Mathematical
Inequalities
& Applications
Volume 15, Number 1 (2012), 61–67

ON A JENSEN–HOSSZÚ EQUATION, II

ZYGFRYD KOMINEK AND JUSTYNA SIKORSKA

(*Communicated by Z. Páles*)

Abstract. We solve functional equation of the form

$$f(x+y-xy)+f(xy)=2f\left(\frac{x+y}{2}\right)$$

in the class of functions transforming the unit interval into the space of all reals. We also prove that this equation is stable in the Hyers-Ulam's sense.

1. Introduction

It is well-known that in the class of functions transforming the closed or open unit interval I as well as the set of all reals $\mathbb{R}$ into the set of all reals the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right)=f(x)+f(y)$$

and the Hosszú functional equation

$$f(x+y-xy)+f(xy)=f(x)+f(y)$$

are equivalent and the general solution has the form $f(x)=a(x)+c$, $x\in I$, where a is an additive function and c is an arbitrary constant [3]. Recall that $a:\mathbb{R}\to\mathbb{R}$ is an additive function if it satisfies the following Cauchy functional equation

$$a(x+y)=a(x)+a(y), \qquad x,y\in\mathbb{R}.$$

We will consider the functional equation in which the left-hand side has the same form as in the Hosszú equation and the right-hand side coincides with the left-hand side of the Jensen equation, i.e., the following functional equation

$$f(x+y-xy)+f(xy)=2f\left(\frac{x+y}{2}\right), \qquad x,y\in I. \tag{1}$$

We will prove that equation (1) is also equivalent to the Hosszú (and for the same reason to the Jensen) equation and, moreover, (1) is stable in the sense of Hyers and Ulam. In

Mathematics subject classification (2010): 39B82, 39B62, 26A61.
Keywords and phrases: Jensen functional equation, Hosszú functional equation, Hyers-Ulam stability, approximation.

Paper MIA-15-06

[2] it is proved that equation (1) in the class of functions acting from $\mathbb{R}$ into itself is stable (in the sense of Hyers and Ulam). Note also that L. Losonczi [5] (cf. also [7]) have proved the Hyers-Ulam stability of the Hosszú functional equation in the class of functions transforming the set of all reals into itself, while Jacek Tabor [6] have proved that this equation is not stable in the class of functions transforming the unit real interval into $\mathbb{R}$.

2. Results

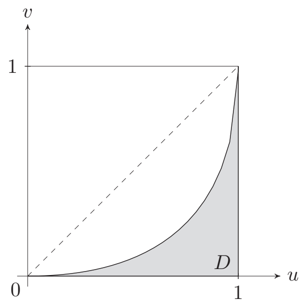
We start with the easier case when $I=[0,1]$. Assume that $\delta \geqslant 0$ is a fixed number and $f:[0,1]\to\mathbb{R}$ is a solution of the following inequality

$$\left| f(x+y-xy)+f(xy)-2f\left(\frac{x+y}{2}\right)\right| \leqslant \delta, \quad x,y\in[0,1]. \tag{2}$$

It is easy to see that the function $\tilde{f}(x):=f(x)-f(0)$, $x\in\mathbb{R}$ also satisfies (2). Thus without loss of generality we may assume that

$$f(0)=0. \tag{3}$$

Take an arbitrary $(u,v)\in D:=\{(u,v)\in[0,1]^2;\ (u+v)^2-4v\geqslant 0\}$.



Then the quadratic equation of the form $x^2-(u+v)x+v=0$ has solutions:

$$x=\frac{u+v-\sqrt{(u+v)^2-4v}}{2} \qquad \text{and} \qquad y=\frac{u+v+\sqrt{(u+v)^2-4v}}{2}.$$

It is not hard to check that $x,y\in[0,1]$. It follows from the equalities

$$x+y=u+v, \qquad \text{and} \qquad xy=v$$

that

$$u=x+y-xy \qquad \text{and} \qquad v=xy$$

and hence

$$\left| f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right| \leqslant \delta \tag{4}$$

for every $(u,v) \in D$. Putting $y=0$ in (4) we get

$$\left| f(x) - 2f\left(\frac{x}{2}\right) \right| \leqslant \delta, \quad x \in [0,1]. \tag{5}$$

For arbitrary non-negative integer n let us put

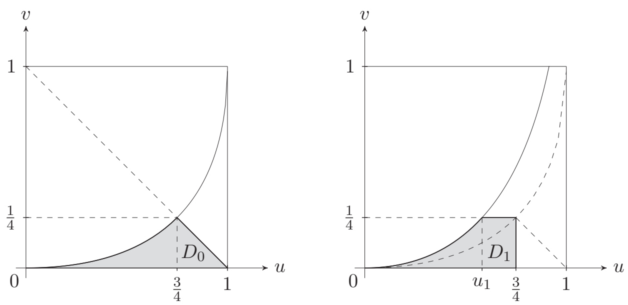
$$D_0 := \{(u,v) \in D; u+v \leqslant 1\}, \quad P := \left[0, \tfrac{3}{4}\right] \times \left[0, \tfrac{1}{4}\right],$$

$$D_{n+1} := \left\{(u,v) \in P; \left(u, \frac{v}{2}\right) \in D_n \right\}.$$

Observe that

$$\begin{cases} D_0 \subset D, \\ \text{if } (u,v) \in D_{n+1}, \text{ then } \left(u, \dfrac{v}{2}\right) \in D_n, \\ \text{if } (u,v) \in D_{n+1}, \text{ then } \left(u + \dfrac{v}{2}, \dfrac{v}{2}\right) \in D_n \cup D_0. \end{cases} \tag{6}$$

Below, we show the drafts of sets D_0 and D_1.



Putting $u_n := \frac{1}{\sqrt{2}^n} - \frac{1}{4 \cdot 2^n}$ for $n \in \mathbb{N} \cup \{0\}$, one can easily check that the rectangle $\left[u_n, \frac{3}{4}\right] \times \left[0, \frac{1}{4}\right]$ is a subset of D_n. Because of the inequality $u_6 < \frac{1}{8}$ we infer that

$$\left[\frac{1}{8}, \frac{1}{4}\right]^2 \subset D_6. \tag{7}$$

Moreover, by the triangle inequality for any $u, v \in D_0 \cup P$ we have

$$\begin{aligned} \left| f(u) + f(v) - 2f\left(\frac{u+v}{2}\right) \right| \leqslant & \left| f(v) - 2f\left(\frac{v}{2}\right) \right| \\ & + \left| 2f\left(\frac{u + \frac{v}{2}}{2}\right) - f\left(u + \frac{v}{2}\right) \right| \\ & + \left| f\left(\frac{v}{2}\right) + f\left(u + \frac{v}{2}\right) - 2f\left(\frac{u+v}{2}\right) \right| \\ & + \left| f(u) + f\left(\frac{v}{2}\right) - 2f\left(\frac{u + \frac{v}{2}}{2}\right) \right|. \end{aligned} \tag{8}$$

Putting $a_0 = 1$, $a_{n+1} = 2(1+a_n)$, $n \in \mathbb{N}\cup\{0\}$, we will show that for any $n \in \mathbb{N}\cup\{0\}$,

$$\left|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right| \leqslant a_n\delta, \quad (u,v)\in D_n. \tag{9}$$

On account of the inclusion $D_0 \subset D$, it is clear for $n=0$. Assume (9) for an $n \in \mathbb{N}\cup\{0\}$. Then (9) is a consequence of (8), (5) and (6). In particular, by (6), (7) and (9) we have

$$\left|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right| \leqslant a_6\delta, \quad (u,v)\in \left[\frac{1}{8},\frac{1}{4}\right]^2.$$

By [4, Theorem 3] (see also [1, Theorem 3]) there exist an additive function $a:\mathbb{R}\to\mathbb{R}$ and a constant $\lambda\in\mathbb{R}$, $\lambda\geqslant 1$ (not depending on f), such that

$$|f(x)-a(x)| \leqslant \lambda\delta, \quad x\in\left[\frac{1}{8},\frac{1}{4}\right]. \tag{10}$$

According to the inequality

$$\left|f\left(\frac{x}{2}\right)-a\left(\frac{x}{2}\right)\right| \leqslant \left|f\left(\frac{x}{2}\right)-\frac{1}{2}f(x)\right|+\left|\frac{1}{2}(f(x)-a(x))\right|, \quad x\in\left[\frac{1}{8},\frac{1}{4}\right],$$

(5) and (10) we obtain

$$|f(x)-a(x)| \leqslant \frac{\delta}{2}+\frac{1}{2}\lambda\delta \leqslant \lambda\delta, \quad x\in\left[\frac{1}{16},\frac{1}{8}\right].$$

By induction one can prove that

$$|f(x)-a(x)| \leqslant \lambda\delta, \quad x\in\left[0,\frac{1}{4}\right].$$

This together with (5), for arbitrary $x\in[0,1]$, yields

$$\begin{aligned}|f(x)-a(x)| &\leqslant \left|f(x)-2f\left(\frac{x}{2}\right)\right|+\left|2f\left(\frac{x}{2}\right)-4f\left(\frac{x}{4}\right)\right|+\left|4f\left(\frac{x}{4}\right)-4a\left(\frac{x}{4}\right)\right|\\ &\leqslant \delta+2\delta+4\lambda\delta=(3+4\lambda)\delta.\end{aligned}$$

Thus we have proved the following theorem.

THEOREM 1. *If $f:[0,1]\to\mathbb{R}$ satisfies inequality* (2)*, then there exist an additive function $a:\mathbb{R}\to\mathbb{R}$ and a constant $\mu\in\mathbb{R}$ such that*

$$|f(x)-f(0)-a(x)| \leqslant \mu\delta, \quad x\in[0,1].$$

Putting $\delta=0$, as a simple consequence of Theorem 1 we obtain the following theorem.

THEOREM 2. *A function* $f:[0,1]\to\mathbb{R}$ *is a solution of functional equation* (1) *if and only if there exist an additive function* $a:\mathbb{R}\to\mathbb{R}$ *and a real constant* c *such that* $f(x)=a(x)+c,\ x\in[0,1]$. *In other words, the Jensen-Hosszú functional equation and the Jensen functional equation are equivalent in the class of functions transforming interval* $[0,1]$ *into* $\mathbb{R}$.

If a function $f:(0,1)\to\mathbb{R}$ satisfies inequality (2) for all $x,y\in(0,1)$ then we can prove an analogue results to Theorems 1 and 2. But in this case we have some difficulties to obtain condition (5). Thus we use a slightly different method to prove this theorem.

THEOREM 3. *Let* $f:(0,1)\to\mathbb{R}$ *be a function satisfying* (2) *for all* $x,y\in(0,1)$. *Then there exist an additive function* $a:\mathbb{R}\to\mathbb{R}$ *and real constants* c *and* $\bar{\mu}$ *such that*

$$|f(x)-a(x)-c|\leqslant\bar{\mu}\delta,\quad x\in(0,1).$$

Proof. Let us define

$$\bar{D}:=\{(u,v)\in(0,1)^2;(u+v)^2-4v\geqslant 0\}.$$

Similarly as in the proof of Theorem 1 one can prove that

$$\left|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right|\leqslant\delta,\quad (u,v)\in\bar{D}. \tag{11}$$

It is easy to check that the function F of the form

$$F(x):=f(x)+c_1,\quad x\in(0,1),$$

where $c_1:=f(\frac{1}{2})-2f(\frac{1}{4})$, fulfills

$$\left|F(u)+F(v)-2F\left(\frac{u+v}{2}\right)\right|\leqslant\delta,\quad (u,v)\in\bar{D}. \tag{12}$$

Putting $\varepsilon:=7-4\sqrt{3}$ and taking arbitrary $y\in(0,\varepsilon)$, by virtue of (11) we have

$$\left|f\left(\frac{1-y}{2}\right)+f(y)-2f\left(\frac{1+y}{4}\right)\right|\leqslant\delta,$$

$$\left|f\left(\frac{1-y}{2}\right)+f\left(\frac{y}{2}\right)-2f\left(\frac{1}{4}\right)\right|\leqslant\delta$$

and

$$\left|f\left(\frac{y}{2}\right)+f\left(\frac{1}{2}\right)-2f\left(\frac{1+y}{4}\right)\right|\leqslant\delta.$$

Therefore,

$$\left|f(y)-2f\left(\frac{y}{2}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{1}{4}\right)\right|\leqslant 3\delta,$$

which is equivalent to

$$\left|F(y)-2F\left(\frac{y}{2}\right)\right| \leqslant 3\delta, \quad y\in(0,\varepsilon). \tag{13}$$

For arbitrary $x\in(0,2\varepsilon)$ we can choose a $y\in(0,\varepsilon)$ such that $(\frac{x}{2},\frac{y}{2})\in\bar{D}$, $\frac{x+y}{2}\in(0,\varepsilon)$. Then $(x,y)\in\bar{D}$ and by (12) and (13),

$$\begin{aligned}\left|2F\left(\frac{x}{2}\right)-F(x)\right| &\leqslant \left|-F(x)-F(y)+2F\left(\frac{x+y}{2}\right)\right| \\ &\quad+\left|4F\left(\frac{x+y}{4}\right)-2F\left(\frac{x+y}{2}\right)\right|+\left|F(y)-2F\left(\frac{y}{2}\right)\right| \\ &\quad+\left|2F\left(\frac{x}{2}\right)+2F\left(\frac{y}{2}\right)-4F\left(\frac{x+y}{4}\right)\right| \\ &\leqslant \delta+6\delta+3\delta+2\delta=12\delta.\end{aligned}$$

Repeating this approximation procedure, after finite number of steps (three steps yet) we get the existence of a constant $\lambda\in\mathbb{R}$ such that

$$\left|2F\left(\frac{x}{2}\right)-F(x)\right| \leqslant \lambda\delta, \quad x\in(0,1),$$

thus we have got an analogue of (5). The rest of the proof of Theorem 3 goes along the same lines as in the proof of Theorem 1 (after (5)). Therefore, there exist an additive function $a:\mathbb{R}\to\mathbb{R}$ and a constant $c_2\in\mathbb{R}$ such that

$$|F(x)-a(x)-c_2| \leqslant \bar{\mu}\delta, \quad x\in(0,1).$$

Setting $c=c_2+c_1$, by definition of F we get

$$|f(x)-a(x)-c| \leqslant \bar{\mu}\delta, \quad x\in(0,1).$$

This ends the proof of Theorem 3. □

Setting $\delta=0$ in Theorem 3 we obtain the following

THEOREM 4. *A function* $f:(0,1)\to\mathbb{R}$ *is a solution of functional equation* (1) *if and only if there exist an additive function* $a:\mathbb{R}\to\mathbb{R}$ *and a real constant* c *such that* $f(x)=a(x)+c$, $x\in(0,1)$. *In other words, the Jensen-Hosszú functional equation and the Jensen functional equation are equivalent in the class of functions transforming interval* $(0,1)$ *into* $\mathbb{R}$.

REMARK 1. The estimation constant obtained in Theorem 3 is much greater than the constant obtained in Theorem 1, so we provided first the case where f was defined on the closed unit interval.

REMARK 2. The results will remain true if we change the target space $\mathbb{R}$ for an arbitrary real Banach space. For the approximation (10), however, we can use only [1, Theorem 3].

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(Received April 21, 2009)

Zygfryd Kominek
Institute of Mathematics
Silesian University
Bankowa 14
PL-40-007 Katowice, Poland
e-mail: zkominek@math.us.edu.pl

Justyna Sikorska
Institute of Mathematics
Silesian University
Bankowa 14
PL-40-007 Katowice, Poland
e-mail: sikorska@math.us.edu.pl