpha \in [0,1]$ there exist two positive operators A, B such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$ and a positive linear map Φ such that

$$\Phi(A) \sharp_\alpha \Phi(B) = K(m,M,\alpha)^{-1}\Phi(A \sharp_\alpha B).$$

As a matter of fact, put A, B and Φ be as in Remark 2. If we put

$$r = \frac{\alpha m^\alpha (M-m) - m(M^\alpha - m^\alpha)}{(1-\alpha)(M-m)(M^\alpha - m^\alpha)},$$

then we have $0 < r < 1$. Therefore it follows that

$$\begin{aligned}
\frac{\Phi(A) \sharp_\alpha \Phi(B)}{\Phi(A \sharp_\alpha B)} &= \frac{(m + r(M-m))^\alpha}{m^\alpha + r(M^\alpha - m^\alpha)} \\
&= \left(\frac{\alpha(Mm^\alpha - mM^\alpha)}{(1-\alpha)(M^\alpha - m^\alpha)}\right)^\alpha \frac{(1-\alpha)(M-m)}{m^\alpha M - mM^\alpha} \\
&= K(m,M,\alpha)^{-1}.
\end{aligned}$$

If we put $\alpha = \frac{1}{2}$ in Theorem 3, then we have the following corollary ([7]) since $K(m,M,\frac{1}{2})^{-1} = \frac{\sqrt{M}+\sqrt{m}}{2\sqrt[4]{Mm}}$:

COROLLARY 5. *Let A and B be positive operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m \leqslant M$ and let $\Phi : B(H) \mapsto B(K)$ be a positive linear map. Then*

$$\Phi(A) \sharp \Phi(B) \leqslant \frac{\sqrt{M}+\sqrt{m}}{2\sqrt[4]{Mm}} \Phi(A \sharp B).$$

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