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SCHUR–CONVEXITY OF ČEBIŠEV FUNCTIONAL

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Abstract. In this paper the Čebišev functional $T(f,g;a,b)$ is regarded as a function of two variables

$$T(f,g;x,y)=\frac{1}{y-x}\int_x^y f(t)g(t)dt-\Big(\frac{1}{y-x}\int_x^y f(t)dt\Big)\Big(\frac{1}{y-x}\int_x^y g(t)dt\Big),\ (x,y)\in[a,b]\times[a,b]$$

The property of Schur-covexity (Schur-concavity) of this function is considered. Some applications for the means are pointed out.

1. Introduction

Let I be an interval with nonempty interior and $\mathbf{x}=(x_i,x_2,..,x_n)$ and $\mathbf{y}=(y_i,y_2,..,y_n)$ in I^n be two n-tuples such that $\mathbf{x}\prec\mathbf{y}$, i.e.

$$\sum_{i=1}^{k}x_{[i]}\leqslant\sum_{i=1}^{k}y_{[i]},\ k=1,\dots,n-1$$
$$\sum_{i=1}^{n}x_{[i]}=\sum_{i=1}^{n}y_{[i]},$$

where $x_{[i]}$ denotes the i th largest component in $\mathbf{x}$.

DEFINITION 1. Function $F:I^n\to\mathbb{R}$ is Schur-convex on I^n if

$$F(x_i,x_2,..,x_n)\leqslant F(y_i,y_2,..,y_n)$$

for each two n-tuples $\mathbf{x}$ and $\mathbf{y}$ such that it holds $\mathbf{x}\prec\mathbf{y}$ on I^n.
Function F is Schur-concave on I^n if and only if $-F$ is Schur-convex.

The next lemma gives us a necessery and sufficient condition for verifying the Schur-convexity property of F when $n=2$ ([4, p. 333], [3, p. 57]).

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LEMMA A 1. *Let $F : I^2 \to \mathbb{R}$ be a continuous function on I^2 and differentiable in interior of I^2. Then F is Schur-convex if and only if it is symmetric and it holds*

$$\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)(y-x) \geqslant 0 \tag{1}$$

for all $x, y \in I$, $x \neq y$.

The authors in [1] were inspired by some inequalities concerning gamma and digamma function and proved the following result for the integral arithmetic mean:

THEOREM A 1. *Let f be a continuous function on I. Then*

$$F(x,y) = \frac{1}{y-x}\int_x^y f(t)dt \tag{2}$$
$$F(x,x) = f(x)$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Also, in [1], applications to logarithmic mean are given.

COROLLARY A 1. *The generalized logarithmic mean defined as follows*

$$L_r(x,y) = \left(\frac{y^r - x^r}{r(y-x)}\right)^{\frac{1}{r-1}}, \; x,y > 0 \tag{3}$$
$$L_1(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$$
$$L_0(x,y) = \frac{y-x}{\log y - \log x}$$
$$L(x,x) = x \tag{4}$$

is Schur-convex for $r > 2$ and Schur-concave for $r < 2$.

The Čebišev functional $T(f,g;a,b)$ is defined for two Lebesgue integrable f and g on interval $[a,b] \in \mathbb{R}$ as

$$T(f,g;a,b) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right).$$

We will consider the function $T(x,y) := T(f,g;x,y), (x,y) \in [a,b] \times [a,b]$.
We will use the well-known Čebišev inequality:

THEOREM A 2. *Let f and g be Lebesgue integrable on interval $[a,b]$. If f and g are monotonic in the same sense (in the opposite sense) then*

$$T(f,g;a,b) \geqslant 0 \; (\leqslant 0). \tag{5}$$

In this paper we generalize results in Theorem A 1 and Corollary A 1. As a consequence, a result for the extended generalized logarithmic type mean is pointed out.

2. Results

THEOREM 2.1. *Let f and g be Lebesgue integrable functions on $I = [a,b]$. If they are monotone in the same sense (in the opposite sense) then $T(x,y) := T(f,g;x,y)$, $(x,y) \in [a,b] \times [a,b] \in \mathbb{R}^2$ is Schur-convex (Schur-concave) on $[a,b] \times [a,b]$.*

Proof. There are three cases to be considered according monotonicity of functions.

Case 1. Let f and g be two increasing functions on $[a,b]$ and $x < y$. So, we have $f(x) \leqslant f(t) \leqslant f(y)$ and $g(x) \leqslant g(t) \leqslant g(y)$ and it yields

$$(f(y) - f(t))(f(t) - f(x)) \geqslant 0, \tag{6}$$

$$(g(y) - g(t))(g(t) - g(x)) \geqslant 0, \tag{7}$$

Multiplying these inequalities by $\frac{1}{y-x}$ and integrating over $[x,y]$ produces two inequalities

$$\frac{1}{y-x}\int_x^y f^2(t)dt \leqslant ((f(x)+f(y))\frac{1}{y-x}\int_x^y f(t)dt - f(x)f(y),$$

$$\frac{1}{y-x}\int_x^y g^2(t)dt \leqslant ((g(x)+g(y))\frac{1}{y-x}\int_x^y g(t)dt - g(x)g(y).$$

Then, we can estimate $T(f,f;x,y)$

$$\begin{aligned} T(f,f;x,y) &= \frac{1}{y-x}\int_x^y f^2(t)dt - \left(\frac{1}{y-x}\int_x^y f(t)dt\right)^2 \\ &\leqslant ((f(x)+f(y))\frac{1}{y-x}\int_x^y f(t)dt - f(x)f(y) - \left(\frac{1}{y-x}\int_x^y f(t)dt\right)^2 \\ &= \left(f(y) - \frac{1}{y-x}\int_x^y f(t)dt\right)\left(\frac{1}{y-x}\int_x^y f(t)dt - f(x)\right); \end{aligned} \tag{8}$$

and analogues $T(g,g;x,y)$ as follows

$$\begin{aligned} T(g,g;x,y) &= \frac{1}{y-x}\int_x^y g^2(t)dt - \left(\frac{1}{y-x}\int_x^y g(t)dt\right)^2 \\ &\leqslant \left(g(y) - \frac{1}{y-x}\int_x^y g(t)dt\right)\left(\frac{1}{y-x}\int_x^y g(t)dt - g(x)\right). \end{aligned} \tag{9}$$

The functional $T(f,g;x,y)$ can be expressed as

$$T(f,g;x,y) = \frac{1}{2(y-x)^2}\int_x^y\int_x^y (f(t)-f(s))(g(t)-g(s))dtds.$$

and analogues $T(f,f;x,y)$ and $T(g,g;x,y)$

$$T(f,f;x,y) = \frac{1}{2(y-x)^2}\int_x^y\int_x^y (f(t)-f(s))^2 dtds,$$

$$T(g,g;x,y) = \frac{1}{2(y-x)^2}\int_x^y\int_x^y (g(t)-g(s))^2 dtds.$$

Using Cauchy inequality we obtain the inequality

$$
\begin{aligned}
|T(f,g;x,y)| &\leqslant \frac{1}{2(y-x)^2}\left(\int_x^y\int_x^y (f(t)-f(s))^2 dtds\right)^{\frac{1}{2}}\left(\int_x^y\int_x^y (g(t)-g(s))^2 dtds\right)^{\frac{1}{2}}\\
&= \left(\frac{1}{2(y-x)^2}\int_x^y\int_x^y (f(t)-f(s))^2 dtds\right)^{\frac{1}{2}}\\
&\quad \times\left(\frac{1}{2(y-x)^2}\int_x^y\int_x^y (g(t)-g(s))^2 dtds\right)^{\frac{1}{2}}\\
&= T(f,f;x,y)^{\frac{1}{2}}T(g,g;x,y)^{\frac{1}{2}}.
\end{aligned}
$$

In the rest of the proof we will use the short notation for the integral means: $\overline{f}:=\frac{1}{y-x}\int_x^y f(t)dt$ and $\overline{g}:=\frac{1}{y-x}\int_x^y g(t)dt$.

According (8) and (9) we have the following estimation

$$
\begin{aligned}
|T(f,g;x,y)| &\leqslant [(f(y)-\overline{f})(\overline{f}-f(x))]^{\frac{1}{2}}[(g(y)-\overline{g})(\overline{g}-g(x))]^{\frac{1}{2}}\\
&= [(\overline{f}-f(x))(\overline{g}-g(x))\cdot(f(y)-\overline{f})(g(y)-\overline{g})]^{\frac{1}{2}}.
\end{aligned}
$$

The AG inequality implies

$$|T(f,g;x,y)| \leqslant \frac{1}{2}[(\overline{f}-f(x))(\overline{g}-g(x))+(f(y)-\overline{f})(g(y)-\overline{g})].$$

Applying Theorem A 2 the inequality (5) we obtain

$$T(f,g;x,y) \leqslant \frac{1}{2}[(\overline{f}-f(x))(\overline{g}-g(x))+(f(y)-\overline{f})(g(y)-\overline{g})]. \tag{10}$$

To prove the Schur-convexity of $T(f,g;x,y)$ by Lemma A 1 the inequality (1) it is sufficient to prove $(\frac{\partial T(f,g;x,y)}{\partial y}-\frac{\partial T(f,g;x,y)}{\partial x})(y-x)\geqslant 0,$ for all $x,y\in[a,b],$ since the function $T(x,y):=T(f,g;x,y)$ is evidently symmetric.

Direct calculation yields that

$$
\begin{aligned}
&\left(\frac{\partial T(f,g;x,y)}{\partial y}-\frac{\partial T(f,g;x,y)}{\partial x}\right)(y-x)\\
&= \left\{\frac{1}{y-x}[-2T(f,g;x,y)+f(x)g(x)+f(y)g(y)+2\overline{f}\,\overline{g}\right.\\
&\qquad \left. -f(y)\overline{g}-g(y)\overline{f}+f(x)\overline{g}+g(x)\overline{f}]\right\}(y-x) \qquad (11)\\
&= 2\left\{\frac{1}{2}[(\overline{f}-f(x))(\overline{g}-g(x))+(f(y)-\overline{f})(g(y)-\overline{g})]-T(f,g;x,y)\right\}. \qquad (12)
\end{aligned}
$$

Then, the inequalitiey (10) implies

$$\left(\frac{\partial T(f,g;x,y)}{\partial y}-\frac{\partial T(f,g;x,y)}{\partial x}\right)(y-x)\geqslant 0.$$

We have to remark that for $x > y$ the inequalities in (6) and (7) stil are valid. Furthermore, according the equations in (11) and (12) it is obviously $\left(\frac{\partial T(f,g;x,y)}{\partial y} - \frac{\partial T(f,g;x,y)}{\partial x}\right)(y-x) \geqslant 0$.

Case 2. Suppose that f and g are both decreasing functions on $[a,b]$ and $x < y$.

Since $f(x) \geqslant f(t) \geqslant f(y)$ and $g(x) \geqslant g(t) \geqslant g(y)$ the inequalities in (6) and (7) again are valid and the proof is the same as in Case 1.

If $x > y$ then the conclusion is tha same as in remark in Case 1.

Case 3. Let f be an increasing function and g decreasing function. Note that we can considere Case 1 for function f and $-g$.

According inequality in (10) we have

$$T(f,-g;x,y) \leqslant \frac{1}{2}[(\overline{f} - f(x))(-\overline{g} + g(x)) + (f(y) - \overline{f})(-g(y) + \overline{g})].$$

By definition of $T(f,-g;x,y)$ it holds

$$-T(f,g;x,y) \leqslant -\frac{1}{2}[(\overline{f} - f(x))(\overline{g} - g(x)) + (f(y) - \overline{f})(g(y) - \overline{g})]$$

and finally we obtain the opposit inequality in (10) for functions f and g:

$$T(f,g;x,y) \geqslant \frac{1}{2}[(\overline{f} - f(x))(\overline{g} - g(x)) + (f(y) - \overline{f})(g(y) - \overline{g})]. \tag{13}$$

Similarly as in Case 1, according (11) we conclude that

$$\left(\frac{\partial T(f,g;x,y)}{\partial y} - \frac{\partial T(f,g;x,y)}{\partial x}\right)(y-x) \leqslant 0.$$

and according Lemma A 1 we prove Schur-concavity of Čebišev functional $T(f,g;x,y)$ with (x,y) in $[a,b] \times [a,b] \in \mathbb{R}^2$.

COROLLARY 2.1. *For the generalised logaritmic mean defined by (3) it holds*

(i) if $(r,s) \in (1,\infty) \times (1,\infty) \bigcup (-\infty,1) \times (-\infty,1)$, then

$$G_{rs}(x,y) := L_{r+s-1}^{r+s}(x,y) - L_{r+1}^{r}(x,y) \cdot L_{s+1}^{s}(x,y)$$

is Schur-convex with $(x,y) \in (0,\infty) \times (0,\infty)$;

(ii) if $(r,s) \in (1,\infty) \times (-\infty,1) \bigcup (-\infty,1) \times (1,\infty)$, then $G_{rs}(x,y)$ is Schur-concave with $(x,y) \in (0,\infty) \times (0,\infty)$.

Proof. We use Theorem 1 for a function $f(t) = t^{r-1}$ and $g(t) = t^{s-1}$. Function f and g are both increasing for $r-1 > 0$ and $s-1 > 0$ and both decreasing for $r-1 < 0$ and $s-1 < 0$. Function f and g are monotone in the opposite sense for $r-1 > 0$ and $s-1 < 0$ or $r-1 < 0$ and $s-1 > 0$. □

REMARK 2.1. One attempt to obtain Schur convexity of Čebišev functional is done in [2].

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