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TWO–PARAMETER SUNOUCHI OPERATOR WITH RESPECT TO CHARACTER SYSTEM OF p–SERIES FIELD IN THE KACZMARZ REARRANGEMENT

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Abstract. Let G_p be p-series field. We prove the restricted two-parameter Sunouchi operator T_h^{χ} is bounded from $H_q^{\Box}$ to L_q for $0 < q \leqslant 1$. By means of interpolation and duality argument, this theorem can be extended to Hardy-Lorentz spaces. As a consequence, we prove the restricted Sunouchi operator is of weak type (L_1, L_1).

1. Introduction

The operator T (called Sunouchi operator) was introduced and first investigated by Sunouchi [1], [2] in Walsh-Fourier analysis. For example he showed a characterization for the L_p spaces for $p > 1$ by means of T. Since this characterization fails to hold for $p = 1$, it was of interest to investigate the boundedness of T on a Hardy space. In [3] Simon showed that T is a sublinear bounded map from the dyadic Hardy space H_1 into L_1. Moreover Weisz [4] proved the restricted double Sunouchi operator T_h^{χ} is bounded from $H_q^{\Box}$ to L_q for $0 < q \leqslant 1$ and double Sunouchi operator T^{χ} is bounded from H_q to L_q for $2/3 < q \leqslant 1$, respectively.

The Vilenkin analogue of the Sunouchi operator was given by Gát [5], [6]. He investigated the boundedness of T from (Vilenkin) H_1 into L_1 and proved that if a Vilenkin group has an unbounded structure and H_1 is defined by means of the usual maximal function, then T is not bounded. Furthermore, if we consider a modified H_1 space (introduced by Simon [7]), then a necessary and sufficient condition can be given for a Vilenkin group that $T : H_1 \to L_1$ be bounded. All Vilenkin groups with bounded structure and also certain groups without this boundedness property satisfy the condition given by Gát. Thus, in the so-called bounded case the (H_1, L_1) boundedness of T remains true also for Vilenkin system. In [8] Simon extended this result, by showing the (H_q, L_q)-boundedness of T for all $0 < q \leqslant 1$. Moreover, the equivalence

$$\|f\|_{H_q} \sim \|Tf\|_q \quad (\frac{1}{2} < q \leqslant 1)$$

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was also obtained for f with mean value zero.

For Walsh-Kaczmarz system, G. Gát [9] proved that the Sunouchi operator is of weak type (L_1,L_1), of type (p,p) $(1<p<\infty)$ and of type (H_1,L_1). In this paper we consider the restricted Sunouchi operator of the character system of p-series field in the Kaczmarz rearrangement.

2. Martingale Hardy space

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N}:=\mathbb{P}\cup\{0\}$ and $\mathbb{N}^2$ be its Descartes product $\mathbb{N}\times\mathbb{N}$. An element from $\mathbb{N}^2$ will be denoted by (n,m) or simply by n. Let $2\leqslant p\in\mathbb{N}$ and denote by Z_p the p-th cyclic group. That is Z_p can be represented by the set $\{0,1,\cdots,p-1\}$, where the group operator is the mod p addition and every subset is open. Harr measure on Z_p is given in the way that $\mu(\{j\})=\frac{1}{p}(j\in Z_p)$. Let G_p denote the complete direct product of Z_p's equipped with product topology and product measure μ, then G_p forms a compact Abelian group with Haar measure 1. The elements of G_p are sequences of the form $(x_0,x_1,\cdots,x_k,\cdots)$, where $x_k\in Z_p$ for every $k\in\mathbb{N}$ and the topology of the group G_p is completely determined by the sets

$$I_n(0):=\{(x_0,x_1,\cdots,x_k,\cdots)\in G_m : x_k=0\ (k=0,\cdots,n-1)\}$$

$(I_0(0):=G_p)$. Let $I_n(x):=I_n(0)+x$ $(n\in\mathbb{N})$ and $I_{n,n}(x,y):=I_n(x)\times I_n(y)$. The σ-algebra generated by the rectangles $\{I_{n,n}(x,y):x,y\in G_p\}$ will be denoted by $\Sigma_n(n\in\mathbb{N})$.

The expectation and the conditional expectation operators relative to Σ_n are denoted by E, E_n, respectively. Let $L_q(G_p^2)$ denote the usual Lebesgue space and the norm or quasinorm of this space is defined by $\|f\|_q:=(E|f|^q)^{1/q}$ $(0<q\leqslant\infty)$. For simplicity, we assume that for a function $f\in L_1$ we have $E_0f=0$ $(n\in\mathbb{N})$.

An integrable sequence $(f=f_n,n\in\mathbb{N}^2)$ is said to be a martingale if

1) it is adapted, i.e. f_n is Σ_n measurable for all $n\in\mathbb{N}^2$ and

2) E_nf_m for all $n\leqslant m$, where for $n=(n_1,n_2)$, $m=(m_1,m_2)\in\mathbb{N}^2$, $n\leqslant m$ means that $n_1\leqslant m_1$ and $n_2\leqslant m_2$. For simplicity, we always suppose that for a martingale f we have $f_n=0$ if n_1 or $n_2=0$.

The martingale $f=(f_n,n\in\mathbb{N}^2)$ is said to be L_q bounded if $f_n\in L_q$ and

$$\|f\|_q:=\sup_n\|f_n\|_q<\infty.$$

The diagonal maximal function of a martingale $f=(f_n,n\in\mathbb{N}^2)$ is defined by $f^{\square}:=\sup_n|f_{n,n}|$. It is easy to see that the maximal function can also be given by

$$f^{\square}(x,y)=\sup_{n\in\mathbb{N}}\frac{1}{\mu(I_{n,n}(x,y))}\left|\int_{I_{n,n}(x,y)}fd\mu\right|.$$

The corresponding quadratic variations of a martingale f is introduced with

$$S^{\square}(f):=\Big(\sum_{n\in\mathbb{N}}|f_{n,n}-f_{n-1,n-1}|^2\Big)^{1/2}$$

Obviously,

$$\|f\|_1 \leqslant \|f^\square\|_1.$$

It was proved by Burkholder, Davis and Gundy [13], [14], [15] in the one parameter case and by Brossard [16], [17] and Metraux [18] in the two parameter case that

$$\|S^\square(f)\|_p \sim \|f^\square\|_p$$

for each $0 < p < \infty$, where $\sim$ denotes the equivalence of the norms. The equivalences

$$\|f^*\|_p \sim \|f^\square\|_p \sim \|f\|_p$$

for $1 < p < \infty$ follow from Doob's inequality (see Neveu[19], Cairoli[20]).

For $0 < q, s \leqslant \infty$ the martingale Hardy-Lorentz Space $H^\square_{q,s}$ consist of all martingales for which

$$\|f\|_{H^\square_{q,s}} := \|S^\square(f)\|_{q,s} < \infty.$$

Note that in case $q = s$ the usual definition of Hardy spaces $H^\square_{q,q} = H^\square_q$ is obtained.

3. Character system in the Kaczmarz rearrangement

Let $\Gamma(p)$ denote the character group of G_p. We enumerate the elements of $\Gamma(p)$ as follows: for $k \in \mathbb{N}$ and $x \in G_p$ we denote r_k the k-th generalized Rademacher function

$$r_k(x) := \exp\left(\frac{2\pi \imath x_k}{p}\right) \quad (\imath := \sqrt{-1},\ x \in G_p,\ k \in \mathbb{N}).$$

It is known that

$$\sum_{l=0}^{p-1} r_n^l(x) = \begin{cases} p \text{ if } x_n = 0 \\ 0 \text{ if } x_n \neq 0. \end{cases} \tag{1}$$

Let $n \in \mathbb{N}$. Then n can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k p^k$, $0 \leqslant n_k < p$, $n_k \in \mathbb{N}$. The sequence $(n_0, n_1, \cdots)$ is called the expansion of n with respect to number system based p. We often use the following notations: $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $p^{|n|} \leqslant n < p^{|n|+1}$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j p^j$.

Now, we define the sequence of function $\psi := (\psi_n : n \in \mathbb{N})$ by

$$\psi_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} \quad (x \in G_p,\ n \in \mathbb{N}).$$

We remark that $\Gamma(p) = \{\psi_n : n \in \mathbb{N}\}$ is a complete orthonormal system relate to the normalized Haar measure on G_p.

The character group $\Gamma(p)$ can be given in the Kaczmarz enumeration as follows: $\Gamma(p) = \{\chi_n : n \in \mathbb{N}\}$, where

$$\chi_n(x) := r_{|n|}^{n_{|n|}} \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} \quad (x \in G_p,\ n \in \mathbb{P}), \quad \chi_0(x) = 1 \quad (x \in G_p).$$

The Kronecker product $\alpha_{n,m}$ of two character systems is said to be the two-dimensional character system. Thus

$$\alpha_{n,m}(x,y) := \alpha_n(x)\alpha_m(y),$$

where α is χ or ψ.

The notation

$$r_{n,m}(x,y) := r_n(x)r_m(y)$$

is also used. Let the transformation $\tau_A : G_p \to G_p$ be defined as follows:

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \cdots, x_0, x_A, x_{A+1}, \cdots).$$

The transformation τ_A is measure-preserving and $\tau_A(\tau_A(x)) = x$. By means of the definition τ_A we have

$$\chi_n(x) = r_{|n|}^{n_{|n|}} \psi_{n-n_{|n|}p^{|n|}}(\tau_{|n|}(x)) \quad (x \in G_p,\ n \in \mathbb{N}).$$

Recall that the Dirichlet kernel $D_n^{\alpha} := \sum_{k=0}^{n-1} \alpha_k$ has a closed form

$$D_{p^n}^{\chi}(x) = D_{p^n}^{\psi}(x) = \begin{cases} p^n & \text{if } x \in I_n \\ 0 & \text{if } x \notin I_n, \end{cases} \tag{2}$$

where $x \in G_p, \alpha$ is ψ or χ.

If $f \in L_1$ then the number $\hat{f}(n,m) := E(f\chi_{n,m})$ is said to be the (n,m)-th coefficient of f with respect to system χ. We can extend this definition to martingale as well.

Denote by $s_{n,m}f$ the (n,m)-th partial sum of the Fourier series of a martingale f with respect to character system χ, namely,

$$s_{n,m}f := \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \hat{f}(k,l)\chi_{k,l}$$

It is easy to see that

$$s_{p^n,p^m}f = f_{n,m}.$$

For $n,m \in \mathbb{N}$ and a martingale f the Cesàro mean of order (n,m) of the double Fourier series of f with respect to character system χ is given by

$$\sigma_{n,m}f := \frac{1}{nm}\sum_{k=1}^{n}\sum_{l=1}^{m} s_{k,l}f = \sum_{k=1}^{n}\sum_{l=1}^{m}(1-\frac{k}{n})(1-\frac{l}{m})\hat{f}(k,l)\chi_{k,l}(x,y).$$

The n-th partial sum in the first (resp. second) variable of the Fourier series of the martingale f with respect to character system χ is denoted by $s_n^1 f$ (resp. $s_n^2 f$) and the one dimensional Cesàro operators are denoted by σ_n^1 and σ_n^2. Thus $s_{n,m} = s_n^1 s_m^2$, $\sigma_{n,m} = \sigma_n^1\sigma_m^2$,

$$s_n^1\sigma_m^2 f(x,y) = \frac{1}{m}\sum_{l=1}^{m} s_{n,l}f = \sum_{k=1}^{n}\sum_{l=1}^{m}(1-\frac{l}{m})\hat{f}(k,l)\chi_{k,l}(x,y)$$

and

$$\sigma_n^1 s_m^2 f(x,y) = \frac{1}{n}\sum_{k=1}^{n} s_{k,m}f = \sum_{k=1}^{n}\sum_{l=1}^{m}(1-\frac{k}{n})\hat{f}(k,l)\chi_{k,l}(x,y).$$

Let $K_n^\alpha := \frac{1}{n}\sum_{k=1}^{n} D_k^\alpha$, where α is χ or ψ. The next equality will also be used in our investigations(see [6] or [11]): if $x \in G_p,\ n \in \mathbb{N}$, then

$$K_{p^n}^\psi(x) = \frac{p^n+1}{2p^n}D_{p^n}^\psi(x) + \frac{1}{p^n}\sum_{k=0}^{n}\sum_{i=1}^{p-1}\frac{p^k}{1-r_k((p-i)e_k)}D_{p^n}^\psi(x+ie_k), \tag{3}$$

where $(je_l)_k = 0\ \ (k \neq l)$ and $(je_l)_l = j$. Furthermore, for $x \in I_{K-r}(ve_j)$, $(j = 0,\cdots, K-r-1;\ v = 1,\cdots p-1)$ and $t \in I_{K-r}$,

$$K_{p^n}^\psi(x-t) = \frac{1}{p^n}\frac{p^j}{1-r_j(ve_j)}D_{p^n}(x-t+(p-v)e_j). \tag{4}$$

It is simple to show that in case $f \in L_1$

$$s_{n,m}f(x,y) = \int_{G_p}\int_{G_p} f(t,u)D_n^\chi(x-t)D_m^\chi(y-u)d\mu(t)d\mu(u),$$

$$s^1\sigma_m^2 f(x,y) = \int_{G_p}\int_{G_p} f(t,u)D_n^\chi(x-t)K_m^\chi(y-u)d\mu(t)d\mu(u),$$

$$\sigma_n^1 s_m^2 f(x,y) = \int_{G_p}\int_{G_p} f(t,u)K_n^\chi(x-t)D_m^\chi(y-u)d\mu(t)d\mu(u),$$

and

$$\sigma_{n,m}f(x,y) = \int_{G_p}\int_{G_p} f(t,u)K_n^\chi(x-t)K_m^\chi(y-u)d\mu(t)d\mu(u).$$

4. The boundedness of restricted double Sunouchi operator on $H_q^\Box$

The atomic decomposition is a useful characterization of Hardy spaces. To demonstrate this let us introduce first the concept of an atom.

DEFINITION 1. A bounded measurable function a is an $H_q^\Box$ - atom if there exists a square I such that $\int_I a = 0$, $\|a\|_\infty \leqslant \mu(I)^{-1/q}$ and $\{a \neq 0\} \subset I$.

Motivated by the definition in Móricz, Schipp, Wade [21], we introduce the quasi-local operators. Their definition is weakened and extended here. For each interval I let I^r be the interval for which $I \subset I^r$ and $\mu(I^r) = p^r\mu(I)$. If $I := I_1 \times I_2$ is a rectangle then set $I^r := I_1^r \times I_2^r$.

DEFINITION 2. An operator V which maps the set of martingales into the collection of measurable functions is called $H_q^\Box$-quasi-local if there exist $r \in \mathbb{N}$ and a constant $c_q > 0$ such that

$$\int_{G_p^2\setminus I^r} |Va|^q d\mu \leqslant c_q$$

for every $H_q^{\square}$-atom a where I is the support of the atom.

We say that the operator T is of type (q,q) if $\|Tf\|_q \leqslant c_q\|f\|_q$ for some constant c_q for all $f \in L_q$. T is said to be of weak type (L_1,L_1) if there exists a $c>0$ such that for all $\lambda>0$, $f \in L_1$, the inequalities $\mu(y \in G_p, |Tf(y)|>\lambda) \leqslant c\|f\|_1/\lambda$ holds.

THEOREM A. [26] *Suppose that the operator V is sublinear and $H_q^{\square}$-quasi-local for any $0<q\leqslant 1$. If V is bounded from L_{q_1} to L_{q_1} for any $q_1>1$, then*

$$\|Vf\|_q \leqslant c_q\|f\|_{H_q^{\square}} \qquad (f \in H_q^{\square}).$$

We define one and two-parameter Sunouchi operator of the character system of the p-series field in the Kaczmarz rearrangement as follows:

$$T^{\chi}f := \Big(\sum_{n=0}^{\infty} |s_{p^n}f - \sigma_{p^n}f|^2\Big)^{1/2},$$

$$T^{\chi}f := \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |s_{p^n,p^m}f - s^1_{p^n}\sigma^2_{p^m}f - \sigma^1_{p^n}s^2_{p^m}f + \sigma_{p^n,p^m}f|^2\Big)^{1/2}.$$

In [25] we have verified that in the one parameter case

$$\|T^{\chi}f\|_q \leqslant c\|f\|_q \quad (1<q<\infty, f \in L^q(G_p)). \tag{5}$$

To investigate the Sunouchi operator on Hardy space we need another operator

$$T^{\chi}_{v}f := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} v_{n,m}r_{n,m}(s_{p^n,p^m}f - s^1_{p^n}\sigma^2_{p^m}f - \sigma^1_{p^n}s^2_{p^m}f + \sigma_{p^n,p^m}f),$$

where $r_{n,m}$ are the generalized Rademacher functions and $v:(v_{n,m})$ is a sequence of ± 1. In this section we consider the restricted Sunouchi operator

$$T^{\chi}_{h}f := \Big(\sum_{n,m\in\mathbb{N},|n-m|\leqslant h} |s_{p^n,p^m}f - s^1_{p^n}\sigma^2_{p^m}f - \sigma^1_{p^n}s^2_{p^m}f + \sigma_{p^n,p^m}f|^2\Big)^{1/2}$$

and the similarly defined T^{χ}_{hv} operator where $h \geqslant 0$ is fixed.

LEMMA 1. [4] *Let $1<q<\infty$. Then*

$$\|T^{\chi}_{v}f\|_q \leqslant c_1\|T^{\chi}f\|_q \leqslant c_2\|f\|_q$$

and the same holds for the operator T^{χ}_{hv} and T^{χ}_{h}.

THEOREM 1. (Main) *Let $0<q\leqslant 1$. Then*

$$\|T^{\chi}_{hv}f\|_q \leqslant c_q\|f\|_{H_q^{\square}} \qquad (f \in H_q^{\square}).$$

Proof. By Theorem A and Lemma 1 we have only to prove that the operator T_{hv}^{χ} is $H_q^{\square}$ -quasi-local for every $0 < q \leqslant 1$. Let a be an arbitrary $H_q^{\square}$-atom with support $I \times J$ and $\mu(I) = \mu(J) = p^{-K}$, $K \in \mathbb{N}$. Without loss of generalization, we suppose $I = J = I_K$,

$$T_{hv}^{\chi}a(x,y) = \sum_{n,m \in \mathbb{N}, |n-m| \leqslant h} v_{n,m} r_{n,m}(x,y)$$

$$\times \int_{G_p}\int_{G_p} a(t,u)(D_{p^m}^{\chi}(y-u) - K_{p^m}^{\chi}(y-u))(D_{p^n}^{\chi}(x-t) - K_{p^n}^{\chi}(x-t))d\mu(t)d\mu(u).$$

By the definition of atom and by (2)

$$\int_{G_p}\int_{G_p} a(t,u)(D_{p^m}^{\chi}(y-u) - K_{p^m}^{\chi}(y-u))(D_{p^n}^{\chi}(x-t) - K_{p^n}^{\chi}(x-t))d\mu(t)d\mu(u) = 0$$

if $n < K$ and $m < K$. Therefore we can suppose that $n \geqslant K$ or $m \geqslant K$. Choose $r \in \mathbb{N}$ such that $r-1 < h \leqslant r$. If $n \geqslant K$ then $m \geqslant n-h \geqslant K-r$. Let $n \geqslant K-r$ and $m \geqslant K-r$.

To prove the quasi-locality of T_{hv}^{χ} we have to integrate T_{hv}^{χ} over $G_p^2 \backslash (I^r \times J^r)$.

Step 1: Integrating over $(G_p \backslash I^r) \times J^r$. It is shown in [22] that

$$p^n K_{p^n}^{\chi}(x) = 1 + \sum_{j=0}^{n-1}\sum_{l=1}^{p-1} r_j^l(x) p^j K_{p^j}^{\psi}(\tau_j(x)) + \sum_{j=0}^{n-1} p^j D_{p^j}^{\psi}(x) \sum_{l=1}^{p-1}\sum_{i=0}^{l-1} r_j^i(x).$$

If $j \geqslant K-r$, $x \notin I_{K-r}$ and $t \in I_K$ then $x-t \notin I_K$, i.e by (2) $D_{p^n}^{\chi}(x-t) = 0$.

Denote $I_{K-r}^{m,l} := I_{K-r}(x_0, \cdots, x_m \neq 0, 0, \cdots, 0, x_l \neq 0, \cdots, 0)$, then

$$G_p \backslash I_{K-r} = \bigcup_{l=0}^{K-r-1} \bigcup_{m=-1}^{l-1} \bigcup_{x_0=0}^{p-1} \cdots \bigcup_{x_{m-1}=0}^{p-1} \bigcup_{x_m=1}^{p-1} \bigcup_{x_l=1}^{p-1} I_{K-r}^{m,l}.$$

Let $i \geqslant K-r$ and $x \notin I_{K-r}$. Then by [23] $K_{p^i}^{\psi}(\tau_i(x-t)) \neq 0$ implies that

$$t \in I_i(0, \cdots, 0, x_{K-r}, \cdots x_{i-1}) \quad \text{and} \quad m = l,\ x_0 = x_{m-1} = 0. \tag{6}$$

Let $x \in I_{K-r}^l$, $t \in I_i(0, \cdots, 0, x_{K-r}, \cdots x_{i-1})$. Thus $z := \tau_i(x-t) \in I_i(0, \cdots, 0, x_{i-l}, \cdots 0)$. By (2) and (4) we have

$$K_{p^i}^{\psi}(z) = \frac{1}{p^i}\frac{p^{i-l}}{1 - r_{i-l}(x_{i-l}e_{i-l})} D_{p^i}(z + (p - x_{i-l})e_{i-l}). \tag{7}$$

So by Hölder inequality and Lemma 1 we have

$$\int_{J^r} |T_{hv}^{\chi}a(x,y)|^q d\mu(y) \leqslant \mu(J^r)^{1-q}[\sum_{n \geqslant K-r}\int_{J^r}\int_{G_p} |\sum_{m \geqslant K-r, |n-m| \leqslant h} v_{n,m} r_m(y)$$

$$\times \int_{G_p} a(t,u)(D_{p^m}^{\chi}(y-u) - K_{p^m}^{\chi}(y-u))d\mu(u)|$$

$$\times (D_{p^n}^{\chi}(x-t) - K_{p^n}^{\chi}(x-t))d\mu(t)]^q d\mu(y)$$

$$
\begin{aligned}
&= \mu(J^r)^{1-q}[\sum_{n\geqslant K-r}\int_{J^r}\int_{G_p}|\sum_{m\geqslant K-r,|n-m|\leqslant h} v_{n,m}r_m(y)\int_{G_p} a(t,u)(D^{\chi}_{p^m}(y-u) \\
&\quad -K^{\chi}_{p^m}(y-u))d\mu(u)| \\
&\quad \times(\frac{1}{p^n}\sum_{i=K-r+1}^{n-1} p^i \sum_{s=1}^{p-1} r_i^s(x-t)K^{\psi}_{p^i}(\tau_i(x-t)))d\mu(t)]^q d\mu(y) \\
&\leqslant c\mu(J^r)^{1-q/2}[\sum_{n\geqslant K-r}\int_{G_p}(\int_{G_p}|a(t,y)|^2 d\mu(y))^{1/2} \\
&\quad \times(\frac{1}{p^n}\sum_{i=K-r+1}^{n-1} p^i \sum_{s=1}^{p-1} r_i^s(x-t)K^{\psi}_{p^i}(\tau_i(x-t)))d\mu(t)]^q \\
&\leqslant c\mu(J^r)^{1-q/2}[\sum_{n\geqslant K-r}\frac{1}{p^n}\sum_{i=K-r+1}^{n-1} p^i\int_{(I_i(0,\cdots,0,x_{i-l},\cdots 0)(t)}(\int_{G_p}|a(t,y)|^2 d\mu(y))^{1/2} \\
&\quad \times(\frac{p^{i-l}}{1-r_{i-l}(x_{i-l}e_{i-l})}\sum_{s=1}^{p-1} r_i^s(x-t)d\mu(t)]^q \times 1_{I^l_{K-r}}(x) \\
&\leqslant c\mu(J^r)^{1-q/2}[\sum_{n\geqslant K-r}\frac{1}{p^n}\sum_{i=K-r+1}^{n-1} p^{2i-l}\int_{(I_i(0,\cdots,0,x_{i-l},\cdots 0)(t)}(\int_{G_p}|a(t,y)|^2 d\mu(y))^{1/2} \\
&\quad \times(\sum_{s=0}^{p-1} r_i^s(x-t)-1)d\mu(t)]^q \times 1_{I^l_{K-r}}(x) \\
&\leqslant c\mu(J^r)^{1-q/2}\{\sum_{n\geqslant K-r}\frac{1}{p^n}\sum_{i=K-r+1}^{n-1} p^{i-l}[E_{i+1}(\int_{G_p}|a(x,y)|^2 d\mu(y))^{1/2} \\
&\quad -E_i(\int_{G_p}|a(x,y)|^2 d\mu(y))^{1/2}]\}^q 1_{I^l_{K-r}}(x).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\int_{G_p\setminus I^r}\int_{J^r}|T^{\chi}_{hv}a(x,y)|^q d\mu(y)d\mu(x) \\
&\leqslant c\sum_{l=0}^{K-r-1}\mu(J^r)^{1-q/2}\times\{\sum_{n\geqslant K-r}\frac{1}{p^{nq}}\sum_{i=K-r+1}^{n-1} p^{(i-l)q} \\
&\quad \times\int_{I^r}[E_{i+1}(\int_{G_p}|a(t,y)|^2 d\mu(y))^{1/2}-E_i(\int_{G_p}|a(t,y)|^2 d\mu(y))^{1/2}]\}^q 1_{I^l_{K-r}}(x)d\mu(x) \\
&\leqslant c\sum_{l=0}^{K-r-1}\mu(J^r)^{1-q/2}\times\{\sum_{n\geqslant K-r}\frac{1}{p^n}\sum_{i=K-r+1}^{n-1} p^{(i-l)} \\
&\quad \times\int_{I^r}[E_{i+1}(\int_{G_p}|a(x,y)|^2 d\mu(y))^{1/2}-E_i(\int_{G_p}|a(x,y)|^2 d\mu(y))^{1/2}]\}^2 d\mu(x)\}^{q/2}\mu(I^r)^{1-q/2} \\
&\leqslant c\sum_{l=0}^{K-r-1} p^{-2l}\mu(J^r)^{2-q}\|\sum_{n=K-r}^{\infty}\sum_{i=K-r+1}^{n} p^{i-n}(E_{i+1}B-E_iB)\|_2^q,
\end{aligned}
$$

where $B = (\int_{G_p} |a(x,y)|^2 d\mu(y))^{1/2}$. $l < m$ implies

$$\begin{aligned} & E_0((E_{l+1}B - E_l B)(E_{m+1}\overline{B} - E_m\overline{B})) \\ = & E_0(E_{l+1}((E_{l+1}B - E_l B)(E_{m+1}\overline{B} - E_m\overline{B}))) \\ = & E_0((E_{l+1}B - E_l B)E_{l+1}((E_{m+1}\overline{B} - E_m\overline{B}))) = 0. \end{aligned}$$

From this and Bessel's inequality we have

$$\begin{aligned} & \|(\sum_{n=K-r}^{\infty} |p^{-n} \sum_{i=K-r+1}^{n-1} p^j (E_{j+1}B - E_j B)|^2)^{1/2}\|_2^2 \\ \leqslant & \int_{G_p} \sum_{n=K-r}^{\infty} \sum_{i=K-r+1}^{n-1} p^{2i-2n} |E_{j+1}B - E_j B(x)|^2 dx \\ \leqslant & \int_{G_p} \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} p^{2i-2n} |E_{j+1}a(x) - E_j a(x)|^2 dx \\ \leqslant & \int_{G_p} \sum_{i=0}^{\infty} |E_{i+1}B(x) - E_i B(x)|^2 dx \\ \leqslant & \|B\|_2^2 = \|a\|_2^2. \end{aligned}$$

Therefore we have

$$\begin{aligned} & \int_{G_p \backslash I^r} \int_{J^r} |T_{hv}^{\chi} a(x,y)|^q d\mu(y) d\mu(x) \\ \leqslant & c \sum_{l=0}^{K-r-1} p^{-2l} p^{(-K+r)(2-q)} \|a\|_2^q \\ \leqslant & c \sum_{l=0}^{K-r-1} p^{-2l} p^{(-K+r)(2-q)} \|a\|_\infty^q \mu(I \times J)^q \\ \leqslant & c \sum_{l=0}^{K-r-1} p^{-2l} p^{(-K+r)(2-q)} p^{2K(1-q)} \leqslant c. \end{aligned} \tag{8}$$

Step 2: Integrating over $(G_p \backslash I^r) \times (G_p \backslash J^r)$. Similarly to Step 1 we can show that for $x \in (G_p \backslash I^r)$ and $y \in (G_p \backslash J^r)$

$$|T_{hv}^{\chi} a(x,y)|^q \leqslant [\sum_{n,m \geqslant K-r} \int_I \int_J |a(t,u) K_{p^m}^{\chi}(y-u)) \times K_{p^n}^{\chi}(x-t))| d\mu(t) d\mu(u)]^q$$

$$\leqslant [\sum_{n,m \geqslant K-r} \int_I \int_J |a(t,u) \times (\frac{1}{p^m} \sum_{j=K-r+1}^{m-1} p^j \sum_{s=1}^{p-1} r_j^s(x-t) K_{p^j}^{\psi}(\tau_j(x-t)))$$

$$\times (\frac{1}{p^n} \sum_{i=K-r+1}^{n-1} p^i \sum_{s'=1}^{p-1} r_i^{s'}(x-t) K_{p^i}^{\psi}(\tau_i(x-t))) d\mu(t) d\mu(u)|]^q$$

$$\leqslant [\sum_{n\geqslant K-r}\int_J \sum_{m\geqslant K-r}\int_I |a(t,u)\times(\frac{1}{p^m}\sum_{j=K-r+1}^{m-1}p^j\sum_{s=1}^{p-1}r_j^s(x-t)K_{p^j}^{\psi}(\tau_j(x-t)))d\mu(t)$$
$$\times(\frac{1}{p^n}\sum_{i=K-r+1}^{n-1}p^i\sum_{s'=1}^{p-1}r_i^{s'}(x-t)K_{p^i}^{\psi}(\tau_i(x-t)))d\mu(u)|]^q.$$

By using twice the method of proving of Step 1, we have

$$|T_{hv}^{\chi}a(x,y)|$$
$$\leqslant \sum_{n=K-r}^{\infty}\frac{1}{p^n}\sum_{i=K-r+1}^{n-1}p^{i-l'}\{E_{i+1}[\sum_{m=K-r}^{\infty}\frac{1}{p^m}\sum_{j=K-r+1}^{m-1}p^{j-l}[E_{j+1}|a(x,y)|-E_j(|a(x,y)|)]$$
$$-E_i[\sum_{m=K-r}^{\infty}\frac{1}{p^m}\sum_{j=K-r+1}^{m-1}p^{j-l}[E_{j+1}|a(x,y)|-E_j(|a(x,y)|)]\}1_{I_{K-r}^l}(x)1_{I_{K-r}^{l'}}(y).$$

Therefore

$$\int_{(G_p\backslash I^r)\times(G_p\backslash J^r)}|T_{hv}^{\chi}a(x,y)|^q d\mu(t)d\mu(u)$$
$$\leqslant c\sum_{l'=0}^{K-r-1}p^{-2l'}p^{(-K+r)(1-q/2)}\|\sum_{m=K-r}^{\infty}\frac{1}{p^m}\sum_{j=K-r+1}^{m-1}p^{j-l}[E_{j+1}|a(x,y)|-E_j(|a(x,y)|)\|_2^q$$
$$\leqslant c\sum_{l'=0}^{K-r-1}p^{-2l'}p^{(-K+r)(1-q/2)}\sum_{l=0}^{K-r-1}p^{-2l}p^{(-K+r)(1-q/2)}\|a\|_2^q$$
$$\leqslant cp^{(-K+r)(2-q)}\|a\|_{\infty}^q\mu(I\times J)^{q/2}\leqslant c\sum_{l=0}^{K-r-1}p^{-2l}p^{(-K+r)(2-q)}p^{2K(1-q/2)}\leqslant c_{q,r}. \tag{9}$$

Step 3: Integrating over $I^r\times(G_p\backslash J^r)$. This case is analogous to Step 1. Taking into account (8) and (9) we conclude that

$$\int_{G_p^2\backslash(I^r\times J^r)}|T_{hv}^{\chi}a|^q d\mu\leqslant c_q,$$

which complete the proof. □

The results concerning the T_h^{χ} operator follow easily from this theorem.

COROLLARY 1. *Let* $0<q\leqslant 1$. *Then*

$$\|T_h^{\chi}\|_q\leqslant c_q\|f\|_{H_q^{\square}}\qquad(f\in H_q^{\square}).$$

COROLLARY 2. *Let* $0<q<\infty$ *and* $0<s\leqslant\infty$. *Then*

$$\|T_h^{\chi}f\|_{q,s}\leqslant c_{q,s}\|f\|_{H_{q,s}^{\square}}\qquad(f\in H_{q,s}^{\square}).$$

Especially, if $f\in L^1$, *then*

$$\sup_{\lambda>0}\mu(|T_h^{\chi}f|>\lambda)\leqslant c_1\|f\|_1.$$

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