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SOME MAJORIZATION INEQUALITIES FOR CONVEX FUNCTIONS OF SEVERAL VARIABLES

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Abstract. The purpose of this note is to prove some weak majorization inequalities involving convex functions of several variables. A sub-additive inequality for separately matrix convex functions is also proved.

1. Introduction

For $m \in \mathbf{N}$, let $\mathscr{M}_m$ be the algebra of all $m \times m$ complex matrices, $\Pi_m \subset \mathscr{M}_m$ the set of all (Hermitian) projections in $\mathscr{M}_m$, $\mathscr{S}_m$ the set of positive semi-definite matrices in $\mathscr{M}_m$ and $\mathscr{P}_m$ be the set of positive definite matrices in $\mathscr{M}_m$. Let I be an interval in $\mathbb{R}$. We denote by $\mathscr{M}_m(I)$ the set of all Hermitian members of $\mathscr{M}_m$ whose spectrum is contained in I. By I_m, we denote the identity matrix in $\mathscr{M}_m$. Let $m,n \in \mathbf{N}$, and let I, J be intervals in $\mathbb{R}$. Let f be a real valued function of two real variables x and y, $x \in I$, $y \in J$. Let $A \in \mathscr{M}_m(I)$, $B \in \mathscr{M}_n(J)$ have spectral resolutions $A = \sum_{i=1}^{k} \lambda_i P_i$, $B = \sum_{j=1}^{l} \mu_j Q_j$. Then $f(A,B)$ is the matrix defined as

$$f(A,B) = \sum_{i=1}^{k} \sum_{j=1}^{l} f(\lambda_i, \mu_j) P_i \otimes Q_j$$

(Koranyi [11]). A function $f : I \times J \to \mathbb{R}$ is called matrix convex if

$$f(\alpha(A,C) + (1-\alpha)(B,D)) \leqslant \alpha f(A,C) + (1-\alpha) f(B,D)$$

for all A, $B \in \mathscr{M}_m(I)$, C, $D \in \mathscr{M}_n(J)$, $0 \leqslant \alpha \leqslant 1$ and $m,n \in \mathbf{N}$. A function $f : I \times J \to \mathbb{R}$ is called matrix convex in first variable if

$$f(\alpha(A,C) + (1-\alpha)(B,C)) \leqslant \alpha f(A,C) + (1-\alpha) f(B,C)$$

for all A, $B \in \mathscr{M}_m(I)$, $C \in \mathscr{M}_n(J)$, $0 \leqslant \alpha \leqslant 1$, and $m,n \in \mathbf{N}$. The matrix convexity in the second variable is defined in the same way. A function $f : I \times J \to \mathbb{R}$ is called separately matrix convex if it is matrix convex in each variable separately. The function f is called matrix concave (separately matrix concave) if $-f$ is matrix convex (separately matrix convex). In case $m = n = 1$ we say f is separately convex/separately concave accordingly. If f is positive, then f is called log convex (log concave), if

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$\log f$ is convex (concave). It is immediate that a matrix convex function is separately matrix convex. The function $f(s,t) = \frac{1}{st}$ is matrix convex on $(0,\infty) \times (0,\infty)$ where as the function $g(s,t) = st$ is separately matrix convex on $(-\infty,\infty) \times (-\infty,\infty)$. For more concrete examples of matrix convex functions of several variables the reader is referred to [2, 6, 7, 8, 9]

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be elements in $\mathbb{R}^n$. Let $x^{\downarrow}$ and $x^{\uparrow}$ be the vectors obtained by rearranging the coordinates of x in decreasing and increasing order respectively. The weak submajorization relation $x \prec_w y$ means

$$\sum_{j=1}^{k} x_j^{\downarrow} \leqslant \sum_{j=1}^{k} y_j^{\downarrow}, \ \ 1 \leqslant k \leqslant n.$$

Similarly, the weak supermajorization relation $x \prec^w y$ means

$$\sum_{j=1}^{k} x_j^{\uparrow} \geqslant \sum_{j=1}^{k} y_j^{\uparrow}, \ \ 1 \leqslant k \leqslant n.$$

Let $x, y \in \mathbb{R}_n^+$. Then we define the weak sublog-majorization $x \prec_{wlog} y$ when

$$\prod_{j=1}^{k} x_j^{\downarrow} \leqslant \prod_{j=1}^{k} y_j^{\downarrow}, \ \ 1 \leqslant k \leqslant n.$$

Similarly, the weak superlog-majorization relation $x \prec^{wlog} y$ means

$$\prod_{j=1}^{k} x_j^{\uparrow} \geqslant \prod_{j=1}^{k} y_j^{\uparrow}, \ \ 1 \leqslant k \leqslant n.$$

For a Hermitian $A \in \mathscr{M}_n$, $\lambda_j^{\downarrow}(A)$, $1 \leqslant j \leqslant n$ denotes the eigenvalues of A arranged in the decreasing order. We use the notation $\lambda(A)$ to denote the row vector $(\lambda_1^{\downarrow}(A), \dots, \lambda_n^{\downarrow}(A))$.

Matrix convex and separately matrix convex functions of two and more variables have been studied and characterized in terms of Schwarz inequalities in [1, 6, 7, 8, 9]. In [1], Ando proved that the function $f(s,t) = s^{\alpha} t^{1-\alpha}$, $0 \leqslant \alpha \leqslant 1$ is matrix concave on $(0,\infty) \times (0,\infty)$ and used it to give a simple proof of Lieb's concavity theorem.

In Section 2, we prove some majorization inequalities involving convex functions of two variables. In Section 3, some sub-additive inequalities are proved for separately matrix convex functions. These results can be written for functions of several variables, however we restrict ourselves to the case of two variables for simplicity. The results presented in this paper are motivated by the results in [3, 4, 8, 12].

2. Majorization Inequalities

In this section we shall prove some majorization inequalities. We begin with the following lemma.

LEMMA 2.1. [2] *Let f be a function defined on $I\times J$, $0\in I\cap J$, $A\in \mathscr{M}_m(I)$, $B\in \mathscr{M}_n(J)$. Then*

$$f(U^*AU,V^*BV)=(U\otimes V)^*f(A,B)(U\otimes V)$$

for all unitaries $U\in \mathscr{M}_m$, $V\in \mathscr{M}_n$, $m,n\in N$.

LEMMA 2.2. *Let f be a separately convex function defined on $I\times J$ and $A\in \mathscr{M}_m(I),B\in \mathscr{M}_n(J)$. Then*

$$f(\langle Ax,x\rangle ,\langle By,y\rangle)\ \leqslant\ \langle f(A,B)x\otimes y,x\otimes y\rangle$$

for all unit vectors $x\in C^m,y\in C^n$.

Proof. First assume that A and B are diagonal. Let $A=diag(a_1,a_2,\dots,a_m)$ and $B=diag(b_1,b_2,\dots,b_n)$. Let $x=(x_1,x_2,\dots,x_m)^*$ and $y=(y_1,y_2,\dots,y_n)^*$ be unit vectors. Then

$$\begin{aligned}f(\langle Ax,x\rangle,\langle By,y\rangle)&=f(x^*Ax,y^*By)\\&=f(\sum_{i=1}^{m}|x_i|^2a_i,\sum_{j=1}^{n}|y_j|^2b_j)\\&\leqslant\sum_{i=1}^{m}\sum_{j=1}^{n}|x_i|^2|y_j|^2f(a_i,b_j)\\&=(x\otimes y)^*f(A,B)(x\otimes y)\\&=\langle f(A,B)x\otimes y,x\otimes y\rangle,\end{aligned}$$

using the separate convexity of f and the facts that $\sum_{i=1}^{m}|x_i|^2=1,\ \sum_{j=1}^{n}|y_j|^2=1.$

If A and B are not diagonal then their exist unitary matrices U and V such that $A=U^*D_1U,\ B=V^*D_2V$, where D_1 and D_2 are diagonal matrices. Then

$$\begin{aligned}f(\langle Ax,x\rangle ,\langle By,y\rangle)&=f(\langle U^*D_1Ux,x\rangle ,\langle V^*D_2Vy,y\rangle)\\&=f(\langle D_1Ux,Ux\rangle ,\langle D_2Vy,Vy\rangle))\\&\leqslant\langle f(D_1,D_2)Ux\otimes Vy,Ux\otimes Vy\rangle\\&=\langle f(D_1,D_2)(U\otimes V)(x\otimes y),(U\otimes V)(x\otimes y)\rangle\\&=\langle (U\otimes V)^*f(D_1,D_2)(U\otimes V)(x\otimes y),(x\otimes y)\rangle\\&=\langle f(A,B)x\otimes y,x\otimes y\rangle ,\end{aligned}$$

using first part of the proof and Lemma 2.1 respectively. □

LEMMA 2.3. [5, p.281] *Let $A\in \mathscr{M}_n$ be Hermitian. Then*

$$\sum_{j=1}^{k}\lambda_j^{\downarrow}(A)=\max\sum_{j=1}^{k}\left\langle Au_j,u_j\right\rangle,\ \ k=1,2,\dots,n,$$

where the maximum is taken over all choices of orthonormal vectors $u_1,u_2,\dots,u_k$.

LEMMA 2.4. [4, p.228] *Let $A,\ B\in\mathscr{P}_m$. Then*

$$\lambda(\log A+\log B)\prec_w \lambda(\log(A^{1/2}BA^{1/2})).$$

THEOREM 2.5. *Let f be a convex function defined on $I\times J$ and $A,B\in\mathscr{M}_m(I)$, $C,D\in\mathscr{M}_n(J)$. Then*

$$\lambda\left(f(\alpha A+(1-\alpha)B,\alpha C+(1-\alpha)D)\right)\ \prec_w\ \lambda\left(\alpha f(A,C)+(1-\alpha)f(B,D)\right)$$

for all $0\leqslant\alpha\leqslant 1$.

Proof. Let $\gamma_1,\gamma_2,\ldots,\gamma_m$ and $\mu_1,\mu_2,\ldots,\mu_n$ be the eigenvalues of $\alpha A+(1-\alpha)B$ and $\alpha C+(1-\alpha)D$ respectively with $u_1,u_2,\ldots,u_m$ and $v_1,v_2,\ldots,v_n$ as an orthonormal system of corresponding eigenvectors. Then for each $r=1,\ldots,mn$, there exist pair $(i_r,j_r)\in\{1,\ldots,m\}\times\{1,\ldots,n\}$ such that $\lambda_r^{\downarrow}\big(f(\alpha A+(1-\alpha)B,\alpha C+(1-\alpha)D)\big)=f(\gamma_{i_r},\mu_{j_r})$. As f is convex and therefore separately convex, Lemma 2.2 can be applied to f. Thus using convexity of f and Lemma 2.2 at appropriate places, we have

$$\begin{aligned}
&\sum_{r=1}^{k}\lambda_r^{\downarrow}(f(\alpha A+(1-\alpha)B,\alpha C+(1-\alpha)D))\\
&\qquad=\sum_{r=1}^{k}f(\gamma_{i_r},\mu_{j_r})\\
&\qquad=\sum_{r=1}^{k}f(\langle(\alpha A+(1-\alpha)B)u_{i_r},u_{i_r}\rangle,\langle(\alpha C+(1-\alpha)D)v_{j_r},v_{j_r}\rangle)\\
&\qquad=\sum_{r=1}^{k}f(\alpha(\langle Au_{i_r},u_{i_r}\rangle,\langle Cv_{j_r},v_{j_r}\rangle)+(1-\alpha)(\langle Bu_{i_r},u_{i_r}\rangle,\langle Dv_{j_r},v_{j_r}\rangle))\\
&\qquad\leqslant\sum_{r=1}^{k}\alpha f(\langle Au_{i_r},u_{i_r}\rangle,\langle Cv_{j_r},v_{j_r}\rangle)+(1-\alpha)f(\langle Bu_{i_r},u_{i_r}\rangle,\langle Dv_{j_r},v_{j_r}\rangle)\\
&\qquad\leqslant\sum_{r=1}^{k}\alpha\left\langle f(A,C)u_{i_r}\otimes v_{j_r},u_{i_r}\otimes v_{j_r}\right\rangle+(1-\alpha)\left\langle f(B,D)u_{i_r}\otimes v_{j_r},u_{i_r}\otimes v_{j_r}\right\rangle\\
&\qquad=\sum_{r=1}^{k}\left\langle(\alpha f(A,C)+(1-\alpha)f(B,D))u_{i_r}\otimes v_{j_r},u_{i_r}\otimes v_{j_r}\right\rangle,
\end{aligned}$$

for $1\leqslant k\leqslant mn$. Hence by Lemma 2.3, we have

$$\sum_{r=1}^{k}\lambda\left(f(\alpha A+(1-\alpha)B,\alpha C+(1-\alpha)D)\right)\leqslant\sum_{r=1}^{k}\lambda\left(\alpha f(A,C)+(1-\alpha)f(B,D)\right).$$

This completes a proof. □

COROLLARY 2.6. *Let f be a concave function defined on $I\times J$ and $A,B\in\mathscr{M}_m(I)$, $C,D\in\mathscr{M}_n(J)$. Then*

$$\lambda\left(f\left(\alpha A+(1-\alpha)B,\alpha C+(1-\alpha)D\right)\right)\prec^{w}\lambda\left(\alpha f(A,C)+(1-\alpha)f(B,D)\right)$$

for all $0 \leqslant \alpha \leqslant 1$.

Proof. In Theorem 2.5 changing f to $-f$ yield the desired majorization inequality. □

COROLLARY 2.7. *Let* f *be a log-convex function defined on* $I \times J$ *and* $A, B \in \mathscr{M}_m(I)$, $C, D \in \mathscr{M}_n(J)$. *Then*

$$\lambda\left(f(\alpha A + (1-\alpha)B, \alpha C + (1-\alpha)D)\right) \prec_{wlog} \lambda\left(f(A,C)^{\alpha} f(B,D)^{1-\alpha}\right)$$

for all $0 \leqslant \alpha \leqslant 1$.

Proof. The function $\log f(s,t)$ is a convex function on $I \times J$. Therefore by Theorem 2.5 and Lemma 2.4, we have

$$\begin{aligned}
\lambda(\log f(\alpha A + (1-\alpha)B, \alpha C + (1-\alpha)D)) &\prec_w \lambda(\alpha \log f(A,C) + (1-\alpha)\log f(B,D)) \\
&= \lambda(\log f(A,C)^{\alpha} + \log f(B,D)^{1-\alpha}) \\
&\prec_w \lambda(\log[f(A,C)^{\alpha/2} f(B,D)^{1-\alpha} f(A,C)^{\alpha/2}]). \\
&= \lambda(\log f(A,C)^{\alpha} f(B,D)^{1-\alpha})
\end{aligned}$$

This implies

$$\lambda(f(\alpha A + (1-\alpha)B, \alpha C + (1-\alpha)D) \prec_{wlog} \lambda(f(A,C)^{\alpha} f(B,D)^{1-\alpha}). \quad \square$$

COROLLARY 2.8. *Let* f *be a positive concave function defined on* $I \times J$ *and* $A, B \in \mathscr{M}_m(I)$, $C, D \in \mathscr{M}_n(J)$. *Then*

$$\lambda\left(f(\alpha A + (1-\alpha)B, \alpha C + (1-\alpha)D)\right) \prec^{wlog} \lambda\left(f(A,C)^{\alpha} f(B,D)^{1-\alpha}\right)$$

for all $0 \leqslant \alpha \leqslant 1$.

Proof. Since a positive concave function is log-concave, it follows that the function $f^{-1}(s,t) = \dfrac{1}{f(s,t)}$ is log-convex. Therefore by corollary 2.7, we have

$$\prod_{i=1}^{k} \lambda_i^{\downarrow}\left(f^{-1}(\alpha A + (1-\alpha)B, \alpha C + (1-\alpha)D)\right) \leqslant \prod_{i=1}^{k} \lambda_i^{\downarrow}\left(f(A,C)^{-\alpha} f(B,D)^{-(1-\alpha)}\right),$$

for $1 \leqslant k \leqslant mn$, which implies

$$\prod_{i=1}^{k} \lambda_i^{\downarrow -1}\left(f(A,C)^{-\alpha} f(B,D)^{-(1-\alpha)}\right) \leqslant \prod_{i=1}^{k} \lambda_i^{\downarrow -1}\left(f^{-1}(\alpha A + (1-\alpha)B, \alpha C + (1-\alpha)D)\right).$$

Since $\lambda_i^{\downarrow -1}(R) = \lambda_i^{\uparrow}(R^{-1})$ and $\lambda_i^{\uparrow}(RS) = \lambda_i^{\uparrow}(SR)$ for $R, S \in \mathscr{P}_n$, we have

$$\prod_{i=1}^{k} \lambda_i^{\uparrow}\left(f(A,C)^{\alpha} f(B,D)^{1-\alpha}\right) \leqslant \prod_{i=1}^{k} \lambda_i^{\uparrow}\left(f(\alpha A + (1-\alpha)B, \alpha C + (1-\alpha)D)\right).$$

This completes a proof. □

We prove our next result for separately convex functions (see [10, Theorem 4.3]).

THEOREM 2.9. *Let f be a separately convex function defined on $I\times J$, $0\in I\cap J$, $f(x,0)=0=f(0,y)$ for all $x\in I$, $y\in J$. Then*

$$\lambda\left(f(X^*AX,Y^*BY)\right)\prec_w \lambda\left((X\otimes Y)^* f(A,B)\,(X\otimes Y)\right)$$

for all $A\in\mathscr{M}_m(I),B\in\mathscr{M}_n(J)$ and contractions $X\in\mathscr{M}_m,Y\in\mathscr{M}_n$.

Proof. Let $\gamma_1,\gamma_2,\ldots,\gamma_m$ and $\mu_1,\mu_2,\ldots,\mu_n$ be the eigenvalues of X^*AX and Y^*BY respectively, with $u_1,u_2,\ldots,u_m$ and $v_1,v_2,\ldots,v_n$ as an orthonormal system of corresponding eigenvectors. Then for each $r=1,\ldots,mn$, there exist pair $(i_r,j_r)\in\{1,\ldots,m\}\times\{1,\ldots,n\}$ such that $\lambda_r^{\downarrow}\left(f(X^*AX,Y^*BY)\right)=f(\gamma_{i_r},\mu_{j_r})$. Since $f(x,0)=0=f(0,y)$, we may assume that $||Xu_i||\neq 0,\ ||Yv_j||\neq 0$ for $i=1,2,\ldots,m;\ j=1,2,\ldots,n$. As f is separately convex, Lemma 2.2 can be applied to f. Thus using separate convexity of f, the condition $f(x,0)=0=f(0,y)$ and Lemma 2.2 at appropriate places, we have

$$\begin{aligned}
&\sum_{r=1}^{k}\lambda_r^{\downarrow}(f(X^*AX,Y^*BY))\\
&\quad=\sum_{r=1}^{k} f(\gamma_{i_r},\mu_{j_r})=f\left(\langle X^*AXu_{i_r},u_{i_r}\rangle,\langle Y^*BYv_{j_r},v_{j_r}\rangle\right)\\
&\quad=\sum_{r=1}^{k} f\left(\langle AXu_{i_r},Xu_{i_r}\rangle,\langle BYv_{j_r},Yv_{j_r}\rangle\right)\\
&\quad=\sum_{r=1}^{k} f\left(||Xu_{i_r}||^2\left\langle A\frac{Xu_{i_r}}{||Xu_{i_r}||},\frac{Xu_{i_r}}{||Xu_{i_r}||}\right\rangle,||Yv_{j_r}||^2\left\langle B\frac{Yv_{j_r}}{||Yv_{j_r}||},\frac{Yv_{j_r}}{||Yv_{j_r}||}\right\rangle\right)\\
&\quad=\sum_{r=1}^{k} f\left(||Xu_{i_r}||^2\left\langle A\frac{Xu_{i_r}}{||Xu_{i_r}||},\frac{Xu_{i_r}}{||Xu_{i_r}||}\right\rangle\right.\\
&\qquad\qquad\left.+\left(1-||Xu_{i_r}||^2\right)\cdot 0,||Yv_{j_r}||^2\left\langle B\frac{Yv_{j_r}}{||Yv_{j_r}||},\frac{Yv_{j_r}}{||Yv_{j_r}||}\right\rangle\right)\\
&\quad\leqslant\sum_{r=1}^{k}||Xu_{i_r}||^2 f\left(\left\langle A\frac{Xu_{i_r}}{||Xu_{i_r}||},\frac{Xu_{i_r}}{||Xu_{i_r}||}\right\rangle,||Yv_{j_r}||^2\left\langle B\frac{Yv_{j_r}}{||Yv_{j_r}||},\frac{Yv_{j_r}}{||Yv_{j_r}||}\right\rangle\right)\\
&\qquad\qquad+\left(1-||Xu_{i_r}||^2\right)f\left(0,||Yv_{j_r}||^2\left\langle B\frac{Yv_{j_r}}{||Yv_{j_r}||},\frac{Yv_{j_r}}{||Yv_{j_r}||}\right\rangle\right)\\
&\quad=\sum_{r=1}^{k}||Xu_{i_r}||^2 f\left(\left\langle A\frac{Xu_{i_r}}{||Xu_{i_r}||},\frac{Xu_{i_r}}{||Xu_{i_r}||}\right\rangle,||Yv_{j_r}||^2\left\langle B\frac{Yv_{j_r}}{||Yv_{j_r}||},\frac{Yv_{j_r}}{||Yv_{j_r}||}\right\rangle\right.\\
&\qquad\qquad\left.+(1-||Yv_{j_r}||^2)\cdot 0\right)\\
&\quad\leqslant\sum_{r=1}^{k}||Xu_{i_r}||^2||Yv_{j_r}||^2 f\left(\left\langle A\frac{Xu_{i_r}}{||Xu_{i_r}||},\frac{Xu_{i_r}}{||Xu_{i_r}||}\right\rangle,\left\langle B\frac{Yv_{j_r}}{||Yv_{j_r}||},\frac{Yv_{j_r}}{||Yv_{j_r}||}\right\rangle\right)
\end{aligned}$$

$$
\begin{aligned}
&+||Xu_{i_r}||^2(1-||Yv_j||^2)f\left(\left\langle A\frac{Xu_{i_r}}{||Xu_{i_r}||},\frac{Xu_{i_r}}{||Xu_{i_r}||}\right\rangle,0\right)\\
&\leqslant \sum_{r=1}^{k}||Xu_{i_r}||^2||Yv_{j_r}||^2\left\langle f(A,B)\left(\frac{Xu_{i_r}}{||Xu_{i_r}||}\otimes\frac{Yv_{j_r}}{||Yv_{j_r}||}\right),\frac{Xu_{i_r}}{||Xu_{i_r}||}\otimes\frac{Yv_{j_r}}{||Yv_{j_r}||}\right\rangle\\
&=\sum_{r=1}^{k}\left\langle f(A,B)(Xu_{i_r}\otimes Yv_{j_r}),(Xu_{i_r}\otimes Yv_{j_r})\right\rangle\\
&=\sum_{r=1}^{k}\left\langle f(A,B)(X\otimes Y)(u_{i_r}\otimes v_{j_r}),(X\otimes Y)(u_{i_r}\otimes v_{j_r})\right\rangle\\
&=\sum_{r=1}^{k}\left\langle (X\otimes Y)^*f(A,B)(X\otimes Y)(u_{i_r}\otimes v_{j_r}),(u_{i_r}\otimes v_{j_r})\right\rangle,
\end{aligned}
$$

for $1\leqslant k\leqslant mn$. Hence by Lemma 2.3, we have

$$\lambda\left(f(X^*AX,Y^*BY)\right)\prec_w\lambda\left((X\otimes Y)^*f(A,B)(X\otimes Y)\right).$$

This completes a proof. □

COROLLARY 2.10. *Let f be a separately convex function defined on $I\times J$, $0\in I\cap J$, $f(x,0)=0=f(0,y)$ for all $x\in I$, $y\in J$. Then*

$$\lambda\left(f\left(\sum_{i=1}^{k}x_iA_i,\ \sum_{j=1}^{l}y_jB_j\right)\right)\prec_w\lambda\left(\sum_{i,j=1}^{k,l}f(x_i,y_j)A_i\otimes B_j\right)$$

for $A_1,\dots,A_k\in\mathscr{S}_m$, $B_1,\dots,B_l\in\mathscr{S}_n$ where $\sum_{i=1}^{k}A_i\leqslant I_m$, $\sum_{j=1}^{l}B_j\leqslant I_n$ and $x_i\in I$, $y_j\in J$.

Proof. Taking $X=\begin{bmatrix}X_1&0&\cdots&0\\ \vdots&\ddots&\ddots&\vdots\\ X_k&0&\cdots&0\end{bmatrix}$, $Y=\begin{bmatrix}Y_1&0&\cdots&0\\ \vdots&\ddots&\ddots&\vdots\\ Y_l&0&\cdots&0\end{bmatrix}$ and $A=\begin{bmatrix}A_1&0&\cdots&0\\ 0&A_2&\ddots&\vdots\\ \vdots&\ddots&\ddots&0\\ 0&\cdots&0&A_k\end{bmatrix}$, $B=\begin{bmatrix}B_1&0&\cdots&0\\ 0&B_2&\ddots&\vdots\\ \vdots&\ddots&\ddots&0\\ 0&\cdots&0&B_l\end{bmatrix}$ in Theorem 2.9 it follows that

$$\lambda\left(f(\sum_{i=1}^{k}X_i^*A_iX_i,\sum_{j=1}^{l}Y_j^*B_jY_j)\right)\prec_w\lambda\left(\sum_{i,j=1}^{k,l}(X_i\otimes Y_j)^*f(A_i,B_j)(X_i\otimes Y_j)\right)\qquad(1)$$

for $A_1,\dots,A_k\in\mathscr{M}_m(I)$, $B_1,\dots,B_l\in\mathscr{M}_n(J)$ and $X_1,\dots,X_k\in\mathscr{M}_m$, $Y_1,\dots,Y_l\in\mathscr{M}_n$ where $\sum_{i=1}^{k}X_i^*X_i\leqslant I_m$ and $\sum_{j=1}^{l}Y_j^*Y_j\leqslant I_n$. Replacing A_i, B_j with x_i, y_j and X_i, Y_j

with $A_i^{1/2}$, $B_j^{1/2}$ respectively in (1) we have

$$\lambda\left(f(\sum_{i=1}^{k} A_i^{1/2} x_i A_i^{1/2}, \sum_{j=1}^{l} B_j^{1/2} y_j B_j^{1/2})\right) \prec_w \lambda\left(\sum_{i,j=1}^{k,l} (A_i^{1/2} \otimes B_j^{1/2}) f(x_i, y_j)(A_i^{1/2} \otimes B_j^{1/2})\right).$$

Hence

$$\lambda\left(f(\sum_{i=1}^{k} x_i A_i, \sum_{j=1}^{l} y_j B_j)\right) \prec_w \lambda\left(\sum_{i,j=1}^{k,l} f(x_i, y_j) A_i \otimes B_j\right). \quad \square$$

3. Sub-additive inequalities

To prove sub-additive inequalities for separately matrix convex functions, we use the following lemma.

LEMMA 3.1. *Let f be real valued function defined on $I \times J$, $0 \in I \cap J$ satisfying $f(x,0) = 0 = f(0,y)$, $x \in I$, $y \in J$. Then the following are equivalent:*

(i) f is separately matrix convex.

(ii) $f(PAP, QBQ) \leqslant (P \otimes Q) f(A,B)(P \otimes Q)$ for $A \in \mathscr{M}_m(I)$, $B \in \mathscr{M}_n(J)$ and $P \in \Pi_m$, $Q \in \Pi_n$, $m,n \in N$.

(iii) $f(X^*AX, Y^*BY) \leqslant (X \otimes Y)^* f(A,B)(X \otimes Y)$ for $A \in \mathscr{M}_m(I)$, $B \in \mathscr{M}_n(J)$ and contractions $X \in \mathscr{M}_m$, $Y \in \mathscr{M}_n$, $m,n \in N$.

(iv) $f\left(\sum_{i=1}^{k} X_i^* A_i X_i, \sum_{j=1}^{l} Y_j^* B_j Y_j\right) \leqslant \sum_{i,j=1}^{k,l} (X_i \otimes Y_j)^* f(A_i, B_j)(X_i \otimes Y_j)$ for $A_1, \ldots, A_k \in \mathscr{M}_m(I)$, $B_1, \ldots, B_l \in \mathscr{M}_n(J)$ and $X_1, \ldots, X_k \in \mathscr{M}_m$, $Y_1, \ldots, Y_l \in \mathscr{M}_n$ such that $\sum_{i=1}^{k} X_i^* X_i \leqslant I_m$ and $\sum_{j=1}^{l} Y_j^* Y_j \leqslant I_n$.

(v) $f\left(\sum_{i=1}^{k} x_i A_i, \sum_{j=1}^{l} y_j B_j\right) \leqslant \sum_{i,j} f(x_i, y_j)(A_i \otimes B_j)$, for $A_1, \ldots, A_k \in \mathscr{S}_m$, $B_1, \ldots, B_l \in \mathscr{S}_n$ where $\sum_{i=1}^{k} A_i \leqslant I_m$, $\sum_{i=1}^{l} B_j \leqslant I_n$ for $x_i \in I$, $y_j \in J$.

Proof. Equivalence of first three conditions have been proved in [2]. The proof of $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (v)$ follows as in Corollary 2.10. We prove $(v) \Rightarrow (i)$.

Let $A, B \in \mathscr{M}_m(I)$, $C \in \mathscr{M}_n(J)$ have spectral decompositions $A = \sum_{i=1}^{k} \lambda_i P_i$, $B = \sum_{j=1}^{l} \alpha_j Q_j$, $C = \sum_{t=1}^{r} \gamma_t R_t$ then $\sum_{i=1}^{k} \alpha P_i + \sum_{j=1}^{l} (1-\alpha) Q_j = I_m$. Using (v) we have

$$f(\alpha A + (1-\alpha)B, C) = f(\alpha \sum_{i=1}^{k} \lambda_i P_i + (1-\alpha) \sum_{j=1}^{l} \mu_j Q_j, \sum_{t=1}^{r} \gamma_t R_t)$$

$$
\begin{aligned}
&= f\left(\sum_{i=1}^{k}\lambda_i(\alpha P_i)+\sum_{j=1}^{l}\mu_j((1-\alpha)Q_j),\sum_{t=1}^{r}\gamma_t R_t\right)\\
&\leqslant \alpha\sum_{i,t} f(\lambda_i,\gamma_t)(P_i\otimes R_t)+(1-\alpha)\sum_{j,t} f(\mu_j,\gamma_t)(Q_j\otimes R_t)\\
&= \alpha f(A,C)+(1-\alpha)f(B,C).
\end{aligned}
$$

The matrix convexity in the second variable is proved similarly. □

THEOREM 3.2. *Let f be nonnegative (nonpositive) separately matrix convex function on $[0,\infty)\times[0,\infty)$ with $f(x,0)=0=f(0,y)$, $x,y\in[0,\infty)$ and $A,B\in\mathscr{S}_m$, $C,D\in\mathscr{S}_n$. Then there exists unitaries U, V, W, Z on $\mathscr{M}_m\otimes\mathscr{M}_n$ such that*

$$f(A+B,C+D)\geqslant U^*f(A,C)U+V^*f(A,D)V+W^*f(B,C)W+S^*f(B,D)S.$$

Proof. We can assume that A+B is invertible. Then $X=A^{1/2}(A+B)^{-1/2}$ and $Y=B^{1/2}(A+B)^{-1/2}$ are contractions and $A=X(A+B)X^*$, $B=Y(A+B)Y^*$. Therefore using Lemma 3.1 (iii) and the fact that for any $T\in\mathscr{M}_m$, TT^* and T^*T are unitarily equivalent, we have

$$
\begin{aligned}
f(A,C+D) &= f(X(A+B)X^*,C+D)\\
&\leqslant (X\otimes I_n)f(A+B,C+D)(X\otimes I_n)^*\\
&= U_0 f(A+B,C+D)^{1/2}(X\otimes I_n)^*(X\otimes I_n)f(A+B,C+D)^{1/2}U_0^*
\end{aligned}
$$

for some unitary U_0 and so

$$U_0^*f(A,C+D)U_0\leqslant f(A+B,C+D)^{1/2}(X^*X\otimes I_n)f(A+B,C+D)^{1/2}. \qquad (2)$$

Similarly, we have

$$V_0^*f(B,C+D)V_0\leqslant f(A+B,C+D)^{1/2}(Y^*Y\otimes I_n)f(A+B,C+D)^{1/2} \qquad (3)$$

for some unitary V_0. Adding (2) and (3) we get

$$U_0^*f(A,C+D)U_0+V_0^*f(B,C+D)V_0\leqslant f(A+B,C+D) \qquad (4)$$

using the fact that $X^*X+Y^*Y=I_m$. Similarly, there exists unitaries R, T, L, M such that

$$f(A,C+D)\geqslant R^*f(A,C)R+T^*f(A,D)T$$

and

$$f(B,C+D)\geqslant L^*f(B,C)L+M^*f(B,D)M.$$

Thus it follows from (4) that there exists unitaries U, V, W, S such that

$$f(A+B,C+D)\geqslant U^*f(A,C)U+V^*f(A,D)V+W^*f(B,C)W+S^*f(B,D)S. \quad \square$$

The following corollary follows on applying Theorem 3.2 to $-f$.

COROLLARY 3.3. *Let f be a nonnegative (nonpositive) separately matrix concave function on $[0,\infty)\times[0,\infty)$ with $f(x,0)=0=f(0,y)$ and $A,B\in\mathscr{S}_m$, $C,D\in\mathscr{S}_n$. Then there exists unitaries U, V, W, Z on $\mathscr{M}_m\otimes\mathscr{M}_n$ such that*

$$f(A+B,C+D)\leqslant U^*f(A,C)U+V^*f(A,D)V+W^*f(B,C)W+S^*f(B,D)S.$$

REFERENCES

[1] T. ANDO, *Concavity of certain maps on positive definite matrices and application to Hadamard Products*, Linear Algebra Appl., **26** (1979), 203–241.
[2] J. S. AUJLA, *Matrix convexity of functions of two variables*, Linear Algebra Appl., **194** (1993), 149–160.
[3] J. S. AUJLA AND J. C. BOURIN, *Eigenvalue inequalities for convex and log-convex functions*, Linear Algebra Appl., **424** (2007), 25–35.
[4] J. S. AUJLA AND F. C. SILVA, *Weak majorization inequalities and convex functions*, Linear Algebra Appl., **369** (2003), 217–233.
[5] R. BHATIA, *Matrix Analysis*, Springer Verlag, New York, 1997.
[6] A. L. BROWN AND H. L. VASUDEVA, *The calculus of operator functions and operator convexity*, Diss. Math., **390** (2000), 1–48.
[7] C. DAVIS, *Notions generalizing convexity for functions defined on spaces of matrices*, Proc. Amer. Math. Soc. Symposia (Convexity), **7** (1963) 187–201.
[8] F. HANSEN, *Jensen's operator inequality for functions of two variables*, Proc. Amer. Math. Soc., **125** (1997), 2093–2102.
[9] F. HANSEN, *Operator convex functions of several variables*, Publ. Res. Inst. Math. Sci., **33** (1997), 443–463.
[10] F. HANSEN, *Convex trace functions of several variable*, Linear Algebra Appl., **341** (2002), 309–315.
[11] A. KORANYI, *On some classes of analytic functions of several variables*, Trans. Amer. Math. Soc., **101** (1961), 521–554.
[12] O. E. TIKHNOV, *A note on definition of matrix convex functions*, Linear Agebra Appl., **416** (2006), 773–775.

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