Mathematical
Inequalities
& Applications
Volume 15, Number 1 (2012), 127–135

ESTIMATES FOR BERNSTEIN TYPE OPERATORS

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(*Communicated by T. Erdélyi*)

Abstract. We prove the existence of a sequence of linear positive bounded polynomial operators on $C[0,1]$ which preserve the functions $e_0(x)=1$ and $e_2(x)=x^2$. An extremal property and quantitative estimates are given.

1. Introduction

Let n be a positive integer and Π_n be the space of all algebraic polynomials of degree not greater than n. Let us consider the following Bernstein type operators $L_n : C[0,1]\to\Pi_n$ defined by

$$(L_nf)(x)=\sum_{k=0}^{n}\lambda_{nk}(f)\,p_{nk}(x)\equiv\sum_{k=0}^{n}\lambda_{nk}(f)\binom{n}{k}x^k(1-x)^{n-k}, \tag{1.1}$$

where $\lambda_{nk}:C[0,1]\to\mathbb{R}$ are some linear functionals, $k=0,1,\dots,n$. For $\lambda_{nk}(f)=f\left(\frac{k}{n}\right)$, $k=0,1,\dots,n$, we recover the well-known Bernstein operators $B_n:C[0,1]\to\Pi_n$,

$$(B_nf)(x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

Operators of type (1.1) were introduced in [5, p. 115] with $\lambda_{nk}:C[0,1]\to\mathbb{R}$ positive linear functionals and $\lambda_{nk}(e_0)=1$, $k=0,1,\dots,n$. For $j=0,1,2,\dots$, we denote by e_j the power function $e_j(x)=x^j$, $x\in[0,1]$. The main results of [5] are a direct and an equivalence approximation theorem, formulated with the aid of the second order modulus of smoothness. For (1.1) with $\lambda_{nk}\in C[0,1]^*$ bounded positive linear functionals and $\lambda_{nk}(e_0)=1$ $(k=0,1,\dots,n)$, Felten established direct and equivalence approximation theorems via second order Ditzian-Totik modulus of smoothness (see [4, p. 403 and p. 417]). Recently, Bustamante and Quesada [2] proved a characterization theorem for Bernstein operators using (1.1), where $\lambda_{nk}(f)=\int_0^1 f\,d\nu_{nk}$, $f\in C[0,1]$ and ν_{nk} are positive measures $(k=0,1,\dots,n)$. Their result is the following. If $n>1$ and $x\in[0,1]$ is arbitrary, then

$$x^2\leqslant(B_ne_2)(x)\leqslant(L_ne_2)(x). \tag{1.2}$$

Mathematics subject classification (2010): 41A25, 41A36.

Keywords and phrases: Bernstein operators, polynomial bounded positive operators, modulus of continuity, ordered normed spaces.

Paper MIA-15-13

Moreover, $(B_n e_2)(x) = (L_n e_2)(x)$ for some $x \in (0,1)$ if and only if $B_n = L_n$. This result improves the similar one given in [1, p. 214].

On the other hand, King introduced in [7, p. 204] a new sequence of Bernstein type operators $\{V_n\}$, $V_n : C[0,1] \to C[0,1]$ such that $V_n e_0 = e_0$ and $V_n e_2 = e_2$. In [7] quantitative estimates and connections with summability are discussed. The operators V_n are different from L_n given by (1.1) and they are not polynomial operators, i.e. $V_n f \notin \Pi_n$ for all $f \in C[0,1]$ (see [6, p. 649]). Regarding the operators V_n, in [6, p. 649] is formulated the following open question: can we find another type of linear and positive *polynomial* operators L for which $Le_2 = e_2$?

The goal of the paper is to construct a sequence $\{L_n\}$ of bounded positive linear operators which approximate each continuous function on $[0,1]$ such that $L_n e_0 = e_0$, $L_n e_2 = e_2$ and L_n are polynomial operators for all $n > 1$. The operators L_n will be of type (1.1), where $\lambda_{nk} \in C[0,1]^*$ *are bounded positive linear functionals* $(k = 0,1,\dots,n)$. In this way we solve positively the open question formulated above. Furthermore, similar to (1.2), we establish an extremal property for $B_n e_1$ via $L_n e_1$. The rate of convergence of $\{L_n f\}$ will be estimated by the modulus of continuity, obtaining quantitative estimates.

2. The construction of L_n

Before starting our theorem we recall some definitions and an auxiliary result concerning ordered normed spaces.

A real linear space X is said to be *ordered linear space* if X is equipped with an order relation $\leqslant$ satisfying the conditions: $x,y,z \in X, x \leqslant y \Rightarrow x+z \leqslant y+z$; $x,y \in X, x \leqslant y, \alpha \geqslant 0\, (\alpha \in \mathbb{R}) \Rightarrow \alpha x \leqslant \alpha y$. For any given ordered linear space X we define X_+ to be the set of all positive elements of X, i.e. $X_+ = \{x \in X : 0_X \leqslant x\}$. An ordered linear space X is said to be *ordered normed space* if there exists a norm $\|\cdot\|_X$ on X such that $0_X \leqslant x \leqslant y \Rightarrow \|x\|_X \leqslant \|y\|_X$.

Finally, we have

LEMMA 2.1. [8, p. 82] *Let X be an ordered normed space with $\operatorname{int} X_+ \neq \emptyset$ and Y a normed subspace of X such that $Y \cap \operatorname{int} X_+ \neq \emptyset$. If $\lambda \in Y^*$ is a bounded positive linear functional then there exists a bounded positive linear functional $\tilde{\lambda} \in X^*$ such that $\tilde{\lambda}(x) = \lambda(x)$ for all $x \in Y$.*

Our main result is

THEOREM 2.2. *There exist bounded positive linear operators $L_n : C[0,1] \to \Pi_n$ of type (1.1) such that L_n preserves the functions e_0 and e_2: $L_n e_0 = e_0$ and $L_n e_2 = e_2$, where $n > 1$.*

Proof. The real linear space $C[0,1]$ is an ordered Banach space with the norm $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$ and the natural order relation: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, $x \in [0,1]$. Moreover, for $C[0,1]_+ = \{f \in C[0,1] : 0_{C[0,1]} \leqslant f\}$ we have $\{f \in C[0,1] : \|f - e_0\| < 1\} \subset C[0,1]_+$, and thus $\operatorname{int} C[0,1]_+ \neq \emptyset$. Furthermore, let $Y =$

$\{\alpha e_0+\beta e_1+\gamma e_2:\alpha,\beta,\gamma\in\mathbb{R}\}$. Then Y is a normed subspace of $C[0,1]$ and $e_0\in Y\cap\operatorname{int}C[0,1]_+$. Therefore, by Lemma 2.1, every bounded positive linear functional $\lambda\in Y^*$ has a bounded positive linear extension $\tilde{\lambda}\in C[0,1]^*$.

Now we construct the bounded positive linear functionals $\lambda_{nk}\in Y^*$, $k=0,1,\dots,n$, as follows. We set

$$\begin{aligned}&\lambda_{nk}(e_0)=1, \qquad \lambda_{nk}(e_1)\in\left[\tfrac{k(k-1)}{n(n-1)},\left(\tfrac{k(k-1)}{n(n-1)}\right)^{1/2}\right],\\ &\lambda_{nk}(e_2)=\tfrac{k(k-1)}{n(n-1)},\end{aligned} \tag{2.1}$$

$k=0,1,\dots,n$ and $n>1$. For $P=\alpha e_0+\beta e_1+\gamma e_2\in Y$ and $k=0,1,\dots,n$, we define $\lambda_{nk}(P)=\alpha\lambda_{nk}(e_0)+\beta\lambda_{nk}(e_1)+\gamma\lambda_{nk}(e_2)$.

Obviously λ_{nk} are linear. Moreover, λ_{nk} are positive: if $P(x)\geqslant 0$ for $x\in[0,1]$, then we distinguish the following two cases:

a) $\gamma\geqslant 0$. Then, by (2.1),

$$\lambda_{nk}(P)\geqslant\alpha+\beta\lambda_{nk}(e_1)+\gamma(\lambda_{nk}(e_1))^2=P(\lambda_{nk}(e_1))\geqslant 0;$$

b) $\gamma<0$. Then, in view of (2.1),

$$\begin{aligned}\lambda_{nk}(P)&\geqslant\alpha+\beta\lambda_{nk}(e_1)+\gamma\lambda_{nk}(e_1)\\&=\alpha\,(1-\lambda_{nk}(e_1))+(\alpha+\beta+\gamma)\lambda_{nk}(e_1)\\&=P(0)\,(1-\lambda_{nk}(e_1))+P(1)\lambda_{nk}(e_1)\\&\geqslant 0.\end{aligned}$$

Further, λ_{nk} are bounded on Y $(k=0,1,\dots,n)$. Indeed, the positivity of λ_{nk} and (2.1) imply for all $P\in Y$ that

$$|\lambda_{nk}(P)|\leqslant\lambda_{nk}(|P|)\leqslant\lambda_{nk}(\|P\|e_0)=\|P\|\lambda_{nk}(e_0)=\|P\|.$$

In conclusion we can define the positive linear operators L_n on $C[0,1]$ by

$$(L_nf)(x)=\sum_{k=0}^{n}\tilde{\lambda}_{nk}(f)p_{nk}(x), \qquad x\in[0,1],$$

where $\tilde{\lambda}_{nk}\in C[0,1]^*$ are bounded positive linear functionals such that $\tilde{\lambda}_{nk}(P)=\lambda_{nk}(P)$ for all $P\in Y$. Obviously $L_nf\in\Pi_n$ for all $f\in C[0,1]$ and L_n are bounded, because $\tilde{\lambda}_{nk}\in C[0,1]^*$ bounded positive linear functionals and $\tilde{\lambda}_{nk}(e_0)=\lambda_{nk}(e_0)=1$, $k=0,1,\dots,n$, imply $\|L_nf\|\leqslant\|f\|$, $f\in C[0,1]$.

Further, by simple computation, $\tilde{\lambda}_{nk}(e_0)=\lambda_{nk}(e_0)$ and $\tilde{\lambda}_{nk}(e_2)=\lambda_{nk}(e_2)$ imply

$$(L_ne_0)(x)=\sum_{k=0}^{n}p_{nk}(x)=1$$

and

$$(L_n e_2)(x) = \sum_{k=0}^{n} \frac{k(k-1)}{n(n-1)} p_{nk}(x) = \sum_{k=0}^{n} x^2 p_{n-2,k-2}(x) = x^2,$$

which completes the proof. □

In the next theorem we establish estimates for $L_n e_j,\ j = 1, 2, \ldots$

THEOREM 2.3. *Let $n > 1$. If the operator L_n verifies the conditions of Theorem 2.2, then*

$$\frac{(\lambda_{nk}(e_j))^2}{\lambda_{nk}(e_{j-1})} \leqslant \lambda_{nk}(e_{j+1}) \leqslant \lambda_{nk}(e_j) \leqslant \lambda_{nk}(e_1) \tag{2.2}$$

for all $k = 0, 1, \ldots, n$ and $j = 1, 2, \ldots$;

$$\frac{((L_n e_j)(x))^2}{(L_n e_{j-1})(x)} \leqslant (L_n e_{j+1})(x) \leqslant (L_n e_j)(x) \leqslant (L_n e_1)(x) \tag{2.3}$$

for all $x \in [0, 1]$ and $j = 1, 2, \ldots$

Proof. Because $\lambda_{nk} \in C[0, 1]^*$ are bounded positive linear functionals and $\lambda_{nk}(e_0) = 1,$ we have the representations

$$\lambda_{nk}(f) = \int_0^1 f(t)\, d\mu_{nk}(t), \qquad f \in C[0, 1], \tag{2.4}$$

where the functions μ_{nk} are increasing on $[0, 1]$ and $\int_0^1 d\mu_{nk}(t) = 1,\ k = 0, 1, \ldots, n.$ Then, by Hölder's inequality,

$$\begin{aligned} \lambda_{nk}(e_j) &= \int_0^1 t^j\, d\mu_{nk}(t) = \int_0^1 t^{(j+1)/2} \cdot t^{(j-1)/2}\, d\mu_{nk}(t) \\ &\leqslant \left(\int_0^1 t^{j+1}\, d\mu_{nk}(t) \right)^{1/2} \left(\int_0^1 t^{j-1}\, d\mu_{nk}(t) \right)^{1/2} \\ &= \left(\lambda_{nk}(e_{j+1}) \right)^{1/2} \left(\lambda_{nk}(e_{j-1}) \right)^{1/2} \end{aligned}$$

for $j = 1, 2, \ldots$ Hence

$$\frac{(\lambda_{nk}(e_j))^2}{\lambda_{nk}(e_{j-1})} \leqslant \lambda_{nk}(e_{j+1}) \tag{2.5}$$

for $j = 1, 2, \ldots$ Further, because $P(x) = x^j - x^{j+1} \geqslant 0,\ x \in [0, 1],$ we get, by positivity of λ_{nk} that $\lambda_{nk}(P) = \lambda_{nk}(e_j) - \lambda_{nk}(e_{j+1}) \geqslant 0,$ i.e.

$$\lambda_{nk}(e_{j+1}) \leqslant \lambda_{nk}(e_j) \tag{2.6}$$

for $j = 1, 2, \ldots$ Hence follows that

$$\lambda_{nk}(e_j) \leqslant \lambda_{nk}(e_1), \tag{2.7}$$

where $j = 1,2,\dots$ By combining (2.5), (2.6) and (2.7), we obtain (2.2).

In view of (2.5) and Cauchy-Schwarz inequality, we find

$$\begin{aligned}(L_n e_{j+1})(x) &= \sum_{k=0}^{n} \lambda_{nk}(e_{j+1})\, p_{nk}(x) \geqslant \sum_{k=0}^{n} \frac{(\lambda_{nk}(e_j))^2}{\lambda_{nk}(e_{j-1})}\, p_{nk}(x) \\ &= \sum_{k=0}^{n} \frac{(\lambda_{nk}(e_j) p_{nk}(x))^2}{\lambda_{nk}(e_{j-1}) p_{nk}(x)} \\ &\geqslant \frac{\left(\sum_{k=0}^{n} \lambda_{nk}(e_j) p_{nk}(x)\right)^2}{\sum_{k=0}^{n} \lambda_{nk}(e_{j-1}) p_{nk}(x)} = \frac{((L_n e_j)(x))^2}{(L_n e_{j-1})(x)},\end{aligned} \tag{2.8}$$

where $x \in [0,1]$ and $j = 1,2,\dots$ Further, by (2.6) and (2.7), we get

$$(L_n e_{j+1})(x) \leqslant (L_n e_j)(x) \leqslant (L_n e_1)(x) \tag{2.9}$$

for $x \in [0,1]$ and $j = 1,2,\dots$ Then, (2.8) and (2.9) imply (2.3), which was to be proved. $\square$

3. Main results

For $n > 1$ let $\mathscr{L}(n)$ be the class of all polynomial bounded positive linear operators $L : C[0,1] \to \Pi_n$ defined by (1.1) such that L preserves the functions e_0 and e_2. We have the following extremal property.

THEOREM 3.1. *Let $n > 1$ and $L \in \mathscr{L}(n)$. Then*

$$(Le_1)(x) \leqslant x = (B_n e_1)(x) \tag{3.1}$$

for all $x \in [0,1]$. Moreover, for every $L \in \mathscr{L}(n)$ there exists $x_L \in [0,1]$ such that $(Le_1)(x_L) < x_L$ and there exists $\tilde{L} \in \mathscr{L}(n)$ such that $(Le_1)(x) < (\tilde{L}e_1)(x) \leqslant x$ for all $x \in [0,1]$.

Proof. Let $L \in \mathscr{L}(n) : \ (Lf)(x) = \sum_{k=0}^{n} \lambda_{nk}(f)\, p_{nk}(x),\ x \in [0,1]$, with $Le_0 = e_0$ and $Le_2 = e_2$. Hence, because $\{p_{nk}\}_{k=0}^{n}$ is a basis on Π_n, we find that $\lambda_{nk}(e_0) = 1$ and $\lambda_{nk}(e_2) = \frac{k(k-1)}{n(n-1)}$, $k = 0,1,\dots,n$. But $(\lambda_{nk}(e_1))^2 \leqslant (\lambda_{nk}(e_0))(\lambda_{nk}(e_2))$ in view of (2.2). Thus

$$\lambda_{nk}(e_1) \leqslant \left(\frac{k(k-1)}{n(n-1)}\right)^{1/2}, \quad k = 0,1,\dots,n.$$

Then, by Cauchy-Schwarz inequality,

$$\begin{aligned}(Le_1)(x) &= \sum_{k=0}^{n} \lambda_{nk}(e_1) p_{nk}(x) \leqslant \sum_{k=0}^{n} \left(\frac{k(k-1)}{n(n-1)}\right)^{1/2} p_{nk}(x) \\ &\leqslant \left(\sum_{k=0}^{n} \frac{k(k-1)}{n(n-1)} p_{nk}(x)\right)^{1/2} \left(\sum_{k=0}^{n} p_{nk}(x)\right)^{1/2} \\ &= x,\end{aligned}$$

which is (3.1).

If $(Le_1)(x) = x$ for all $x \in [0,1]$, then $\sum_{k=0}^{n} \lambda_{nk}(e_1) p_{nk}(x) = \sum_{k=0}^{n} \frac{k}{n} p_{nk}(x),\ x \in [0,1]$. Hence $\lambda_{nk}(e_1) = \frac{k}{n},\ k = 0,1,\ldots,n$. As above, $\lambda_{nk}(e_1) \leqslant \left(\frac{k(k-1)}{n(n-1)}\right)^{1/2},\ k = 0,1,\ldots,n$. Then $\frac{1}{n} = \lambda_{n1}(e_1) \leqslant 0$, contradiction. Thus there exists $x_L \in [0,1]$ such that $(Le_1)(x_L) < x_L$.

Now let $L \in \mathscr{L}(n)$ with $(Le_1)(x) = \sum_{k=0}^{n} \lambda_{nk}(e_1) p_{nk}(x),\ x \in [0,1]$ and $\lambda_{nk}(e_0) = 1$, $\lambda_{nk}(e_2) = \frac{k(k-1)}{n(n-1)}$ for $k = 0,1,\ldots,n$. We have

$$\frac{k(k-1)}{n(n-1)} \leqslant \lambda_{nk}(e_1) \leqslant \left(\frac{k(k-1)}{n(n-1)}\right)^{1/2},$$

for $k = 0,1,\ldots,n$ (see (2.2)). We set $\tilde{\lambda}_{nk}(e_0) = 1$,

$$\tilde{\lambda}_{nk}(e_1) = \frac{1}{2}\left(\lambda_{nk}(e_1) + \left(\frac{k(k-1)}{n(n-1)}\right)^{1/2}\right)$$

and $\tilde{\lambda}_{nk}(e_2) = \frac{k(k-1)}{n(n-1)},\ k = 0,1,\ldots,n$. Following the proof of Theorem 2.2, we find that $\tilde{L}$ is a polynomial bounded positive linear operator on $C[0,1]$, where $(\tilde{L}f)(x) = \sum_{k=0}^{n} \tilde{\lambda}_{nk}(f) p_{nk}(x)$. Then

$$\begin{aligned}(\tilde{L}e_1)(x) &= \frac{1}{2}\sum_{k=0}^{n} \left(\lambda_{nk}(e_1) + \left(\frac{k(k-1)}{n(n-1)}\right)^{1/2}\right) p_{nk}(x) \\ &\leqslant \sum_{k=0}^{n} \left(\frac{k(k-1)}{n(n-1)}\right)^{1/2} p_{nk}(x) \leqslant x,\end{aligned}$$

$x \in [0,1]$ and obviously $(Le_1)(x) < (\tilde{L}e_1)(x) \leqslant x,\ x \in [0,1]$, if we suppose for example that $\frac{2}{n(n-1)} < \lambda_{n2}(e_1) < (\frac{2}{n(n-1)})^{1/2}$. □

The next theorem contains quantitative estimates.

THEOREM 3.2. *Let $n>1$ and $\{L_n\}$ be a sequence of polynomial bounded positive linear operators $L_n : C[0,1] \to \Pi_n$ defined by (1.1) such that $L_n e_0 = e_0$ and $L_n e_2 = e_2$.*

(i) *If $\lambda_n := \max\{\frac{k}{n} - \lambda_{nk}(e_1) : k = 0,1,\dots,n\} \to 0$ as $n \to \infty$, then for each $f \in C[0,1]$, the sequence $\{(L_n f)(x)\}$ converges to $f(x)$ uniformly in $x \in [0,1]$, as $n \to \infty$;*

(ii) $\lambda_n \neq o(n^{-2})$;

(iii) *for all $n>1$, $f \in C[0,1]$ and $x \in [0,1]$, we have*

$$|(L_n f)(x) - f(x)| \leqslant (1+\sqrt{2})\,\omega\left(f, (x - (L_n e_1)(x))^{1/2}\right) \tag{3.2}$$

and

$$\|L_n f - f\| \leqslant (1+\sqrt{2})\,\omega(f, \sqrt{\lambda_n}), \tag{3.3}$$

where $\omega(f,\delta) := \sup\{|f(x)-f(y)| : x,y \in [0,1], |x-y| \leqslant \delta\}$, $\delta>0$, is the modulus of continuity of $f \in C[0,1]$.

Proof. (i) Using the properties of L_n, we have

$$\frac{k(k-1)}{n(n-1)} \leqslant \lambda_{nk}(e_1) \leqslant \left(\frac{k(k-1)}{n(n-1)}\right)^{1/2} \leqslant \frac{k}{n},$$

$k=0,1,\dots,n$. Hence, by (3.1),

$$\begin{aligned} 0 \leqslant x - (L_n e_1)(x) &= \sum_{k=0}^{n} \left(\frac{k}{n} - \lambda_{nk}(e_1)\right) p_{nk}(x) \\ &\leqslant \lambda_n \sum_{k=0}^{n} p_{nk}(x) = \lambda_n \end{aligned} \tag{3.4}$$

for $x \in [0,1]$. Because $\lambda_n \to 0$ as $n \to \infty$, we obtain that $\{(L_n e_1)(x)\}$ converges to $e_1(x)$ uniformly in $x \in [0,1]$, $n \to \infty$. Then $L_n e_0 = e_0$, $L_n e_2 = e_2$ and the well-known Korovkin theorem (see e.g. [3, p. 8]) imply the sequence $\{(L_n f)(x)\}$ converges uniformly to $f(x)$ on $[0,1]$, as $n \to \infty$, where $f \in C[0,1]$ is arbitrary.

(ii) We have $L_n e_0 = e_0$ and $L_n e_2 = e_2$. Using Theorem 4.1 of [3, p. 278] translated from $[-1,1]$ to $[0,1]$, we obtain $\lambda_n \neq o(n^{-2})$.

(iii) Let $f \in C[0,1]$ and $\delta > 0$. The modulus of continuity of f has the property $\omega(f,\alpha\delta) \leqslant (1+\alpha)\omega(f,\delta)$, $\alpha>0$. Then, in view of $L_n e_0 = e_0$, the representation

(2.4) and Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
&|(L_nf)(x)-f(x)| \\
&\qquad\leqslant \sum_{k=0}^{n} |\lambda_{nk}(f)-f(x)|p_{nk}(x) \\
&\qquad\leqslant \sum_{k=0}^{n} \int_0^1 |f(t)-f(x)|\,d\mu_{nk}(t)\,p_{nk}(x) \\
&\qquad\leqslant \sum_{k=0}^{n} \int_0^1 \omega(f,|t-x|)\,d\mu_{nk}(t)\,p_{nk}(x) \\
&\qquad\leqslant \omega(f,\delta)\sum_{k=0}^{n} \int_0^1 (1+\delta^{-1}|t-x|)\,d\mu_{nk}(t)\,p_{nk}(x) \\
&\qquad\leqslant \omega(f,\delta)\left\{1+\delta^{-1}\sum_{k=0}^{n} \int_0^1 |t-x|\,d\mu_{nk}(t)\,p_{nk}(x)\right\} \\
&\qquad\leqslant \omega(f,\delta)\left\{1+\delta^{-1}\left(\sum_{k=0}^{n} \int_0^1 (t-x)^2\,d\mu_{nk}(t)\,p_{nk}(x)\right)^{1/2}\right\} \\
&\qquad= \omega(f,\delta)\left\{1+\delta^{-1}\left((L_ne_2)(x)-2x(L_ne_1)(x)+x^2(L_ne_0)(x)\right)^{1/2}\right\} \\
&\qquad= \omega(f,\delta)\left\{1+\delta^{-1}(2x^2-2x(L_ne_1)(x))^{1/2}\right\} \\
&\qquad\leqslant \omega(f,\delta)\left\{1+\frac{\sqrt{2}}{\delta}(x-(L_ne_1)(x))^{1/2}\right\}. \qquad (3.5)
\end{aligned}
$$

If $\delta=(x-(L_ne_1)(x))^{1/2}$, $x\in[0,1]$, then (3.5) imply (3.2). Taking into account (3.4) and (3.5), we obtain (3.3) for $\delta=\sqrt{\lambda_n}$. In both cases we find the uniform convergence of $\{(L_nf)(x)\}$ to $f(x)$ on $[0,1]$, whenever $\lambda_n\to 0$ as $n\to\infty$. □

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(Received March 17, 2010)

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