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SOME SUBORDINATION PROPERTIES OF GENERALIZED JUNG–KIM–SRIVASTAVA INTEGRAL OPERATOR

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Abstract. The object of this paper is to discuss some interesting properties of the integral operator

$$\mathscr{P}^{\alpha} f(z) = \frac{(p+1)^{\alpha}}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt, \quad (\alpha > 0),$$

for the class of all analytic functions $f(z)$ of the form $f(z) = z + \sum_{n=p+1}^{\infty} a_n z^n$, for $z \in \Delta = \{z \in \mathbb{C} : |z| < 1\}$. For $p = 1$, this integral operator was introduced and studied by Jung, Kim and Srivastava in [2].

1. Introduction

Let $\mathscr{A}(p)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1}$$

which are analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathscr{A}(p)$, let $\mathscr{P}^{\alpha} f : \mathscr{A}(p) \longrightarrow \mathscr{A}(p)$ defined by

$$\begin{aligned} \mathscr{P}^{\alpha} f(z) &= \frac{(p+1)^{\alpha}}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)dt, \\ &= z^p + \sum_{n=p+1}^{\infty} \left(\frac{p+1}{n+1}\right)^{\alpha} a_n z^n, \quad (\alpha > 0,\ f(z) \in \mathscr{A}(p)). \end{aligned} \tag{2}$$

It is implied from (2) that

$$z(\mathscr{P}^{\alpha} f(z))' = (p+1)\mathscr{P}^{\alpha-1} f(z) - \mathscr{P}^{\alpha} f(z). \tag{3}$$

When $p = 1$, the identity (3) is given by Jung, Kim and Srivastava [2].

Motivated by the recent work by the authors in [5] we are interested to study the generalized operator $\mathscr{P}^{\alpha} f$ and establish some more interesting properties by using differential subordination which are presumedly new.

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2. Definitions and Lemmas

DEFINITION 4. [1] Let $f(z)$ defined by (1) and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, $p \in \mathbb{N}$, then the Hadamard product of $f(z)$ and $g(z)$ denoted by $(f * g)(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, \quad (z \in \Delta,\ p \in \mathbb{N}).$$

DEFINITION 5. [1] Let $f(z)$ and $g(z)$ be analytic in Δ. We say that $f(z)$ is subordinate to $g(z)$, written by $f \prec g$ or $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in Δ which satisfies $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. If $g(z)$ is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

To prove our result we require the following lemmas.

LEMMA 6. [6] *Let $h(z)$ be analytic and convex univalent in Δ, $h(0) = 1$, and let $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be analytic in Δ. If*

$$g(z) + \frac{zg'(z)}{c} \prec h(z),$$

then for $c \neq 0$ and $\operatorname{Re} c \geqslant 0$,

$$g(z) \prec \frac{c}{z^c} \int_0^z t^{c-1} f(t)dt, \quad z \in \Delta.$$

Let $P(\gamma)$ $(0 \leqslant \gamma < 1)$ denote the class of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ which are analytic in Δ and satisfy the condition $\operatorname{Re}(p(z)) > \gamma$ for $z \in \Delta$.

LEMMA 7. [10] *Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P(\gamma)$ $(0 \leqslant \gamma < 1)$. Then*

$$\operatorname{Re}(p(z)) > 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|}, \quad z \in \Delta.$$

LEMMA 8. [9] *The function $(1-z)^\gamma \equiv e^{\gamma \log(1-z)}$, $\gamma \neq 0$, is univalent in Δ if and only if γ is in the closed disk $|\gamma - 1| \leqslant 1$ or in the closed disk $|\gamma + 1| \leqslant 1$.*

LEMMA 9. [7] *Let $q(z)$ be univalent in Δ and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)(\phi(q(z)))$, $h(z) = \theta(q(z) + Q(z))$ and suppose that*

1. *$Q(z)$ is starlike(univalent) in Δ;*

2. $\operatorname{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right\} > 0, \quad (z \in \Delta).$

If $p(z)$ *is analytic in* Δ, *with* $p(0) = q(0)$, $p(\Delta) \subset D$, *and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$ *and* $q(z)$ *is the best dominant.*

For real or complex numbers $a,\ b,\ c\ (c \neq 0, -1, -2, \dots)$, the Gaussian Hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots,$$

which is absolutely convergent in Δ and hence represents an analytic function in the unit disk Δ (see [[12], chapter 14]).

The following identities are well-known.

LEMMA 10. [12] *For real or complex numbers* $a,\ b,\ c\ (c \neq 0, -1, -2, \dots)$, *we have*

1. ${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0$

2. ${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)$

3. ${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1(a,c-b;c;\frac{z}{z-1})$

4. $(a+1){}_2F_1(1,a,a+1;z) = (a+1) + az{}_2F_1(1,a+1,a+2;z)$.

3. Main Results

THEOREM 3.1. *Let* $\alpha > 1$, $\lambda < 1$, *and let* $-1 \leqslant B_i < A_i \leqslant 1$ *for* $i = 1,2$. *If functions* $f_i(z) \in \mathscr{A}(p)(i = 1,2)$ *satisfy the condition*

$$(1-\lambda)\frac{\mathscr{P}^{\alpha-1}f_i(z)}{z^p} + \lambda\frac{\mathscr{P}^{\alpha}f_i(z)}{z^p} \prec h(A_i,\ B_i;\ z), \tag{11}$$

then

$$\frac{\mathscr{P}^{\alpha}F(z)}{z^p} \prec h(1-2\gamma,\ -1;\ z),$$

where

$$F(z) = (f_1 * f_2)(z) \tag{12}$$

and

$$\gamma = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}{}_2F_1\left(1,1,\frac{p+1}{1-\lambda}+1,\frac{1}{2}\right)\right]. \tag{13}$$

The result is sharp when $B_1 = B_2 = -1$.

Proof. Suppose that $f_i(z) \in \mathscr{A}(p)$ $(i=1,2)$, satisfy the condition (11). Let

$$p_i(z) = (1-\lambda)\frac{\mathscr{P}^{\alpha-1} f_i(z)}{z^p} + \lambda \frac{\mathscr{P}^{\alpha} f_i(z)}{z^p}, \quad (i=1,2). \tag{14}$$

Then we have $p_i(z) \in P(\alpha_i)$, where

$$\alpha_i = \frac{1-A_i}{1-B_i}, \quad (i=1,2).$$

From (3) and (14), it follows that

$$\mathscr{P}^{\alpha} f_i(z) = \frac{p+1}{1-\lambda} z^{p-\left(\frac{p+1}{1-\lambda}\right)} \int_0^z t^{\frac{p+\lambda}{1-\lambda}} p_i(t)dt. \tag{15}$$

Now we let

$$F(z) = (f_1 * f_2)(z).$$

After a simple computation, we get

$$\mathscr{P}^{\alpha} F(z) = \frac{p+1}{1-\lambda} z^{p-\left(\frac{p+1}{1-\lambda}\right)} \int_0^z t^{\frac{p+\lambda}{1-\lambda}} p(t)dt, \tag{16}$$

where

$$p(z) = (p_1 * p_2)(z) = (1-\lambda)\frac{\mathscr{P}^{\alpha-1} F(z)}{z^p} + \lambda \frac{\mathscr{P}^{\alpha} F(z)}{z^p} \tag{17}$$

and

$$p_1(z) \in P(\alpha_1) \ \text{ and } \ p_3(z) = \frac{p_2(z)-\alpha_2}{2(1-\alpha_2)} + \frac{1}{2} \in P\left(\frac{1}{2}\right). \tag{18}$$

By Herglotz formula

$$(p_1 * p_2)(z) \in P(\alpha_3),$$

where

$$\alpha_3 = 1 - 2(1-\alpha_1)(1-\alpha_2). \tag{19}$$

By (16), (17), (18), (19), Lemma 7 and Lemma 10, we have

$$\begin{aligned}
\operatorname{Re}\left(\frac{\mathscr{P}^{\alpha} F(z)}{z^p}\right) &= \frac{p+1}{1-\lambda}\int_0^1 u^{\frac{p+\lambda}{1-\lambda}} \operatorname{Re}\,(p_1 * p_2)(uz)du \\
&\geqslant \frac{p+1}{1-\lambda}\int_0^1 u^{\frac{p+\lambda}{1-\lambda}} \left(2\alpha_3 - 1 + \frac{2(1-\alpha_3)}{1+u|z|}\right) du \\
&> \frac{p+1}{1-\lambda}\int_0^1 u^{\frac{p+\lambda}{1-\lambda}} \left(2\alpha_3 - 1 + \frac{2(1-\alpha_3)}{1+u}\right) du \\
&= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1 - \frac{p+1}{1-\lambda}\int_0^1 \frac{u^{\frac{p+\lambda}{1-\lambda}}}{1+u} du\right) \\
&= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\,{}_2F_1\left(1,1,\frac{p+1}{1-\lambda}+1,\frac{1}{2}\right)\right].
\end{aligned} \tag{20}$$

When $B_1 = B_2 = -1$, we consider the function $f_i(z) \in \mathscr{A}(p)(i = 1,2)$, which satisfy the condition (11) and is given by

$$\mathscr{P}^{\alpha} f_i(z) = \frac{p+1}{1-\lambda} z^{p-\left(\frac{p+1}{1-\lambda}\right)} \int_0^z t^{\frac{p+\lambda}{1-\lambda}} \left(\frac{1+A_i t}{1-t}\right) dt.$$

Then from (16)

$$\frac{\mathscr{P}^{\alpha} F(z)}{z^p} = \frac{p+1}{1-\lambda} \int_0^1 u^{\frac{p+\lambda}{1-\lambda}} \left(1-(1+A_1t)(1+A_2t)+\frac{(1+A_1t)(1+A_2t)}{1-uz}\right) du.$$

This implies

$$\begin{aligned} \frac{\mathscr{P}^{\alpha} F(z)}{z^p} &\to 1-(1+A_1)(1+A_2)\left(1-\frac{p+1}{1-\lambda}\int_0^1 \frac{u^{\frac{p+\lambda}{1-\lambda}}}{1+u} du\right) \\ &= 1-(1+A_1)(1+A_2)\left[1-\frac{1}{2}{}_2F_1\left(1,1,\frac{p+1}{1-\lambda}+1,\frac{1}{2}\right)\right]. \qquad (21) \end{aligned}$$

Hence the result. □

COROLLARY 3.1. *Let $\alpha > 1$, $\lambda < 1$, and let $-1 \leqslant B_i < A_i \leqslant 1$ for $i = 1,2$. If functions $f_i(z) \in \mathscr{A}(1)$ $(i = 1,2)$ satisfy*

$$(1-\lambda)\frac{\mathscr{P}^{\alpha-1} f_i(z)}{z} + \lambda \frac{\mathscr{P}^{\alpha} f_i(z)}{z} \prec h(A_i,\ B_i;\ z), \qquad (22)$$

then

$$\frac{\mathscr{P}^{\alpha} F(z)}{z} \prec h(1-2\gamma,\ -1;\ z),$$

where

$$F(z) = (f_1 * f_2)(z) \qquad (23)$$

and

$$\gamma = \left[1-\frac{1}{2}{}_2F_1\left(1,1,\frac{2}{1-\lambda}+1,\frac{1}{2}\right)\right]. \qquad (24)$$

The result is sharp when $B_1 = B_2 = -1$.

THEOREM 3.2. *Let $\alpha > 1$, $\lambda < 1$ and $-1 \leqslant B < A \leqslant 1$. If $f(z) \in \mathscr{A}(p)$ satisfies*

$$(1-\lambda)\frac{\mathscr{P}^{\alpha-1} f(z)}{z^p} + \lambda \frac{\mathscr{P}^{\alpha} f(z)}{z^p} \prec h(A,\ B;\ z), \qquad (25)$$

then for $m \in \mathbb{N}$, $z \in \Delta$

$$\mathrm{Re}\left(\frac{\mathscr{P}^{\alpha} f(z)}{z^p}\right)^{\frac{1}{m}} > \delta^{1/m}, \qquad (26)$$

where

$$\delta = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1,1+\frac{p+1}{1-\lambda}, \frac{B}{B-1}) & \text{if } B \neq 0 \\ 1 - \frac{A}{1+\left(\frac{1-\lambda}{p+1}\right)} & \text{if } B = 0. \end{cases} \tag{27}$$

The result is sharp.

Proof. Let

$$g(z) = \frac{\mathscr{P}^{\alpha} f(z)}{z^p}, \tag{28}$$

for $f(z) \in \mathscr{A}(p)$. The function $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ is analytic in Δ. By (2), (25) and (28), we obtain

$$g(z) + \frac{1-\lambda}{p+1} z g'(z) \prec h(A,\ B;\ z). \tag{29}$$

It follows from Lemma 6 and (29), that

$$g(z) \prec \frac{p+1}{1-\lambda} z^{-\frac{p+1}{1-\lambda}} \int_0^z t^{\frac{p+1}{1-\lambda}-1} \left(\frac{1+At}{1+Bt}\right) dt$$

or

$$\frac{\mathscr{P}^{\alpha} f(z)}{z^p} = \frac{p+1}{1-\lambda} \int_0^1 u^{\frac{p+1}{1-\lambda}-1} \left(\frac{1+Auw(z)}{1+Buw(z)}\right) du, \tag{30}$$

where $w(z)$ is analytic in Δ with $w(0)=0$ and $|w(z)|<1$ for $z \in \Delta$. As $-1 \leqslant B < A \leqslant 1$, it follows from (30), that

$$\mathrm{Re} \frac{\mathscr{P}^{\alpha} f(z)}{z^p} > \frac{p+1}{1-\lambda} \int_0^1 u^{\frac{p+1}{1-\lambda}-1} \left(\frac{1-Au}{1-Bu}\right) dt \tag{31}$$

$$= \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1,1+\frac{p+1}{1-\lambda}, \frac{B}{B-1}) & \text{if } B \neq 0 \\ 1 - \frac{A}{1+\left(\frac{1-\lambda}{p+1}\right)} & \text{if } B = 0. \end{cases}$$

Because $\mathrm{Re}\left(w^{\frac{1}{m}}\right) \geqslant (\mathrm{Re}\, w)^{\frac{1}{m}}$ for $\mathrm{Re}\, w > 0$ and $m \geqslant 1$, hence (26) follows from (31).

To show the sharpness of (26), we take $f(z) \in \mathscr{A}(p)$ defined by

$$\frac{\mathscr{P}^{\alpha} f(z)}{z^p} = \frac{p+1}{1-\lambda} \int_0^1 u^{\frac{p+1}{1-\lambda}-1} \left(\frac{1+Auz}{1+Buz}\right) dz. \tag{32}$$

For the above function, we find that

$$(1-\lambda)\frac{\mathscr{P}^{\alpha-1} f(z)}{z^p} + \lambda \frac{\mathscr{P}^{\alpha} f(z)}{z^p} = \frac{1+Az}{1+Bz},$$

and

$$\frac{\mathscr{P}^{\alpha} f(z)}{z^p} \to \frac{p+1}{1-\lambda} \int_0^1 u^{\frac{p+1}{1-\lambda}-1} \left(\frac{1-Au}{1-Bu}\right) du \tag{33}$$

as $z \to -1$. Hence the result. □

COROLLARY 3.2. *Let* $\alpha > 1$, $\lambda < 1$ *and* $-1 \leqslant B < A \leqslant 1$. *If* $f(z) \in \mathscr{A}(1)$ *satisfies*

$$(1-\lambda)\frac{\mathscr{P}^{\alpha-1} f(z)}{z} + \lambda \frac{\mathscr{P}^{\alpha} f(z)}{z} \prec h(A,\, B;\, z), \tag{34}$$

then

$$\operatorname{Re}\left(\frac{\mathscr{P}^{\alpha} f(z)}{z}\right)^{\frac{1}{m}} > (\delta^*)^{1/m}, \tag{35}$$

where

$$\delta^* = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1,1+\frac{2}{1-\lambda},\frac{B}{B-1}) & \text{if } B \neq 0 \\ 1 - \frac{A}{1+\left(\frac{1-\lambda}{2}\right)} & \text{if } B = 0. \end{cases} \tag{36}$$

The result is sharp.

For a function $f(z) \in \mathscr{A}(p)$, the generalized Bernardi Libera-Livingstone [4] integral operator

$$J_c : \mathscr{A}(p) \to \mathscr{A}(p),$$

defined by

$$J_c f(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt, \quad (c > -p,\ p \in \mathbb{N}). \tag{37}$$

THEOREM 3.3. *Let* $\alpha > 1$, $c > -p$, *and* $-1 \leqslant B < A \leqslant 1$. *Suppose that* $f(z) \in \mathscr{A}(p)$ *and* $J_c f(z)$ *is given by* (37). *If*

$$(1-\lambda)\frac{\mathscr{P}^{\alpha} f(z)}{z^p} + \lambda \frac{\mathscr{P}^{\alpha} J_c f(z)}{z^p} \prec h(A,\, B;\, z).$$

then

$$\operatorname{Re}\left(\frac{\mathscr{P}^{\alpha} f(z)}{z^p}\right)^{\frac{1}{m}} > (\beta^*)^{1/m},$$

where

$$\beta^* = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1,1+\frac{c+p}{1-\lambda},\frac{B}{B-1}) & \text{if } B \neq 0 \\ 1 - \frac{A}{1+\left(\frac{1-\lambda}{c+p}\right)} & \text{if } B = 0. \end{cases} \tag{38}$$

The result is sharp.

Proof. From (37), we have

$$z(\mathscr{P}^{\alpha}J_c f(z))' = (c+p)\mathscr{P}^{\alpha}f(z) - c\mathscr{P}^{\alpha}J_c f(z). \tag{39}$$

Let

$$g(z) = \frac{\mathscr{P}^{\alpha}J_c f(z)}{z^p}. \tag{40}$$

Then from (37), (39), (40) it follows that

$$(1-\lambda)\frac{\mathscr{P}^{\alpha}f(z)}{z^p} + \lambda\frac{\mathscr{P}^{\alpha}J_c f(z)}{z^p} = g(z) + \frac{1-\lambda}{c+p}zg'(z) \prec h(A,\, B;\, z), \quad z \in \Delta. \qquad \square$$

Following the steps of the proof of Theorem 3.2 we get the result.

COROLLARY 3.3. *Let $\alpha > 1$, $c > -1$, and $-1 \leqslant B < A \leqslant 1$. Suppose that $f(z) \in \mathscr{A}(1)$ and $J_c f(z)$ is given by (37). If*

$$(1-\lambda)\frac{\mathscr{P}^{\alpha}f(z)}{z} + \lambda\frac{\mathscr{P}^{\alpha}J_c f(z)}{z} \prec h(A,\, B;\, z).$$

Then

$$\mathrm{Re}\left(\frac{\mathscr{P}^{\alpha}f(z)}{z}\right)^{\frac{1}{m}} > (\gamma^*)^{1/m}, \tag{41}$$

where

$$\gamma^* = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1,1+\frac{c+1}{1-\lambda},\frac{B}{B-1}) & \text{if } B \neq 0 \\ 1 - \frac{A}{1+\left(\frac{1-\lambda}{c+1}\right)} & \text{if } B = 0. \end{cases} \tag{42}$$

The result is sharp.

THEOREM 3.4. *Let $\alpha > 1$, $0 \leqslant \rho < 1$. Let γ be a complex number with $\gamma \neq 0$ and satisfy either $|2\gamma(1-\rho)(p+1)-1| \leqslant 1$ or $|2\gamma(1-\rho)(p+1)+1| \leqslant 1$. If $f(z) \in \mathscr{A}(p)$ satisfies the condition*

$$\mathrm{Re}\left\{\frac{\mathscr{P}^{\alpha-1}f(z)}{\mathscr{P}^{\alpha}f(z)}\right\} > \rho, \tag{43}$$

then

$$\left(\frac{\mathscr{P}^{\alpha}f(z)}{z^p}\right)^{\gamma} \prec \frac{1}{(1-z)^{2\gamma(1-\rho)(p+1)}} = q(z), \quad (z \in \Delta). \tag{44}$$

The result is sharp.

Proof. Let

$$g(z) = \left(\frac{\mathscr{P}^{\alpha}f(z)}{z^p}\right)^{\gamma}, \quad (z \in \Delta). \tag{45}$$

Then from (3), (43) and (45), we get

$$1+\frac{zg'(z)}{\gamma(p+1)g(z)} \prec \frac{1+(1-2\rho)z}{1-z}, \quad (z\in\Delta). \tag{46}$$

If we take

$$q(z)=\frac{1}{(1-z)^{2\gamma(1-\rho)(p+1)}},\ \theta(w)=1 \text{ and } \phi(w)=\frac{1}{(p+1)\gamma w}.$$

Then $q(z)$ is univalent by the assertion and lemma 8. Here $q(z)$, $\theta(w)$ and $\phi(w)$ satisfy the conditions of lemma 9. Since

$$Q(z)=zq'(z)\phi(q(z))=\frac{2(1-\rho)z}{1-z}$$

is univalent and starlike in Δ and

$$h(z)=\theta(q(z))+Q(z),$$

hence the condition (1) and (2) of lemma 9 are satisfied. Hence the result follows from (46), lemma 8 and lemma 9. □

COROLLARY 3.4. *Let $\alpha>1$, $0\leqslant\rho<1$. Let γ be a complex number with $\gamma\neq 0$ and satisfy either $|4\gamma(1-\rho)-1|\leqslant 1$ or $|4\gamma(1-\rho)+1|\leqslant 1$. If $f(z)\in\mathscr{A}(1)$ satisfies the condition*

$$\operatorname{Re}\left\{\frac{\mathscr{P}^{\alpha-1}f(z)}{\mathscr{P}^{\alpha}f(z)}\right\}>\rho, \tag{47}$$

then

$$\left(\frac{\mathscr{P}^{\alpha}f(z)}{z}\right)^{\gamma} \prec \frac{1}{(1-z)^{4\gamma(1-\rho)}}=q(z), \quad (z\in\Delta). \tag{48}$$

The result is sharp.

COROLLARY 3.5. *Let $\alpha>1$, $0\leqslant\rho<1$. Let γ be a real number with $\gamma\geqslant 1$. If $f(z)\in\mathscr{A}$ satisfies the condition (43), then*

$$\left(\frac{\mathscr{P}^{\alpha}f(z)}{z}\right)^{\frac{1}{4\gamma(1-\rho)}} > 2^{-\frac{1}{\gamma}}, \quad (z\in\Delta). \tag{49}$$

The result is sharp.

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REFERENCES

[1] P.L. DUREN, *Univalent functions*, Springer-Verlag, Berlin-New York, 1983.

[2] I.B. JUNG, Y.C, KIM AND H.M. SRIVASTAVA, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math Anal Appl., **176** (1993), 138–147.

[3] Y. KOMATU, *On a one-parameter additive family of operators defined on analytic functions regular in the unit disk*, Bull. Fac. Sci. Enrgy. Chuo Uni., **22** (1979), 1–22; see also Analytic Functions (J. Lawrynowicz, Editor), pp. 292–300, Springer-Verlag Berlin, Heidelberg and New York, 1980.

[4] R. J. LIBERA, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16** (1965), 755–758.

[5] J.L. LIU, *Notes on Jung-Kim-Srivastava integral operator*, J. Math Anal Appl., **294**, 1 (2004), 96–103.

[6] S.S. MILLER AND P.T. MOCANU, *Differential subordinations and univalent functions*, Michigan Math. J., **28** (1981), 157–171.

[7] S.S. MILLER AND P.T. MOCANU, *On some classes of first order differential subordination*, Michigan Math. J., **32** (1985), 185–195.

[8] S.S. MILLER AND P.T. MOCANU, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65** (1978), 535–545.

[9] M.S. ROBERTSON, *Certain classes of starlike functions*, Michigan Math. J., **32** (1985), 135–140.

[10] H.M. SRIVASTAVA AND S. OWA(EDS.), *Current Topics in Analytic Function Theory*, World Scientific, Singapore, 1992.

[11] H. M. SRIVASTAVA AND S. OWA, *A certain one parameter additive family of operators defined on analytic functions*, J. Math Anal. Appl., **118** (1986), 80–87.

[12] E.T. WHITTAKER AND G.N. WATSON, *A course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and Analytic Functions; with an Account of the Principal Transcdental Functions*, 4th edition, Cambridge University Press, Cambridge, 1927.

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