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GENERAL INEQUALITIES FOR MULTIPOINT PADÉ APPROXIMANTS TO A STIELTJES FUNCTION EXPANDED AT REAL POINTS

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Abstract. In this paper we establish the general inequalities for diagonal and subdiagonal multipoint Padé approximants to a Stieltjes function f in terms of power expansion of f on the real line. The inequalities derived produce the best upper and lower bounds on f with respect to the given coefficients of Stieltjes series. As an example of applications sequences of upper and lower Padé bounds converging to the effective dielectric constant of a random array of spheres are evaluated.

1. Introduction

The properties of one- two- and three-point Padé approximants to a Stieltjes function, say f, were extensively investigated in recent years. The obtained results valid in a real domain read: (i) sequences of diagonal and subdiagonal one-point Padé approximants to an expansion of f at 0 form upper and lower bounds converging to f, cf. [1, 9, 15]; (ii) sequences of diagonal two-point Padé approximants to expansions of f at 0 and ∞ also form upper and lower bounds converging to f, cf. [6, 13]; (iii) sequences of diagonal three-point Padé approximants to expansions of f at $-1,\ 0,\ \infty$ form as previously upper and lower bounds converging to f, cf. [12]. The Padé bounds (i), (ii) and (iii) on f are the best ones with respect to the given finite number of Stieltjes series coefficients.

The aim of this paper is to establish the general inequalities for N-point Padé approximants to a Stieltjes function f in terms of a given finite number of coefficients of power expansions of f at real points $z_1, z_2, \ldots, z_N$, where $z_j < z_N \leqslant \infty$, $j = 1, 2, \ldots, N-1$. The inequalities obtained confirm via simple mathematical formulae the special bounding properties of multipoint point Padé approximants to the Stieltjes functions reported earlier in [1, 3, 4, 5, 6, 12, 13] and predict new ones as well. As an example of practical applications the sequences of upper and lower Padé bounds converging to the effective dielectric constant of a random array of spheres are evaluated.

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2. Preliminaries

In this section the basic notions such as regular Stieltjes functions f, Stieltjes series representing f, inclusion regions of allowed values of f, error bounds for f, unified linear fractional transformation of f, multipoint fractional expansions of f and others are introduced.

2.1. Basic estimates of regular Stieltjes functions

We begin our consideration with the Stieltjes integral representation given by

$$\varphi(s) = \int_0^\infty \frac{d\gamma_\infty(v)}{1+sv}, \quad d\gamma_\infty(v) > 0, \quad \varphi(0) = \eta, \quad 0 < \eta \leqslant \infty. \tag{1}$$

By making the change of variables $s = z+1$ and $v = \frac{u}{1-u}$ we obtain

$$\varphi(z+1) = \int_0^\infty \frac{d\gamma_\infty(v)}{1+(z+1)v} = \int_0^1 \frac{d\gamma(u)}{1+zu} = \chi(z), \quad \chi(-1) = \eta, \tag{2}$$

where

$$d\gamma(u) = (1-u)d\gamma_\infty\left[\frac{u}{1-u}\right] > 0. \tag{3}$$

The formula (3) transforms the measure $d\gamma_\infty(v)$-defined in the infinite interval $[0,\infty]$ to the measure $d\gamma(u)$-defined in the finite one $[0,1]$. Now without loss of generality we assume $\eta = 1$. For any value of $\eta \in (0,\infty]$ the considerations are analogous. On account of that we can limit our studies to the regular Stieltjes functions f representing in the theory, for example of inhomogeneous media, the effective transport coefficients of composites such as electric and thermal conductivities, dielectric constants,magnetic permittivities, diffusion coefficients [10].

DEFINITION 1. The N power expansions

$$\begin{gathered} f(z) = \textstyle\sum_{i=0}^{p_j-1} c_{ij}(z-z_j)^i + O((z-z_j)^{p_j}), \qquad j = 1,2,\dots,N, \\ c_{ij} = c_i(z_j) = \frac{f^{(i)}(z_j)}{i!}, \qquad f_1^{(i)}(z_j) = \left.\frac{d^i f(z)}{d^i(z)}\right|_{z=z_j}, \qquad i = 0,1,\dots,p_j-1 \end{gathered} \tag{4}$$

of the function

$$f(z) = \int_0^1 \frac{d\gamma(u)}{1+zu}, \quad \gamma(0) = 0, \quad d\gamma(u) \geqslant 0, \quad z \in \mathbb{C}\backslash[-\infty,-1) \tag{5}$$

satisfying the condition

$$f(-1) = 1 \tag{6}$$

we call regular Stieltjes series (4)–(6) representing the regular Stieltjes function (5)–(6).

Due to (4)–(6) the regular Stieltjes series possess radii of convergence at least 1 and take at $z=-1$ the values 1. From (5)–(6), it follows: f is real symmetric, i.e. it takes complex conjugate values when the variable z is complex-conjugated $f(z^*) = [f(z)]^*$. Furthermore, if z_s and z_r^* are conjugated numbers then coefficients c_{is} and c_{ir}^* of (4) are complex-conjugated ones. With Definition 1 are connected inseparably the sets:

- Of non-decreasing functions γ

$$\Gamma_P = \Gamma_{z_1,z_2,\ldots,z_N}^{p_1,p_2,\ldots,p_N};_{-1}^{+1} = \left\{\gamma;\ c_{ij} \text{ given},\ \gamma(u)\ \text{ fulfil (4)–(6)}\right\}. \tag{7}$$

- Of non-increasing functions of Stieltjes f:

$$\Phi_P = \Phi_{z_1,z_2,\ldots,z_N}^{p_1,p_2,\ldots,p_N};_{-1}^{+1} = \left\{f;\ f(\cdot) = \int_0^1 \frac{d\gamma(u)}{1+(\cdot)u},\quad \gamma\in\Gamma_P\right\}. \tag{8}$$

- Of admissible values of $f(z_0)$ called the inclusion region of $f(z_0)$:

$$\Phi_P(z_0) = \Phi_{z_1,\ldots,z_N}^{p_1,\ldots,p_N};_{-1}^{+1}(z_0) = \left\{f(z_0);\ f(z_0) = \int_0^1 \frac{d\gamma(u)}{1+z_0u},\ \gamma\in\Gamma_P\right\}. \tag{9}$$

- Of a boundary of admissible values of $f(z_0)$ called the error bound for $f(z_0)$

$$\phi_P(z_0) = \phi_{z_1,z_2,\ldots,z_N}^{p_1,p_2,\ldots,p_N};_{-1}^{+1}(z_0) = \partial\Phi_P(z_0). \tag{10}$$

The column $\frac{p_j}{z_j}$ denotes the numbers of known coefficients p_j of power expansions of f at z_j, $j=1,2,\ldots,N$, while $\frac{+1}{-1}$ informs that $f(-1) = 1$. Parameter $P = \sum_{j=1}^N p_j + 1$ determines the number of available information about f given by (4) and (6). It is worth noting that if $f'(z_0)\in\Phi_P(z_0)$ and if $f''(z_0)\in\Phi_P(z_0)$ and if $0\leqslant\varsigma\leqslant 1$ then

$$\left[\varsigma f'(z_0) + (1-\varsigma)f''(z_0)\right] \in \Phi_P(z_0). \tag{11}$$

Inclusions regions $\Phi_P(z_0)$ of admissible values $f(z_0)$ of a Stieltjes function f are convex. Now our aim is:

PROBLEM 2. *By starting from $\sum_{j=1}^N p_j$ coefficients of Stieltjes series (4) and the equality (6) we evaluate an inclusion region $\Phi_P(z_0)$ estimating $f(z_0)$ at $z_0 \neq z_j$, $j = 1,2,\ldots,N$.*

2.2. Unified continued fraction expansion of a regular Stieltjes functions

We solve Problem 2 by means of a U-linear fractional transformation developed in [14]. The U- transformation relates the Stieltjes function f_1 with the Stieltjes ones f_2 and f_3, ..., and f_{N+1} via the relations

$$\begin{gathered} zf_1(z) = z_1f_1(z_1) + \frac{f_1(z_1)(z-z_1)}{1+z\theta_2f_2(z)},\ zf_2(z) = z_2f_2(z_2) + \frac{f_2(z_2)(z-z_2)}{1+z\theta_3f_3(z)},\ldots, \\ zf_j(z) = z_jf_j(z_j) + \frac{f_j(z_j)(z-z_j)}{1+z\theta_{j+1}f_{j+1}(z)},\ \ldots,\ zf_N(z) = z_jf_j(z_j) + \frac{f_N(z_N)(z-z_N)}{1+z\theta_{N+1}f_{N+1}(z)}, \end{gathered} \tag{12}$$

where θ_{j+1} are chosen in such a way that $f_j(-1)=1,\ j=1,2,...,N+1$. Here for the sake of simplicity we assume that

$$-1<z_1<z_2<...<z_{N-1}<z_N. \tag{13}$$

Relations (12) lead to the U-continued fraction representation of $zf_1(z)$

$$zf_1(z)=z_1f_1(z_1)+\cfrac{f_1(z_1)(z-z_1)}{1+\theta_2\left(z_2f_2(z_2)+\cfrac{f_2(z_2)(z-z_2)}{1+\theta_3\left(\cdots+\cfrac{\cdots}{1+\theta_{N+1}zf_{N+1}(z)}\right)}\right)}. \tag{14}$$

It is convenient to rewrite (14) as follows

$$\begin{gathered} zf_1(z)=\left(z_1f_1(z_1)+\tfrac{f_1(z_1)(z-z_1)}{1+\theta_2}\right)_\times\left(z_1f_1(z_1)+\tfrac{f_1(z_1)(z-z_1)}{1+\theta_2}\right)\\ \cdots_\times\left(f_N(z_N)+\tfrac{f_N(z_N)(z-z_N)}{1+\theta_N}\right)_\times zf_{N+1}(z). \end{gathered} \tag{15}$$

The following abbreviations of (15)

$$zf_1(z)=\bigvee_{k=1}^{N}\left(z_kf_k(z_k)+\frac{f_k(z_k)(z-z_k)}{1+\theta_{k+1}}\right)_\times zf_{N+1}(z), \tag{16}$$

or

$$zf_1(z)=U_N(z)_\times zf_{N+1}(z),\quad U_N(z)=\bigvee_{k=1}^{N}\left(z_kf_k(z_k)+\frac{f_k(z_k)(z-z_k)}{1+\theta_{k+1}}\right) \tag{17}$$

will be used in the sequel. Here N denotes the number of floors (dashes) of the U-continued fraction, see(14).

3. Allowed range of values of a Stieltjes function

In this subsection we evaluate the inclusion region $\Phi_P(z)$ estimating $f_1(z)$ at any point z of the cut $(-\infty,-1)$ complex plane. To this end we use the U-continued fraction rewritten in a general form (cf. (16))

$$\begin{gathered} zf_1(z)=\left(\bigvee_{k=1}^{N}\bigvee_{j=P_{k-1}+1}^{P_k}\left(z_kf_j(z_k)+\frac{f_j(z_k)(z-z_k)}{1+\theta_{j+1}}\right)\right)_\times zf_P(z),\\ P_0=0,\ P_j=\textstyle\sum_{i=1}^{j}p_j,\quad P=P_N+1,\quad j=1,2,...,N. \end{gathered} \tag{18}$$

The terms $f_k(z_k)$ and θ_{k+1}, $k=1,2,...,N$ are determined uniquely by the coefficients of the power series (4) and the condition (6). It is proved in [14]: if $f_1(z)$ is a regular Stieltjes function then $f_P(z)$ is a regular Stieltjes one as well. Hence the Problem 2 reduces to (cf. (18)):

Table 1: *Coefficients of continued fraction expansion of a Stieltjes function $f_1(z)$ from power series (23)*

k	1	2	3	4	5	6	7	8
$f_k(z_k)$	0.5	0.5	0.5	0.5	0.4142	0.2899	0.2612	0.2403
θ_{k+1}	0.5	0.5	0.5	0.5	0.4142	0.2899	0.2612	0.2403

PROBLEM 3. *By starting from the equality $f_P(-1) = 1$ only, we evaluate an inclusion region $\Phi_1(z)$ estimating $f_P(z)$ at any point z of the cut $(-\infty, -1)$ complex plane (cf. (9)).*

Since a set $\Phi_1(z)$ is convex, it suffices to compute the boundary $\phi_1(z)$ of $\Phi_1(z)$ given by (10). Such a computations were carried out by Baker [1, Chapter 17, Eq.17.16, R=1]. He obtained

$$\phi_1(z) = \{w \in \mathbb{C} : w = F_1(z,u);\ -1 \leqslant u \leqslant 1\},$$
$$F_1(z,u) = \begin{cases} u+1, & -1 \leqslant u \leqslant 0, \\ \dfrac{1-u}{1+zu}, & 0 \leqslant u \leqslant 1. \end{cases} \tag{19}$$

By replacing in (18) $f_1(z)$ by $F_P(z,u)$ and $f_P(z)$ by $F_1(z,u)$ we come to the bounding function $F_P(z,u)$ (cf. $(19)_2$)

$$F_P(z,u) = \frac{1}{z} \bigvee_{k=1}^{N} \left(\bigvee_{j=P_{k-1}+1}^{P_k} \left(z_k f_j(z_k) + \frac{f_j(z_k)(z - z_k)}{1 + \theta_{j+1}} \right) \right)_{\times} zF_1(z,u), \tag{20}$$

determining the error bounds $\phi_P(z)$ for $f_1(z)$

$$\phi_P(z) = \{w \in \mathbb{C} : w = F_P(z,u);\ -1 \leqslant u \leqslant 1\}. \tag{21}$$

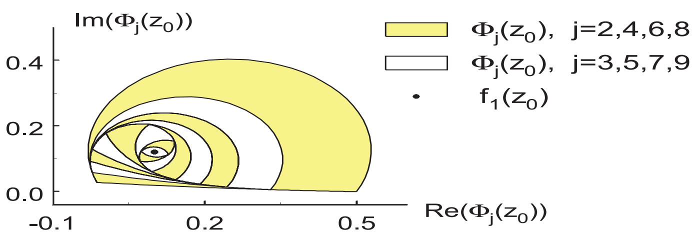

Figure 1: *Sequence of inclusion regions $\Phi_2(z_0)$, $\Phi_3(z_0), \ldots, \Phi_9(z_0)$ of a Stieljes function $f_1(z_0)$ calculated from Stieltjes series*

From the definition (9), it follows at once the inclusion relations

$$f_1(z) \in \Phi_P(z) \subset \Phi_{P-1}(z) \subset, \ldots, \subset \Phi_1(z). \tag{22}$$

leading to:

CONCLUSION 4. The inclusion region $\Phi_P(z)$ forms the best estimate of $f_1(z)$ obtainable using only the given P power series coefficients, and that the use of additional power ones (higher P) improves the estimates $\Phi_P(z)$ of $f_1(z)$.

In order to illustrate the Conclusion 4 we take as an input data the Stieltjes series $f_1(z)$ expanded at $z_1=0,\ z_2=1,\ z_3=5,\ z_4=7$ and $z_5=9$

$$f_1(z)=\tfrac{1}{2}-\tfrac{1}{8}z+\tfrac{1}{16}z^2-\tfrac{5}{128}z^3+O(z^4);\quad f_1(z)=0.4142+O(z-1),$$
$$f_1(z)=0.2899+O(z-5);\quad f_1(z)=0.2612+O(z-7);\quad f_1(z)=0.2403+O(z-9). \tag{23}$$

By substituting (23) to (20) and (20) to (21) we obtain the bounding functions $F_P(z_0,u)$ and consequently the error bounds $\phi_P(z_0)$, $z_0=-15-i30,\ \ P=2,3,...,9$. For example $F_9(z_0,u)$ is equal to (see Table 1).

$$F_9(z_0,u)=\tfrac{1}{z}\bigvee_{k=1}^{5}\left(\bigvee_{j=P_{k-1}+1}^{P_k}\left(z_k f_j(z_k)+\frac{f_j(z_k)(z_0-z_k)}{1+\theta_{j+1}}\right)\right)_{\times} zF_1(z_0,u), \tag{24}$$
$$P_0=0,\ P_1=4,\ P_2=5,\ P_3=6,\ P_4=7,\ P_5=8.$$

The results of the numerical calculations of $\Phi_P(z_0)$, $z_0=-15-i30$, $P=2,3,...,9$ are depicted in Fig. 1.

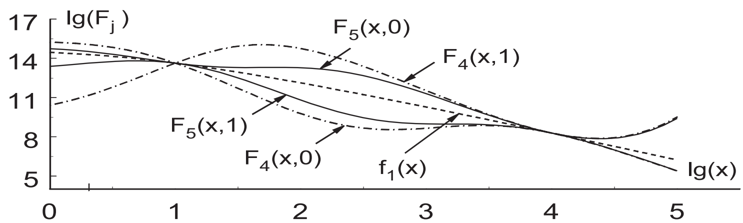

Figure 2: *Sequences of Padé approximants* $F_4(x,0)$, $F_4(x,1)$, $F_5(x,1)$, *and* $F_5(x,0)$, *forming upper and lower bounds on a Stieltjes function* $f_1(x)=-\ln(0.5(x+2))/x/\ln(0.5)$, *see, inequalities* (51) *and* (52).

3.1. Real domain

The real domain is a particular case of the complex one. The elementary bounding function $F_1(z,u)$ $(19)_2$ reduces to

$$F_1(x,u)=1-u,\quad 0\leqslant u\leqslant 1, \tag{25}$$

while $F_P(x,u)$ and $\phi_P(x)$ takes the forms (cf. (20)

$$F_P(x,u)=\frac{1}{x}\bigvee_{k=1}^{N}\left(\bigvee_{j=P_{k-1}+1}^{P_k}\left(z_k f_j(z_k)+\frac{f_j(z_k)(x-z_k)}{1+\theta_{j+1}}\right)\right)_{\times} x(1-u), \tag{26}$$
$$0\leqslant u\leqslant 1,\qquad P_0=0,\ P_j=\textstyle\sum_{i=1}^{j}p_j,\quad P=P_N+1,\ j=1,2,...,N.$$

Table 2: *First seven coefficients of a power expansion of a Stieltjes function* $(q(x)-1)/x$ *representing an effective dielectric constant* $q(x)$ *of a random array of spheres (cf. (57))*

j	1	2	3	4	5	6	7
c_j	0.3	−0.07	0.018780	−0.005903	0.002174	−0.000918	0.000430

and (cf. (26))

$$\phi_P(x) = \Phi_P(x) = \{w \in R : w = F_P(x,u);\ 0 \leqslant u \leqslant 1\}. \tag{27}$$

From (27), it follows that $\phi_P(x)$ is the interval with the ends determined by the Padé approximants $F_P(x,1)$ and $F_P(x,0)$. Thus we conclude:

CONCLUSION 5. The Stieltjes function $f_1(x)$ lies between the Padé approximants $F_P(x,1)$ and $F_P(x,0)$ to (4) and (4)–(6), respectively.

Now we establish the general inequalities for a Stieltjes function $f_1(x)$ and its Padé approximants $F_P(x,1)$ and $F_P(x,0)$. We start from the relations

$$\lim_{x\to\infty} F_P(x,0) = 0 \text{ for } P \text{ even and } \lim_{x\to\infty} F_P(x,0) > 0 \text{ for } P \text{ odd}. \tag{28}$$

Due to Conclusion 5 from (28) follows

$$(-1)^P F_P(x,0) < (-1)^P f_1(x) < (-1)^P F_P(x,1) \text{ if } z_N < x < \infty. \tag{29}$$

Since $f_1(z_N) = F_P(z_N,0) = F_P(z_N,1)$ then

$$(-1)^{P-1} F_P(x,0) < (-1)^{P-1} f_1(x) < (-1)^{P-1} F_P(x,1) \text{ if } z_{N-1} < x < z_N. \tag{30}$$

Since $f_1(z_{N-1}) = F_P(z_{N-1},0) = F_P(z_{N-1},1)$ we have

$$(-1)^{P-2} F_P(x,0) < (-1)^{P-2} f_1(x) < (-1)^{P-2} F_P(x,1) \text{ if } z_{N-2} < x < z_{N-1}. \tag{31}$$

Further restrictions $f(z_{N-1}) = F_P(z_{N-1},0) = F_P(z_{N-1},1)$ lead to

$$(-1)^{P-3} F_P(x,0) < (-1)^{P-3} f_1(x) < (-1)^{P-3} F_P(x,1) \text{ if } x_{N-3} < x < x_{N-2}. \tag{32}$$

By analyzing the relations (29)–(32) we deduce the general inequalities relating the Stieltjes function $f_1(x)$ to its Padé approximants $F_P(x,1)$ and $F_P(x,0)$

$$(-1)^{L_P(x)} F_P(x,0) \leqslant (-1)^{L_P(x)} f_1(x) \leqslant (-1)^{L_P(x)} F_P(x,1), \tag{33}$$

where

$$L_P(x) = \textstyle\sum_{j=1}^{N} p_j H(x-z_j) + 1, \quad H(x) = 0 \text{ if } x < 0 \text{ or } H(x) = 1 \text{ if } x \geqslant 0 \tag{34}$$

is a piecewise function depending on the input information x, z_j and p_j, $j = 1,2,...,N$, see (4) and (6).

Now we show, that the inequalities (33)–(34) are valid for the Padé approximants $F_P(x,1)$ and $F_P(x,0)$ to $f_1(x)$ expanded at $z_N=\infty$ as well. To this end we consider the formal equalities

$$f_1(x)=\frac{1}{x}\sum_{i=0}^{\infty}d_i^{\infty}\left(\frac{1}{x}\right)^i=\frac{1}{x}\sum_{i=0}^{\infty}d_i^{\infty}\left(\frac{1}{x}-\frac{1}{z_N}\right)^i=\sum_{i=0}^{\infty}c_{iN}\left(x-z_N\right)^i,\quad z_N\to\infty. \tag{35}$$

The relations (35) impose on the coefficients c_{iN} and d_j^{∞}, $j=1,2...,i$ the restrictions

$$\frac{\partial^n}{\partial x^n}\left[\sum_{i=0}^{\infty}c_{iN}\left(x-z_N\right)^i\right]_{x=z_N}=\frac{\partial^n}{\partial x^n}\left[\frac{1}{x}\sum_{i=0}^{\infty}d_i^{\infty}\left(\frac{1}{x}-\frac{1}{z_N}\right)^i\right]_{x=z_N},\quad n=0,1,\dots. \tag{36}$$

Hence the Padé approximants $F_P(x,1)$, $F_P(x,0)$ to (cf. $(35)_2$)

$$f_1(x)=\frac{1}{x}\sum_{i=0}^{\infty}d_i^{\infty}\left(\frac{1}{x}\right)^i \tag{37}$$

and the Padé ones $F_P^{z_N}(x,1), F_P^{z_N}(x,0)$ to (cf. $(37)_4$)

$$f_1^{z_N}(x)=\sum_{i=0}^{\infty}c_{iN}(d_0^{\infty},d_1^{\infty},...,d_i^{\infty})\left(x-z_N\right)^i \tag{38}$$

satisfy the relations

$$F_P(x,0)=\lim_{z_N\to\infty}F_P^{z_N}(x,0),\quad F_P(x,1)=\lim_{z_N\to\infty}F_P^{z_N}(x,1). \tag{39}$$

For example the general relations (35)–(39) transform the power series given by

$$f_1(x)=1+O(x+1),\quad f_1(x)=\frac{1}{x}\left(1-\frac{10}{3}\left(\frac{1}{x}\right)+O\left(\left(\frac{1}{x}\right)^2\right)\right) \tag{40}$$

to the power ones

$$f_1(x)=1+O(x+1),\quad f_1^{z_N}(x)=\frac{1}{z_N}+\frac{10-3z_N}{3z_N^3}(x-z_N)+O((x-z_N)^2). \tag{41}$$

From (41), it follows the distribution function (cf. (34))

$$L_2=H(x+1)+H(x-z_N),\ \ x<z_N. \tag{42}$$

The Padé approximants $F_2(x,0)$ to (40) and the Padé ones $F_2^{z_N}(x,0)$ to (41) equal to

$$F_2(x,0)=\frac{1}{x+2}\ \text{ and }\ F_2^{z_N}(x,0)=\frac{1}{1+(x+1)\frac{z_N-1}{z_N+1}} \tag{43}$$

confirm the relations (39). Moreover from (42) and (33)–(34) we obtain the inequality

$$f_1(x)<\frac{x}{x+2},\quad x\geqslant -1. \tag{44}$$

valid for any Stieltjes function $f_1(x)$ satisfying (40).

4. Particular cases of the general inequality

Consider now the sequences of Padé approximants $F_{P-j}(x,1)$ and $F_{P-j}(x,0)$, $j=1,2,...,P-1$. They satisfy the inequalities (cf. (33)–(34))

$$(-1)^{L_{P-j}(x)}F_{P-j}(x,0) \leqslant (-1)^{L_{P-j}(x)}f_1(x) \leqslant (-1)^{L_{P-j}(x)}F_{P-j}(x,1), \quad j=1,2,...,P-1. \tag{45}$$

Due to Conclusion 4 the formulae (45) transform to:

If $L_{P-j}(x)=L_P(x)-j$ then

$$\begin{gathered}(-1)^{L_P(x)}F_{P-2}(x,0) \leqslant (-1)^{L_P(x)}F_{P-1}(x,1) \leqslant (-1)^{L_P(x)}F_P(x,0) \leqslant (-1)^{L_P(x)}f_1(x)\\ (-1)^{L_P(x)}f_1(x) \leqslant (-1)^{L_P(x)}F_P(x,1) \leqslant (-1)^{L_P(x)}F_{P-1}(x,0) \leqslant (-1)^{L_P(x)}F_{P-2}(x,1).\end{gathered} \tag{46}$$

If $L_{P-j}(x)=L_P(x)$ then

$$\begin{gathered}(-1)^{L_P(x)}F_{P-2}(x,0) \leqslant (-1)^{L_P(x)}F_{P-1}(x,0) \leqslant (-1)^{L_P(x)}F_P(x,0) \leqslant (-1)^{L_P(x)}f_1(x)\\ (-1)^{L_P(x)}f_1(x) \leqslant (-1)^{L_P(x)}F_P(x,1) \leqslant (-1)^{L_P(x)}F_{P-1}(x,1) \leqslant (-1)^{L_P(x)}F_{P-2}(x,1).\end{gathered} \tag{47}$$

As an example illustrating the inequalities (46) and (47) we consider the power expansions

$$\begin{gathered}f_1(x)=0.2584962+0.01382716(x-10)+O\left((x-10)^2\right),\\ f_1(x)=0.001228800+0.1084559\cdot 10^{-6}(x-10^4)+O\left((x-10^4)^2\right),\end{gathered} \tag{48}$$

of the Stieltjes function

$$f_1(x)=-\frac{1}{x}\frac{\ln(0.5(x+2))}{\ln(0.5)}, \qquad f_1(-1)=1. \tag{49}$$

For (48) we have (cf. (34))

$$L_5(x)=H(x+1)+2H(x-10)+2H(x-10^4). \tag{50}$$

The function (50) substituted to (46) and (47) yield

$$F_4(x,0) \leqslant F_5(x,1) \leqslant f_1(x) \leqslant F_5(x,0) \leqslant F_4(x,1) \;\text{ if }\; x\in(10,10^4) \tag{51}$$

and

$$F_4(x,1) \leqslant F_5(x,1) \leqslant f_1(x) \leqslant F_5(x,0) \leqslant F_4(x,0) \;\text{ if }\; x\in(1,10). \tag{52}$$

Fig. 1 presents the Padé approximants $F_4(x,0)$, $F_4(x,1)$ $F_5(x,0)$ and $F_5(x,1)$ calculated from power series (48).

The following particular sequences of the multipoint Padé approximants $F_P(x,0)$ to a Stieltjes function $f_1(x)$ are investigated in [4, Theorem 6.1] and [5, Theorem 12.1]

$$F_{4m+1}(x,0),\; L_{4m+1}(x)=2m+1 \text{ and } F_{4m+3}(x,0),\;\; L_{4m+3}(x)=2m+2. \tag{53}$$

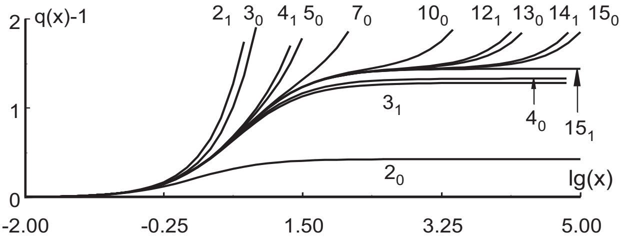

Figure 3: *Sequence of lower* $2_0, 3_1, 4_0, ..., 15_1$ *and upper* $2_1, 3_0, 4_1, .., 15_0$ *Padé bounds converging to the effective dielectric constant* $q(x) - 1$ *of a random array of spheres of volume fraction* $\varphi = 0.3$. *The notations for Padé approximants* $K_0 = xF_K(x, 0)$ *and* $K_1 = xF_K(x, 1)$, $K = 2, 3, ..., 15$ *are introduced.*

By substituting (53) to (33) we obtain

$$(-1)^{2m+2} F_{4m+3}(x, 0) < (-1)^{2m+2} f_1(x) < (-1)^{2m+1} F_{4m+1}(x, 0). \tag{54}$$

Due to Conclusion 4 from (54), it follows the particular inequalities

$$\begin{gathered} f_1(x) < \; ... \; F_{13}(x, 0) < F_9(x, 0) < F_5(x, 0) < F_1(x, 0) \\ F_3(x, 0) < F_7(x, 0) < F_{11}(x, 0) < F_{15}(x, 0) < ... < f_1(x). \end{gathered} \tag{55}$$

predicted by the Theorems [4, Theorem 6.1] and [5, Theorem 12.1], where

$$F_{4m+1}(x, 0) = \frac{\sigma_{2m}(x)}{\varphi_{2m}(x)}, \quad F_{4m+3}(x, 0) = \frac{\sigma_{2m+1}(x)}{\varphi_{2m+1}(x)}, \; \alpha < x < \beta. \tag{56}$$

5. Example of practical applications

In order to apply the general inequalities (33)–(34) to practical calculations we consider a power expansion of a Stieltjes function

$$f_1(x) = \varphi - \frac{1}{3}\varphi(1-\varphi)x + \sum_{j=2}^{P-1} c_j x^j + O(x^P), \quad f_1(-1) \leqslant 1 \tag{57}$$

representing the effective dielectric constant $q(x)$ of random array of spheres via the relation

$$f_1(x) = \frac{q(x) - 1}{x}, \quad x = h - 1, \quad h = \frac{\mu_2}{\mu_1}. \tag{58}$$

Here μ_1 and μ_2 are dielectric constants of a matrix and spheres, while φ denotes an inclusion volume fraction. Coefficients c_j, $j = 2, 3, ...$ are calculated by the method reported in [7, 8, 11]. The first seven ones are gathered in Table 2. Recurrence relations (12) are used to calculate the Padé bounds $F_P(x, 1)$ and $F_P(x, 0)$, $P = 2, 3..., 15$ to power series (57). The products $xF_P(x, 1)$ and $xF_P(x, 0)$, $P = 2, 3..., 15$ representing the effective dielectric constant $q(x) - 1$ of a random array of spheres are depicted in Fig. 3, see (58). Note that Padé bounds $F_P(x, 1)$ and $F_P(x, 0)$, $x > 0$, $P = 2, 3..., 15$ satisfy the fundamental inequalities (46).

6. Summary and final remarks

It has been established the general inequalities (33)–(34) for multipoint Padé approximants to a regular Stieltjes function f expanded to Taylor-series at a finite number of real points. The inequalities derived allow us to specify Padé approximants $F_P(x,1)$ and $F_P(x,0)$ as upper and lower bounds on f in dependence on the input parameters x, p_j and x_j, $j = 1,2,...,N$ characterizing the Stieltjes series (4). In order to do it, it suffices to construct from (4) and (6) a distribution function (34) and substitute it to the relations (33). Numerical examples of applications of the general inequalities (33)–(34) and their particular cases (46) and (47) has been presented, see Fig. 2 and 3.

We incorporate into estimates of a Stieltjes function f a finite number of coefficients of power expansions of f available at arbitrary number of real points $x_1, x_2, ..., z_N$, while Baker [1, 2] deals with one power expansion of f at 0 and the set of N discrete values $f(x_1)$, $f(x_2), ...,$ $f(z_N)$ only. On account of that his bounds are particular cases of the Padé ones produced by the inequalities (33)–(34). The bounding properties of multipoint Padé approximants reported earlier in [3, 4, 5] are also particular cases of the inequalities (33)–(34).

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