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BOUNDS FOR LINEAR FUNCTIONALS ON MONOTONE FUNCTIONS IN L^p–SPACES

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Abstract. We consider the $L^p[a,b]$ space of functions which are integrable in the p-th power on a finite interval $[a,b]$, for $1 \leqslant p < \infty$. We establish optimal bounds on continuous linear functionals over this space, imposing the restrictions on elements of the space, which are assumed to be nondecreasing, integrable to zero, with the unit norm. We mention some applications of the bounds in the probability and statistics.

1. Introduction

Let $1 \leqslant p < \infty$. Consider the space of functions $L^p[a,b]$ which are integrable in the p-th power on a finite interval $[a,b]$. It is known (see Dunford and Schwartz, (1958, Theorem 1, p. 286)) that every continuous linear functional on this space has the following form

$$T_h(g) = \int_a^b g(x)h(x)dx, \quad g \in L^p[a,b], \tag{1.1}$$

where h is an arbitrarily chosen element of $L^q[a,b]$, for $q = p/(p-1)$ if $p > 1$ and $q = \infty$ if $p = 1$, with $L^\infty[a,b]$ defined below. This theorem allows us to identify the dual space of $L^p[a,b]$ with the space $L^q[a,b]$. Using the Hölder inequality (see, eg., Mitrinović, (1970)), the norm of the functional (1.1) equals to

$$||T_h||_p = \sup_{0 \neq g \in L^p[a,b]} \frac{T_h(g)}{||g||_p} = \sup_{||g||_p = 1} T_h(g) = ||h||_q = \left(\int_a^b |h(x)|^q dx \right)^{1/q}. \tag{1.2}$$

By $L^\infty[a,b]$ we denote the space of essentially bounded functions with the norm

$$||g||_\infty = \text{ess} \sup_{a \leqslant x \leqslant b} |g(x)|.$$

In contrast to the case $1 \leqslant p < \infty$, the dual space of $L^\infty[a,b]$ is much more complicated. It includes $L^1[a,b]$ in the sense that every element $g \in L^1[a,b]$ determines functional

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(1.1). However, formula (1.2) remains valid for $p=\infty$, even though $L^1[a,b]$ does not represent the whole dual space of $L^\infty[a,b]$.

We would like to improve the bound (1.2), imposing some restrictions on functions g. We assume that they belong to the family

$$\mathscr{C}_p = \{g \in L^p[a,b] : g \nearrow, T_1(g) = 0, ||g||_p = 1\} \tag{1.3}$$

of nondecreasing elements of $L^p[a,b]$ with the unit norm and integral equal to zero. We establish

$$|||T_h|||_p = \sup_{g\in\mathscr{C}_p} T_h(g) = \sup_{g\in\mathscr{C}_p} \int_a^b g(x)h(x)dx \tag{1.4}$$

for arbitrarily fixed $0 \neq h \in L^q[a,b]$.

There are probabilistic and statistical interpretations of (1.4) and evaluation of this integral is of special interest to statisticians. For instance, the evaluations of linear combinations of order statistics $\sum_{i=1}^{n} c_i X_{i:n}$ are interesting, with a given real coefficient vector $\tilde{c} = (c_1,\ldots,c_n)$, where $X_{i:n}$ stands for the i-th smallest value among $X_1,\ldots,X_n$, and $X_1,\ldots,X_n$ are iid random variables with a common, arbitrary, possibly continuous distribution function F. We have

$$\mathbb{E}\sum_{i=1}^{n} c_i \frac{X_{i:n}-\mu}{\sigma_p} = \int_0^1 \frac{F^{-1}(x)-\mu}{\sigma_p} f_{\tilde{c}:n}(x)dx, \tag{1.5}$$

where μ is the expectation of X_1, σ_p^p is the p-th central absolute moment of X_1, and

$$f_{\tilde{c}:n}(x) = \sum_{i=1}^{n} c_i n \binom{n-1}{i-1} x^{i-1}(1-x)^{n-i},$$

denotes the respective linear combination of densities of i-th order statistics from the standard uniform distribution. We can also use the results concerning (1.4) in order to determine the bounds on the expectation of record values. We define upper records R_n, based on an infinite sequence of iid continuous random variables, as a value which is greater than the previously determined record, where $R_0 = X_1$. Therefore

$$\mathbb{E}\frac{R_n-\mu}{\sigma_p} = \int_0^1 \frac{F^{-1}(x)-\mu}{\sigma_p} f_n(x)dx,$$

where

$$f_n(x) = \frac{[-\ln(1-x)]^n}{n!},$$

(see David and Nagaraja (2003)). Further examples of statistical functionals can be found in Rychlik (2001).

Notice that the lower bound on (1.1) over the set (1.3) can be easily obtained using the upper bound on another functional, because

$$\inf_{g\in\mathscr{C}_p} T_h(g) = -\sup_{g\in\mathscr{C}_p} T_{-h}(g).$$

Therefore we confine ourselves to upper bounds. Notice that

$$\inf_{g\in\mathscr{C}_p} T_h(g) \neq -\sup_{g\in\mathscr{C}_p} T_h(g),$$

which means that upper and lower bounds over the set (1.3) are not symmetric about zero, in contrast to the general ones (1.2). Moreover, the supremum and infimum may have the same sign, and in particular $\sup_{g\in\mathscr{C}_p} T_h(g) \leqslant 0$ for some h. The methods of establishing positive and negative upper bounds are totally different. Corresponding bounds for sequences were considered by Rychlik (1992), and Goroncy and Rychlik (2006), respectively.

2. Positive bounds

Positive bounds (1.2) in case $p = 2$ were presented in Rychlik (2007). Here we present the generalizations of his results for $1 \leqslant p \leqslant \infty$.

Notice that

$$T_h(g) = \int_a^b g(x)[h(x)-c]dx \leqslant ||h-c||_q,$$

for arbitrary real constant c, because $g \in \mathscr{C}_p$ integrates to zero. In case $p = q = 2$ it is easy to show, that

$$\inf_{c\in\mathbb{R}} ||h-c||_2 = ||h_0||_2,$$

where

$$h_0(x) = h(x) - \frac{1}{b-a}\int_a^b h(t)dt, \tag{2.1}$$

is the projection of h onto the linear subspace of functions orthogonal to constants. If h is nondecreasing, then the equality in the following inequality

$$T_h(g) \leqslant ||h_0||_2, \quad g \in \mathscr{C}_2, \tag{2.2}$$

holds if

$$g(x) = \frac{h_0(x)}{||h_0||_2} \in \mathscr{C}_2. \tag{2.3}$$

In the opposite case the inequality (2.2) can be improved by using the following inequality, which was proved in a more general version by Moriguti (1953).

THEOREM 1. *Let f be the integrable function on some interval $[a,b]$ and let $F(x) = \int_a^x f(t)dt$, $a \leqslant x \leqslant b$ be its antiderivative. Moreover, let $\underline{F}$ be the supremum of all the convex functions on $[a,b]$, which are not greater than F. Let $\underline{f}$ be the right-continuous derivative of $\underline{F}$. Then we have the following inequality*

$$\int_a^b g(x)f(x)dx \leqslant \int_a^b g(x)\underline{f}(x)dx$$

for every nondecreasing functions g, for which both integrals exist. The equality holds iff g is constant on every open interval, on which $F > \underline{F}$.

For $p=2$, we put $f=h_0$, and we obtain the bound and attainability conditions by replacing h_0 by $\underline{h}_0$ in (2.2) and (2.3) (cf. Rychlik (2007)). Below we consider the generalization of this result for $1<p<\infty$.

Denote the antiderivative of h by

$$H(x)=\int_a^x h(t)dt,$$

and the antiderivative of (2.1) by

$$H_0(x)=\int_a^x h_0(t)dt=\int_a^x h(t)dt-\frac{x-a}{b-a}\int_a^b h(t)dt=H(x)-\frac{x-a}{b-a}H(b), \qquad a\leqslant x\leqslant b. \tag{2.4}$$

Notice that $H_0(a)=H_0(b)=0$.

THEOREM 2. *Let $1<p<\infty$ and $h\in L^q[a,b]$. For every $g\in\mathscr{C}_p$ we have the following inequality*

$$\int_a^b g(x)h(x)dx\leqslant\left(\int_a^b|\underline{h}_0(x)-c_p|^q dx\right)^{1/q}=||\underline{h}_0-c_p||_q, \tag{2.5}$$

where $\underline{h}_0$ is the right-continuous derivative of the greatest convex minorant $\underline{H}_0$ of the antiderivative H_0 of function h_0, and c_p is some constant.

If the antiderivative (2.4) of function (2.1) is negative for some $a<x<b$, then $\underline{h}_0$ is not a constant function and there exists c_p and a unique $d\in(a,b)$ such that $\underline{h}_0(d-)\leqslant c_p\leqslant\underline{h}_0(d+)$ and

$$\int_a^d[c_p-\underline{h}_0(x)]^{q/p}dx=\int_d^b[\underline{h}_0(x)-c_p]^{q/p}dx.$$

Moreover, the right hand-side of (2.5) is positive and the equality is attained by the unique function

$$g_0(x)=\frac{|\underline{h}_0(x)-c_p|^{q/p}}{||\underline{h}_0-c_p||_q^{q/p}}\operatorname{sgn}\{\underline{h}_0(x)-c_p\}. \tag{2.6}$$

Proof. By using the Moriguti inequality (see Theorem 1), and the Hölder inequality (see Mitrinović, (1970)) we get

$$\begin{aligned}\int_a^b g(x)h(x)dx&=\int_a^b g(x)h_0(x)dx\\ &\leqslant\int_a^b g(x)\underline{h}_0(x)dx=\int_a^b g(x)(\underline{h}_0(x)-c)dx \qquad (2.7)\\ &\leqslant\left(\int_a^b|g(x)|^p dx\right)^{1/p}\left(\int_a^b|\underline{h}_0(x)-c|^q\right)^{1/q} \qquad (2.8)\\ &=||\underline{h}_0(x)-c||_q, \qquad (2.9)\end{aligned}$$

for an arbitrary constant c. Provided $\underline{h}_0$ is non-constant, such a value c exists and satisfies

$$\underline{h}_0(a) < \underline{h}_0(d-) \leqslant c \leqslant \underline{h}_0(d+) < \underline{h}_0(b),$$

for some $d \in (a,b)$. Note that

$$\int_{\underline{h}_0(x)<c} (c - \underline{h}_0(x))^{q/p} dx - \int_{\underline{h}_0(x)>c} (\underline{h}_0(x) - c)^{q/p} dx$$

is a continuous, strictly increasing function of argument c that takes both positive and negative values and c_p is the unique point where it crosses the axes.

If H_0 is negative somewhere in (a,b), then function $\underline{h}_0$ is not constant, and so the denominator of (2.6) is nonzero. Hence (2.6) is a well defined function and one can easily check that it belongs to (1.3). It is the only function with the unit norm, which ensures the equality in the Hölder inequality in (2.8). Functions (2.6) and $\underline{h}_0$ are constant on the same intervals. Hence (2.6) ensures the equality in the Moriguti inequality in (2.7) as well. □

REMARK 1. Inequality (2.5) holds for any constant c, not just c_p. The existence of (2.6) shows that c_p is the choice of c that minimizes the right-hand side of (2.5), so that this bound is optimal.

Notice that if $\underline{h}_0$ is a constant function, then in Theorem 2 we obtain the trivial zero bound.

Now we present two Theorems concerning the cases $p = 1$ and $p = \infty$.

THEOREM 3. *Let $h \in L^\infty[a,b]$. For every $g \in \mathscr{C}_1$ we have the following bound*

$$\int_a^b g(x)h(x)dx \leqslant \frac{1}{2}[\underline{h}_0(b) - \underline{h}_0(a)]. \tag{2.10}$$

Moreover, if for some $a < x < b$ we have $H_0(x) < 0$, then the bound (2.10) is strictly positive, attained when

$$\alpha = \sup\{x : \underline{h}_0(x) = \underline{h}_0(a)\} > a, \tag{2.11}$$

$$\beta = \inf\{x : \underline{h}_0(x) = \underline{h}_0(b)\} < b, \tag{2.12}$$

by function

$$g_{\alpha,\beta}(x) = \begin{cases} -\frac{1}{2(\alpha-a)}, & a < x < \alpha, \\ 0, & \alpha < x < \beta, \\ \frac{1}{2(b-\beta)}, & \beta < x < b. \end{cases} \tag{2.13}$$

If $\alpha = a$ or/and $\beta = b$, then the inequality in (2.10) is strict but equality is approached by the sequences of functions (2.13) with $\alpha \searrow a$ or/and $\beta \nearrow b$.

Proof. Again, for every $g \in \mathscr{C}_1$, by Theorem 1 and the Hölder inequality for $p=1$ we have

$$\int_a^b g(x)h(x)dx = \int_a^b g(x)h_0(x) \leqslant \int_a^b g(x)\underline{h}_0(x) = \int_a^b g(x)(\underline{h}_0(x)-c)$$
$$\leqslant \int_a^b |g(x)|dx \cdot \operatorname{ess}\sup_{a<x<b} |\underline{h}_0(x)-c| = ||\underline{h}_0 - c||_\infty, \tag{2.14}$$

where c is an arbitrary constant. Note that since $\underline{h}_0$ is nondecreasing, we have

$$||\underline{h}_0 - c||_\infty \geqslant (\underline{h}_0(b) - \underline{h}_0(a))/2. \tag{2.15}$$

The bound in (2.14) is optimal when the equality in (2.15) holds, that is for $c = c_1 = (\underline{h}_0(a) + \underline{h}_0(b))/2$.

In general, the equality in (2.14) holds when g is positive if

$$\underline{h}_0 - c_1 = \operatorname{ess}\sup_{a<x<b} |\underline{h}_0(x) - c_1|,$$

negative if

$$\underline{h}_0 - c_1 = -\operatorname{ess}\sup_{a<x<b} |\underline{h}_0(x) - c_1|,$$

and zero if

$$\underline{h}_0 - c_1 \in \left(-\operatorname{ess}\sup_{a<x<b} |\underline{h}_0(x) - c_1| \,,\ \operatorname{ess}\sup_{a<x<b} |\underline{h}_0(x) - c_1| \right).$$

The last case can not arise because g is assumed to have norm equal to 1.

The simplest case is when $\underline{h}_0$ is constant on the ends of interval $(0,1)$, that is if (2.11) and (2.12) hold. Then the requirement $g \in \mathscr{C}_1$ implies (2.13). Note that this also ensures the equality in the Moriguti inequality.

If $\underline{h}_0$ is not constant on the ends of interval $(0,1)$, then the equality in (2.14) can not be attained. It can only be approached by the sequences of functions (2.13) with $\alpha \searrow a$ or/and $\beta \nearrow b$. $\square$

THEOREM 4. *Let $h \in L^1[a,b]$ and $m = \frac{a+b}{2}$. For every $g \in \mathscr{C}_\infty$ we have the following bound*

$$\int_a^b g(x)h(x)dx \leqslant -2\underline{H}_0(m). \tag{2.16}$$

Moreover, if for some $a < x < b$ we have $H_0(x) < 0$, then the bound (2.16) is strictly positive, attained by an arbitrary element g of C_∞ defined as follows. If $H_0(m) = \underline{H}_0(m)$ then $g(x) = \mathbf{1}_{(m,b)}(x) - \mathbf{1}_{(a,m)}(x)$. Otherwise, let (a',b') be the component of the open set $\{x : H_0(x) > \underline{H}_0(x)\}$ that contains m and then $g(x) = \mathbf{1}_{(b',b)}(x) + \frac{a'-a+b'-b}{b'-a'}\mathbf{1}_{(a',b')}(x) - \mathbf{1}_{(a,a')}(x)$.

Proof. Analogously, for every $g \in \mathscr{C}_\infty$, we have

$$\int_a^b g(x)h(x)dx = \int_a^b g(x)h_0(x) \leqslant \int_a^b g(x)\underline{h}_0(x) = \int_a^b g(x)(\underline{h}_0(x) - c)$$
$$\leqslant \operatorname*{ess\,sup}_{a<x<b} |g(x)| \cdot \int_a^b |\underline{h}_0(x) - c|dx = ||\underline{h}_0 - c||_1, \tag{2.17}$$

where c is an arbitrary constant. Taking $m = \frac{a+b}{2}$ and $c = \underline{h}_0(m)$ gives,

$$||\underline{h}_0 - c||_1 = \int_a^m (c - \underline{h}_0(x))dx + \int_m^b (\underline{h}_0(x) - c)dx = -2\underline{H}_0(m),$$

and implies (2.16).

If $H_0(x) < 0$ for some x, then by convexity, $\underline{H}_0(m) < 0$ as well, so the right-hand side of (2.16) is positive. Since $\underline{H}_0$ is the greatest convex minorant of H_0, either $H_0(m) = \underline{H}_0(m)$ or else the graph of $\underline{H}_0$ near m is a line segment whose endpoints are on the graph of H_0. That is, there exists an interval (a', b'), containing m, such that $H_0(a') = \underline{H}_0(a')$, $H_0(b') = \underline{H}_0(b')$, and

$$\underline{H}_0(m) = \frac{b' - m}{b' - a'} H_0(a') + \frac{m - a'}{b' - a'} H_0(b').$$

In the former case, the function $g(x) = \mathbf{1}_{(m,b)}(x) - \mathbf{1}_{(a,m)}(x)$, which is in $\mathscr{C}_\infty$, produces equality in (2.16). In the latter case, taking $g(x) = \mathbf{1}_{(b',b)}(x) + \frac{a'-a+b'-b}{b'-a'}\mathbf{1}_{(a',b')}(x) - \mathbf{1}_{(a,a')}(x)$ (also in $\mathscr{C}_\infty$) gives,

$$\int_a^b g(x)h(x)dx = \int_a^b g(x)h_0(x) = -H_0(b') + \frac{a' - a + b' - b}{b' - a'}(H_0(b') - H_0(a')) - H_0(a')$$
$$= -2\left(\frac{b' - m}{b' - a'} H_0(a') + \frac{m - a'}{b' - a'} H_0(b')\right) = -2\underline{H}_0(m).$$

Thus, the equality in (2.16) is attained for this g. □

3. Nonpositive bounds

Here we present the results possibly improving zero bound (2.5), for the functionals h, for which the antiderivative (2.4) of function (2.1) is nonnegative, and in consequence its greatest convex minorant $\underline{H}_0$ and the derivative $\underline{h}_0$ of the minorant are zero.

Nonpositive bounds in case $p = 2$ were described by Rychlik (2007, Theorems 3, 4). Below we present the generalizations of his results for $1 \leqslant p \leqslant \infty$.

THEOREM 5. *Let* $1 \leqslant p < \infty$ *and* $h \in L^q[a,b]$ *be fixed. If* $H_0(x) \geqslant 0$ *for every* $a \leqslant x \leqslant b$ *and* $H_0(\theta) = 0$ *for some* $a < \theta < b$*, then for every function* $g \in \mathscr{C}_p$ *we have the following inequality*

$$\int_a^b g(x)h(x)dx \leqslant 0.$$

The equality holds for the following two-valued function

$$f_\theta(x) = \begin{cases} -\frac{b-\theta}{[(\theta-a)^p(b-\theta)+(\theta-a)(b-\theta)^p]^{1/p}}, & a < x < \theta, \\ \frac{\theta-a}{[(\theta-a)^p(b-\theta)+(\theta-a)(b-\theta)^p]^{1/p}}, & \theta < x < b. \end{cases} \tag{3.1}$$

Proof. By the assumption $\underline{h}_0(x) = 0,\ a \leqslant x \leqslant b$. Using the Moriguti inequality (see Theorem 1), we have

$$\int_a^b g(x)h(x)dx = \int_a^b g(x)h_0(x)dx \leqslant \int_a^b g(x)\underline{h}_0(x)dx = 0.$$

According to this theorem, the equality is attained for function g which is constant on intervals (a,θ) and (θ,b), with a possible positive jump in θ. The only nondecreasing function with the unit norm which is integrable to zero and satisfies these conditions is given by (3.1). □

Now suppose that the continuous function (2.4) is strictly positive on an open interval (a,b).

THEOREM 6. *Let* $1 \leqslant p \leqslant \infty$ *and* $h \in L^q[a,b]$ *be fixed. If* $H_0(x) > 0$ *for all* $a < x < b$*, then for every function* $g \in \mathscr{C}_p$ *we have the following inequality*

$$\int_a^b g(x)h(x)dx \leqslant \begin{cases} -\inf\limits_{a<x<b} \frac{H_0(x)(b-a)}{[(x-a)^p(b-x)+(x-a)(b-x)^p]^{1/p}}, & 1 \leqslant p < \infty, \\ 0, & p = \infty. \end{cases} \tag{3.2}$$

If $1 \leqslant p < \infty$*, then the equality in the above inequality is attained for the function of the form* (3.1)*, where* $\theta \in (a,b)$ *is the point at which the infimum of the right-hand side of* (3.2) *is attained. If the infimum is attained in limit as* $x \searrow a$ $(x \nearrow b)$*, then the equality in* (3.2) *is attained in limit for sequences of functions* (3.1) *with* $\theta \searrow a$ $(\theta \nearrow b)$*.*

If $p = \infty$*, then the equality is attained in limit by the function*

$$f_\theta(x) = \begin{cases} -1, & a < x < \theta, \\ \frac{\theta-a}{b-\theta}, & \theta < x < b, \end{cases} \quad a < \theta \leqslant \frac{a+b}{2},$$
$$f_\theta(x) = \begin{cases} -\frac{b-\theta}{\theta-a}, & a < x < \theta, \\ 1, & \theta < x < b, \end{cases} \quad \frac{a+b}{2} \leqslant \theta < b, \tag{3.3}$$

if $\theta \searrow a$ *or* $\theta \nearrow b$*.*

Proof. Again, using Theorem 1 we get

$$T_h(g) = \int_a^b g(x)h(x)dx \leqslant \int_a^b g(x)\underline{h}_0(x) = 0, \quad g \in \mathscr{C}_p. \tag{3.4}$$

Because $H_0(x) > \underline{H}_0(x) = 0$, $a < x < b$, then the equality in (3.4) is attained if $g(x)$ is constant on (a,b). However none of the elements of (1.3) is constant, because $\int_a^b g(x)dx = 0$ implies $g(x) = 0$, which is contradictory to the condition $||g||_p = 1$. Therefore we have a sharp inequality in (3.4).

Suppose $1 \leqslant p \leqslant \infty$ and fix an $h \in L^q[a,b]$ satisfying $H_0(x) > 0$ for each $x \in (a,b)$. For every $t \in (a,b)$, define g_t by

$$g_t(x) = \begin{cases} -\frac{b-t}{b-a}, & a < x < t, \\ \frac{t-a}{b-a}, & t < x < b. \end{cases}$$

Note that $g_t/||g_t||_p \in \mathscr{C}_p$, it is the f_t defined in (3.1). For any $g \in \mathscr{C}_p$, dg is a non-negative (Lebesgue-Stieltjes) measure, so

$$\begin{aligned} g(x) &= \frac{1}{b-a}\left(\int_a^x [g(x)-g(s)]ds - \int_x^b [g(s)-g(x)]ds\right) \\ &= \frac{1}{b-a}\left(\int_a^x \int_{(s,x]} dg(t)ds - \int_x^b \int_{(x,s]} dg(t)ds\right) \\ &= \int_{(a,x)} \frac{t-a}{b-a}dg(t) - \int_{(x,b)} \frac{b-t}{b-a}dg(t) = \int_{(a,b)} g_t(x)dg(t). \end{aligned}$$

If $f \in L^q[a,b]$, then fg is integrable, and an interchange of the order of integration shows

$$\int_a^b fg = \int_{(a,b)} \left(\int_a^b fg_t\right) dg(t) \leqslant \left(\sup_{a<t<b} \frac{1}{H_0(t)} \int_a^b fg_t\right) \int_{(a,b)} H_0(t)dg(t).$$

Taking supremum over all f in the unit ball of $L^q[a,b]$ gives,

$$1 \leqslant \left(\sup_{a<t<b} \frac{||g_t||_p}{H_0(t)}\right) \int_{(a,b)} H_0(t)dg(t),$$

or, equivalently,

$$-\int_{(a,b)} H_0(t)dg(t) \leqslant -\inf_{a<t<b} \frac{H_0(t)}{||g_t||_p}.$$

To see that this is (3.2), observe that $\int_a^b h_0 g_t = -\int_a^t h_0 = -H_0(t)$, so

$$\int_a^b hg = \int_a^b h_0 g = \int_{(a,b)} \left(\int_a^b h_0 g_t \right) dg(t) = -\int_{(a,b)} H_0(t) dg(t).$$

The continuity of $H_0(t)/||g_t||_p$ shows that the infimum in (3.2) is obtained either when $g = g_t/||g_t||_p$ for some $t \in (a,b)$ or as a limit of these as $t \to a$ or $t \to b$. When $p = \infty$, $||g||_p \leqslant 1$ so the infimum is zero. □

THEOREM 7. *Let $h \in L^1[a,b]$. If $H_0(x) \geqslant 0$, $a < x < b$, and there exists $a < \theta < b$ such that $H_0(\theta) = 0$, then*

$$\int_a^b g(x)h(x)dx \leqslant 0, \quad g \in \mathscr{C}_\infty.$$

The equality is attained by the function defined in (3.3).

The proof is analogous to the proofs of Theorems 5 and 6.

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