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ESTIMATE FOR THE DISCRETE TIME HEDGING ERROR OF THE AMERICAN OPTION ON A DIVIDEND–PAYING STOCK

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Abstract. This work is devoted to the discrete time hedging of the American option on a dividend-paying stock with a convex payoff, the particular case of which is American call option. Perfect hedging requires continuous trading in time and knowledge of the partial derivative of the value function of the American option in the underlying asset. Neither one can trade continuously in time nor the closed-form expression of the value function of the American option is known.

Several approximation methods have been developed for the calculation of this unknown value function. We justify in this work that having at hand any such nonnegative uniform approximation, it is possible to construct a discrete time hedging strategy the value process of which uniformly approximates the value process of the corresponding continuous time perfect hedging portfolio.

1. Introduction

Let $(\Omega,\mathscr{F},P^R)$ be a probability space and $B=(B_t)_{0\leqslant t\leqslant T}$ a standard Wiener process on it, where P^R stands for actual (i.e. real world) probability measure. We will assume that the time horizon T is finite and denote by $F^B=(\mathscr{F}_t^B)_{0\leqslant t\leqslant T}$ the P^R-completion of the natural filtration of $(B_t)_{0\leqslant t\leqslant T}$.

On the filtered probability space $(\Omega,\mathscr{F},\mathscr{F}_t^B,P^R), 0\leqslant t\leqslant T$, we consider a financial market with two assets m_t, $0\leqslant t\leqslant T$, the price of a unit of a money market account at time t, and S_t, $0\leqslant t\leqslant T$, the value at time t of the share of a stock modeled as a generalized geometric Brownian motion that pays dividends continuously over time at a rate $\delta(t)$, $0\leqslant t\leqslant T$ per unit time. The evolution of these assets obeys the following ordinary and stochastic differential equations

$$dm_t=r(t)m_t dt,\ m_0=1,\ \ 0\leqslant t\leqslant T, \tag{1.1}$$

$$dS_t=b(t)S_t dt+\sigma(t)S_t dB_t-\delta(t)S_t dt,\ \ S_0>0,\ \ 0\leqslant t\leqslant T. \tag{1.2}$$

We assume that $(b(t),\mathscr{F}_t^B)_{0\leqslant t\leqslant T}$ is certain progressively measurable process, the deterministic time-varying interest rate $r(t)$, the volatility $\sigma(t)$ and the dividend rate $\delta(t)$

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are continuously differentiable functions of time and the following requirements are satisfy

$$0 \leqslant r(t) \leqslant \bar{r}, \ \ |b(t)| \leqslant \bar{r}, \ \ 0 < \underline{\sigma} \leqslant \sigma(t) \leqslant \bar{\sigma}, \ \ 0 \leqslant \delta(t) \leqslant \bar{\delta}, \tag{1.3}$$

$$|r(t) - r(s)| + |\sigma(t) - \sigma(s)| + |\delta(t) - \delta(s)| \leqslant K|t - s|, \tag{1.4}$$

where $s,t \in [0,T]$ and $\bar{r}$, $\underline{\sigma}$, $\bar{\sigma}$, $\bar{\delta}$ and K are some positive constants.

In this work we investigate the discrete time hedging problem for the American option on a divided-paying stock written on the underlying asset $(S_t)_{0 \leqslant t \leqslant T}$ with arbitrary nonnegative finite convex payoff function $g(x)$ which satisfies the condition

$$|g'(x-)| \leqslant C, \ \ x > 0, \tag{1.5}$$

where C is certain positive constant.

The typical examples are American put and call options with payoffs $g(x) = (L - x)^+$ and $g(x) = (x - L)^+$ respectively, where L is the exercise price.

In previous paper [4] the case of put (but not a call) option on a non-dividend paying stock has been considered. In recent work we essentially include the important case of unbounded payoff functions the typical example of which is American call option.

If $D(t)$ denotes the number of shares held at time t by the writer of the option for the perfect hedging in continuous time of the underlying stock then by the delta-hedging rule we have $D(t) = \frac{\partial v}{\partial x}(t, S_t)$ (see, for example, Karatzas and Shreve 1998 [9]), where $v(t,x)$ denotes the value function of the American option and $\frac{\partial v(t,x)}{\partial x}$ its partial derivative with respect to x. Here we observe that for the perfect hedging the writer of the option must trade continuously in time and also requires the knowledge of the partial derivative of the value function $v(t,x)$, but the explicit form neither of the function $v(t,x)$ nor of its partial derivative is known in most cases of practical importance for American option valuation problem.

As we know several approximation techniques are developed for the calculation of this unknown value function (for example, finite difference methods developed in Glowinski, Lions and Trémolières 1981 [2]; Wilmott, Dewynne and Howison 1993 [13]; Jaillet, Lamberton and Lapeyre 1990 [6]). The rate of convergence of the uniform schemes to the value function of the optimal stopping problem which gives at the same time the rate of convergence of the uniform approximation to the American option value function is established in Jakobsen 2003 [7].

Take any nonnegative continuous in x uniform approximation $v_h(t,x)$ to the unknown value function $v(t,x)$ of the American option at the equidistant rebalancing times $t_k = k \cdot \Delta$, $\Delta = \frac{T}{n}$, $k = 0,1,2,\dots,n$ (for example, the Bermudan approximation), where h is certain small parameter indicating the error of approximation. In particular, we assume that the following bound is valid uniformly in k, $k = 0,1,2,\dots,n$,

$$\sup_{x \geqslant 0} |v_h(t_k,x) - v(t_k,x)| \leqslant C_1 h. \tag{1.6}$$

Here C_1 is some nonnegative constant depending on parameters of our model $\bar{r}$, $\underline{\sigma}$, $\bar{\sigma}$, $\bar{\delta}$, K, T and the payoff function $g(x)$. We naturally assume that $v_h(T,x) = g(x)$.

Our discrete time hedging strategy consists in the following. For each function $v_h(t_{k-1},x),\ x \geqslant 0,\ k=1,2,...,n,$ consider first its lower convex envelope $\check{v}_h(t_{k-1},x)$, $x \geqslant 0,\ k=1,2,...,n,$ that is, the maximal convex function dominated by the given function $v_h(t_{k-1},x)$ and then its left-hand derivative

$$\varphi_h(t_{k-1},x) = \frac{\partial \check{v}_h(t_{k-1},x-)}{\partial x},\ x>0,\ k=1,2,...,n. \tag{1.7}$$

Now the discrete time hedge $D_{\Delta,h}(t),\ 0 \leqslant t < T$ can be defined in the following manner

$$D_{\Delta,h}(t) = \begin{cases} 0, & \text{if } 0 \leqslant t < t_1, \\ \varphi_h(t_{k-1},S_{t_{k-1}}), & \text{if } t_{k-1} \leqslant t < t_k,\ k=2,3,...,n, \end{cases} \tag{1.8}$$

where Δ is related to the time discretization, $\Delta = t_{k+1}-t_k,\ k=0,1,...,n-1.$

We claim that this is the required discrete time hedging strategy the value process of which uniformly approximates the corresponding continuous time portfolio value process.

Recall the relation between our constructed discrete time hedge $D_{\Delta,h}(t)$ and its self-financing portfolio's value process $\Pi_{\Delta,h}(t)$. Suppose the writer starts with initial capital $\Pi(0)=v(0,S_0)$ and rebalances his portfolio at each time instant $t_k,\ k=1,2,...,n-1,$ and holds $D_{\Delta,h}(t_k)$ number of shares in stock and invests the remainder of the portfolio's value $(\Pi_{\Delta,h}(t_k)-D_{\Delta,h}(t_k)\cdot S_{t_k})$ in a money market account.

During the time interval (t_k,t_{k+1}) the investor is inactive, though his portfolio's value changes at arbitrary time t, $t_k \leqslant t < t_{k+1}$. As the investor holding the stock reinvests the dividends it is easy to show that the following equality does hold

$$\Pi_{\Delta,h}(t) = \exp\left[\int_0^t r(u)du\right]\left(\Pi(0)+\int_0^t D_{\Delta,h}(u)\delta(u)\tilde{S}_u du + \int_0^t D_{\Delta,h}(u)d\tilde{S}_u\right),\ 0 \leqslant t \leqslant T, \tag{1.9}$$

where $\tilde{S}_t = e^{-\int_0^t r(u)du}\cdot S_t,\ 0 \leqslant t \leqslant T$ is the discounted stock price per share at time t.

Let us denote by

$$\theta_t = \frac{b(t)-r(t)}{\sigma(t)}$$

the market price of risk. For arbitrary $A \in \mathscr{F}_T^B$, define

$$P(A) = \int_A Z_T dP^R,$$

where

$$Z_t = \exp\left(-\int_0^t \theta_u dB_u - \frac{1}{2}\int_0^t \theta_u^2 du\right),\ 0 \leqslant t \leqslant T.$$

Define the process

$$W_t = B_t + \int_0^t \theta_u du,\ 0 \leqslant t \leqslant T,$$

by Girsanov's theorem of change of probability measure the new process $W = (W_t)_{0 \leqslant t \leqslant T}$ will be a Brownian motion with respect to new risk neutral probability measure $P(A)$. So we may rewrite (1.2) in the following manner

$$dS_t = \mu(t) S_t dt + \sigma(t) S_t dW_t,\ S_0 > 0,\ 0 \leqslant t \leqslant T, \tag{1.10}$$

where

$$\mu(t) = r(t) - \delta(t),\ 0 \leqslant t \leqslant T.$$

The discounted stock price $\tilde{S}_t$ satisfies

$$d\tilde{S}_t = \sigma(t)\tilde{S}_t dW_t - \delta(t)\tilde{S}_t dt, \quad \tilde{S}_0 = S_0,\ \ 0 \leqslant t \leqslant T. \tag{1.11}$$

It is easy to show that, under the risk-neutral measure P, the discounted continuous time delta-hedging portfolio's value process $e^{-\int_0^t r(u)du} \cdot \Pi(t)$, $0 \leqslant t \leqslant T$, is a martingale and the following equality is valid

$$\Pi(t) = \exp\left[\int_0^t r(u)du\right]\left(\Pi(0) + \int_0^t D(u)\delta(u)\tilde{S}_u du + \int_0^t D(u)d\tilde{S}_u\right),\ 0 \leqslant t \leqslant T. \tag{1.12}$$

The error due to the discrete time hedging of the American option is given by the expectation

$$E^R \sup_{0 \leqslant t \leqslant T} |\Pi_{\Delta,h}(t) - \Pi(t)|. \tag{1.13}$$

By Girsanov's theorem the two expectations are related by the formula

$$E^R Y = E\left(Z_T^{-1} Y\right),$$

where Y be an $\mathscr{F}_T^B$-measurable random variable.

This formula implies

$$\begin{aligned} E^R Y &\leqslant \left(E Z_T^{-2}\right)^{\frac{1}{2}} \left(E Y^2\right)^{\frac{1}{2}} \\ &\leqslant \exp\left(6\, \frac{\bar{r}^2}{\underline{\sigma}^2}\, T\right) \left(E Y^2\right)^{\frac{1}{2}}. \end{aligned}$$

The application of the above inequality to the discrete time hedging error (1.13) implies

$$E^R \sup_{0 \leqslant t \leqslant T} |\Pi_{\Delta,h}(t) - \Pi(t)| \leqslant \exp\left(6\, \frac{\bar{r}^2}{\underline{\sigma}^2}\, T\right) \left[E \sup_{0 \leqslant t \leqslant T} \left|\Pi_{\Delta,h}(t) - \Pi(t)\right|^2\right]^{\frac{1}{2}},$$

which, taking into account (1.9) and (1.12), yields

$$
\begin{aligned}
&E^R \sup_{0\leqslant t\leqslant T} |\Pi_{\Delta,h}(t)-\Pi(t)| \\
&\leqslant \exp\left(\bar{r}T+\frac{6\bar{r}^2T}{\underline{\sigma}^2}\right)\left[E\sup_{0\leqslant t\leqslant T}\left|\int_0^t \big(D_{\Delta,h}(u)-D(u)\big)\,\delta(u)\tilde{S}_u du\right.\right. \\
&\quad \left.\left. +\int_0^t \big(D_{\Delta,h}(u)-D(u)\big)\,d\tilde{S}_u\right|^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

Successive application of (1.11) and the classical Doob's maximal inequality yields an initial estimate for the discrete time hedging error of the American option

$$
\begin{aligned}
&E^R \sup_{0\leqslant t\leqslant T} |\Pi_{\Delta,h}(t)-\Pi(t)| \\
&\leqslant 2\,\bar{\sigma}\,\exp\left(\bar{r}T+\frac{6\bar{r}^2T}{\underline{\sigma}^2}\right)\left[E\int_0^T \big(D_{\Delta,h}(u)-D(u)\big)^2 S_u^2 du\right]^{\frac{1}{2}}.
\end{aligned}
\tag{1.14}
$$

To justify our result we have to estimate the last square integral with respect to the risk-neutral measure P. We will apply weighted square integral inequality for the difference of derivatives of two convex functions from [5] and for this purpose the assumption of the convexity of the payoff function $g(x)$ turn out to be crucial. It is well-known that (see, for example, El Karoui, Jeanblanc-Picqué, and Shreve [1] or Hobson [3]) for both European and American contingent claims with convex payoffs the price of the contingent claims $v(t,x)$ is a convex function of the price of the stocks. This fact leads us to apply the above mentioned integral inequality.

We are ready now to introduce our main results the proofs of which are given in Section 3.

Let us introduce the discrete time hedge $D_\Delta(t)$, $0\leqslant t\leqslant T$ by means of the continuous time delta-hedge $D(t)$ in a natural manner

$$
D_\Delta(t)=D(t_{k-1}),\ \text{if}\ t_{k-1}\leqslant t<t_k,\ k=1,2,...,n. \tag{1.15}
$$

For continuous time delta-hedge $D(t)$ and discrete time hedge $D_\Delta(t)$, we have the following result:

PROPOSITION 1. *The following estimate does hold*

$$
E\int_0^T (D(t)-D_\Delta(t))^2 S_t^2 dt \leqslant c\cdot\ln\frac{T}{\Delta}\,\Delta, \tag{1.16}
$$

if $\Delta=\frac{T}{n}\leqslant 1$, $n=2,3,...$ *where* c *is non-negative constant depending on the parameters* $\bar{r}$, $\bar{\delta}$, $\underline{\sigma}$, $\bar{\sigma}$, C, T, K, $g(1)$ *and* S_0.

The key result states that the value process (1.9) of our constructed discrete time hedging strategy (1.8) uniformly approximates the corresponding continuous time portfolio value process (1.12).

PROPOSITION 2. *Let $v(t,x)$ denotes the unknown value function of the American option where the payoff be any arbitrary finite convex function satisfying the requirement (1.5). Suppose we have at hand some continuous in x nonnegative uniform approximation $v_h(t,x)$ to $v(t,x)$ at the equidistant rebalancing times $t_k = k\cdot\Delta,\ \Delta = \frac{T}{n}$, $n = 2,3,\dots;\ k = 0,1,\dots,n$, such that the bound (1.6) holds.*

Then for the discrete time hedging error the following estimate is valid

$$E^R \sup_{0\leqslant t\leqslant T} |\Pi_{\Delta,h}(t) - \Pi(t)| \leqslant b\left(\ln\frac{T}{\Delta}\right)^{\frac{1}{2}} (h+\Delta)^{\frac{1}{2}}, \tag{1.17}$$

if $\Delta = \frac{T}{n} \leqslant 1,\ h\leqslant 1$, where b is positive constant depending on parameters $\bar{r},\ \bar{\delta},\ \bar{\sigma}$, $\underline{\sigma}$, C_1, T, $g(1)$, K, C and S_0.

REMARK 1. The error of approximation h and the parameter Δ should be dependent, for example $h = \Delta^{\mu},\ \mu > 0$ or $\Delta = h^{\nu},\ \nu > 0$, then we see that the right-hand side of the estimate (1.17) tends to zero as $\ln\frac{T}{\Delta}\cdot\Delta$ and $\ln\frac{T}{\Delta}\cdot h$ converge to zero.

In the next result we apply weighted square integral inequality stated in Hussain, Pečarić and Shashiashvili [5] for the difference of derivatives of two finite convex functions in order to obtain the weighted energy inequality for an arbitrary unknown finite convex function. This kind of inequalities can be used in order to obtain discrete time hedging error estimates.

Let $F(x)$ be arbitrary unknown finite convex function on an infinite interval $[0,\infty)$ satisfying the requirement (1.5). Suppose we are given some continuous uniform approximation $F_h(x)$, $x\geqslant 0$ of the unknown function $F(x)$ then we will have

$$\sup_{x\geqslant 0} |F_h(x) - F(x)| \to 0 \ \text{ as } \ h\to 0. \tag{1.18}$$

We will introduce the family of nonnegative twice continuously differentiable weight functions $H(x)$, $0 < x < \infty$, which satisfy the conditions

$$\lim_{x\to 0+} H(x) = 0,\ \lim_{x\to\infty} H(x) = 0,\ \lim_{x\to 0+} H'(x) = 0,\ \lim_{x\to\infty} xH'(x) = 0, \tag{1.19}$$

and

$$\int_0^{\infty} (x+1)\left|H''(x)\right| dx < \infty, \tag{1.20}$$

and we come to the following result:

PROPOSITION 3. *Let $F(x)$ be arbitrary unknown continuous finite convex function defined on an infinite interval $[0,\infty)$ satisfying the requirement (1.5) and suppose we have at hand its some continuous uniform approximation $F_h(x),\ x\geqslant 0$. Consider the lower convex envelope $\check{F}_h(x)$ (that is, the maximal convex function dominated by the given function $F_h(x)$). Then for the unknown left-hand derivative $F'(x-),\ x > 0$,* *the following energy estimate through* $\breve{F}_h'(x-)$, $x>0$ *does hold*

$$\int_0^\infty \left(\breve{F}_h'(x-)-F'(x-)\right)^2 H(x)dx$$
$$\leqslant \frac{3}{2}\sup_{x\geqslant 0}|F_h(x)-F(x)|\int_0^\infty \left(|\breve{F}_h(x)|+|F(x)|\right)\left|H''(x)\right|dx, \tag{1.21}$$

where $H(x)$, $0<x<\infty$, *is any nonnegative twice continuously differentiable weight function satisfying the conditions (*1.19*) and (*1.20*).*

2. Refinement of some regularity properties of pricing functions for American options

This section is devoted to the refinement of the well-known results concerning the regularity of the American option value function given in Jaillet, Lamberton, and Lapeyre [6]. We make more precise these properties where the payoff $g(x)$ is arbitrary finite convex function and satisfies the requirement (1.5). These results will be used in order to obtain the discrete time hedging error estimate (1.17).

It is well-known that the American option valuation problem is related to the corresponding optimal stopping problem (see, for example, Karatzas and Shreve, Chapter 2 [9]) of the diffusion process in the following manner

$$v(t,x)=\sup_{\tau\in\mathscr{T}_{t,T}} E\left[\exp\left(-\int_t^\tau r(v)dv\right)g(S_\tau(t,x))\right],\ x\geqslant 0,\ 0\leqslant t\leqslant T, \tag{2.1}$$

where $\mathscr{T}_{t,T}$ denotes the set of all stopping times τ such that $t\leqslant\tau\leqslant T$, and the stochastic process $S_u(t,x), t\leqslant u\leqslant T$, satisfies the same stochastic differential equation

$$dS_u(t,x)=\mu(u)S_u(t,x)du+\sigma(u)S_u(t,x)dW_u,\ t\leqslant u\leqslant T, \tag{2.2}$$

with the initial condition $S_t(t,x)=x$, $x\geqslant 0$, where

$$\mu(u)=r(u)-\delta(u),\ t\leqslant u\leqslant T.$$

The unique solution $(S_u(t,x),\mathscr{F}_u)_{t\leqslant u\leqslant T}$ of this equation is given by the exponential

$$S_u(t,x)=x\ \exp\left[\int_t^u\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+\int_t^u\sigma(v)dW_v\right],\ t\leqslant u\leqslant T. \tag{2.3}$$

Introduce the new stochastic process $(X_u(t,x),\mathscr{F}_u)_{t\leqslant u\leqslant T}$

$$X_u(t,y)=y+\int_t^u\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+\int_t^u\sigma(v)dW_v, \tag{2.4}$$

$t\leqslant u\leqslant T$, $-\infty<y<\infty$. It is easy to show that

$$S_u(t,x)=\exp\left[X_u(t,\ln x)\right],\ t\leqslant u\leqslant T,\ x>0, \tag{2.5}$$

and for arbitrary stopping time τ such that $t \leqslant \tau \leqslant T$ we have

$$g(S_\tau(t,x)) = \psi(X_\tau(t,\ln x)),$$

where $\psi(y) = g(e^y)$, $-\infty < y < \infty$, is the new payoff function.

Let us define the corresponding optimal stopping problem

$$u(t,y) = \sup_{\tau \in \mathscr{T}_{t,T}} E\left[\exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,y))\right], \tag{2.6}$$

with $0 \leqslant t \leqslant T$ and $-\infty < y < \infty$, then we find

$$v(t,x) = u(t,\ln x),\ x > 0,\ 0 \leqslant t \leqslant T. \tag{2.7}$$

We introduce the following lemma:

LEMMA 1. *Let $g(x)$, $x \geqslant 0$ be a nonnegative arbitrary finite convex function satisfying the requirement (1.5). Then the new payoff function defined by $\psi(y) = g(e^y)$, $-\infty < y < \infty$ is locally Lipschitz continuous, that is,*

$$|\psi(y_2) - \psi(y_1)| \leqslant C\, e^{y_2}\, (y_2 - y_1),\ -\infty < y_1 \leqslant y_2 < \infty. \tag{2.8}$$

Proof. It is well-known that any convex function is locally absolutely continuous (see, for example, Royden, Page 114 [11]), hence

$$g(x_2) - g(x_1) = \int_{x_1}^{x_2} g'(u-)du,\ 0 < x_1 \leqslant x_2 < \infty, \tag{2.9}$$

where $g'(u-)$ denotes the left-hand derivative at point u of the convex function $g(x)$.

Now consider the difference $\psi(y_2) - \psi(y_1)$, using the expression (2.9) and condition (1.5), we can write

$$\begin{aligned} |\psi(y_2) - \psi(y_1)| &= |g(e^{y_2}) - g(e^{y_1})| \\ &= \left|\int_{e^{y_1}}^{e^{y_2}} g'(u-)du\right| \\ &\leqslant C\left(e^{y_2} - e^{y_1}\right), \end{aligned}$$

and Lemma 1 follows from the latter estimate by using the mean value theorem. □

By the scaling property of the Brownian motion we can express the value function $u(t,y)$ of the optimal stopping problem (2.6) as follows (see Jaillet, Lamberton and Lapeyre [6])

$$\begin{aligned} u(t,y) = \sup_{\tau \in \mathscr{T}_{0,1}} E\Bigg[\exp\left(-\int_t^{t+\tau(T-t)} r(v)dv\right)\psi\Bigg(y + \int_t^{t+\tau(T-t)}\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv \\ + \int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v\Bigg)\Bigg],\ 0 \leqslant t \leqslant T,\ -\infty < y < \infty, \end{aligned} \tag{2.10}$$

where $\mathscr{T}_{0,1}$ denotes the set of all stopping times τ with respect to the filtration $(\mathscr{F}_u)_{0\leqslant u\leqslant 1}$ taking values in $[0,1]$.

Let us come to the following result:

PROPOSITION 4. *The value function $u(t,y)$, $0\leqslant t\leqslant T$, $-\infty<y<\infty$ of the optimal stopping problem (2.6) is Locally Lipschitz continuous in the argument y i.e.*

$$|u(t,y)-u(t,z)|\leqslant D\, e^{|y|+|z|}\ |y-z|,\ y,z\in\mathbb{R},\ 0\leqslant t\leqslant T, \tag{2.11}$$

where D is some nonnegative constant depending on parameters $\bar{r}$, $\bar{\sigma}$, $\bar{\delta}$, C and T.

Proof. Fix any τ in $\mathscr{T}_{t,T}$ and $y,z\in\mathbb{R}$, we can write

$$\begin{aligned}
&\left|E\ \exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,y))-E\ \exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,z))\right|\\
&\leqslant C\,E[\exp(|X_\tau(t,y)|+|X_\tau(t,z)|)\ |X_\tau(t,y)-X_\tau(t,z)|]\\
&=C|y-z|\,E\ \exp\left(\left|y+\int_t^\tau\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+\int_t^\tau\sigma(v)dW_v\right|\right.\\
&\quad\left.+\left|z+\int_t^\tau\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+\int_t^\tau\sigma(v)dW_v\right|\right)\\
&\leqslant C|y-z|\ \exp\left(|y|+|z|+2T\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)\right)E\exp\left(2\left|\int_t^\tau\sigma(v)dW_v\right|\right),
\end{aligned} \tag{2.12}$$

where we have used Lemma 1 and $\bar{\mu}=\bar{r}+\bar{\delta}$.

Using the inequality

$$e^{|x|}\leqslant e^x+e^{-x},\ -\infty<x<\infty, \tag{2.13}$$

we can write

$$\begin{aligned}
&\exp\left(2n\left|\int_t^\tau\sigma(v)dW_v\right|\right)\leqslant\exp\left(2n\int_t^\tau\sigma(v)dW_v\right)+\exp\left(-2n\int_t^\tau\sigma(v)dW_v\right)\\
&=\exp\left(2n\int_t^\tau\sigma(v)dW_v-\frac{1}{2}\int_t^\tau(2n\ \sigma(v))^2dv\right)\exp\left(\int_t^\tau 2n^2\sigma^2(v)dv\right)\\
&\quad+\exp\left(-2n\int_t^\tau\sigma(v)dW_v-\frac{1}{2}\int_t^\tau(-2n\ \sigma(v))^2dv\right)\exp\left(\int_t^\tau 2n^2\sigma^2(v)dv\right),
\end{aligned}$$

for any positive integer n.

Using the values

$$E\exp\left(2n\int_t^\tau\sigma(v)dW_v-\frac{1}{2}\int_t^\tau(2n\ \sigma(v))^2dv\right)=1$$

and

$$E\exp\left(-2n\int_t^\tau\sigma(v)dW_v-\frac{1}{2}\int_t^\tau(-2n\ \sigma(v))^2dv\right)=1$$

we obtain explicitly

$$E\ \exp\left(2n\left|\int_t^\tau \sigma(v)dW_v\right|\right) \leqslant 2\ \exp\left(2n^2\int_t^T \sigma^2(v)dv\right) \leqslant 2\ \exp\left(2n^2\bar{\sigma}^2 T\right). \qquad (2.14)$$

Using (2.14), with $n=1$, inequality (2.12) yields

$$\left|E\ \exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,y)) - E\ \exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,z))\right| \leqslant D|y-z|\ \exp(|y|+|z|)\ ,\ y,z\in\mathbb{R},\ 0\leqslant t\leqslant T,$$

where D is nonnegative constant depending on the parameters $\bar{r}$, $\bar{\sigma}$, $\bar{\delta}$, C and T only.

As the difference between the supremums is less or equal than the supremum of differences, we arrive to our required result. □

Denote

$$Y_\tau(t,y) \equiv y + \int_t^{t+\tau(T-t)}\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv + \int_0^\tau \sqrt{T-t}\ \sigma(t+v(T-t))dW_v,$$

where $\tau\in\mathscr{T}_{0,1}$, $0\leqslant t\leqslant T$, $-\infty<y<\infty$, then expression (2.10) can be written as

$$u(t,y) = \sup_{\tau\in\mathscr{T}_{0,1}} E\left[\exp\left(-\int_t^{t+\tau(T-t)} r(v)dv\right)\psi(Y_\tau(t,y))\right]. \qquad (2.15)$$

Before proving the next result, we estimate several expressions.

Using equality (2.9) and condition (1.5) on the convex function $g(x)$, we can write

$$\begin{aligned}\psi(x) &= g(e^x) - g\left(e^0\right) + g\left(e^0\right) \\ &= \int_{e^0}^{e^x} g'(u-)du + g(1) \\ &\leqslant C\ \left|e^x - e^0\right| + g(1) \\ &\leqslant C\ (e^x+1) + g(1).\end{aligned} \qquad (2.16)$$

Therefore, for any $y \in \mathbb{R},\ 0 \leqslant \tau \leqslant 1,\ 0 \leqslant t \leqslant T,$ we have

$$
\begin{aligned}
&E\psi(Y_\tau(t,y)) \\
&\leqslant C\left[1 + E\,\exp\left(y + \left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)\tau(T-t) + \int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v\right)\right] + g(1) \\
&\leqslant C\left[1 + \exp\left(y + \left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)T\right)\,E\exp\left(\int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v\right.\right. \\
&\qquad \left.\left. - \frac{1}{2}\int_0^\tau (T-t)\,\sigma^2(t+v(T-t))dv + \frac{1}{2}\int_0^\tau (T-t)\,\sigma^2(t+v(T-t))dv\right)\right] + g(1) \\
&\leqslant C\left[1 + \exp\left(y + \left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)T\right)\,E\exp\left(\int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v\right.\right. \\
&\qquad \left.\left. - \frac{1}{2}\int_0^\tau (T-t)\,\sigma^2(t+v(T-t))dv\right)\exp\left(\frac{1}{2}\int_0^1 (T-t)\,\sigma^2(t+v(T-t))dv\right)\right] + g(1),
\end{aligned}
$$

from here we can write

$$
\begin{aligned}
&E\psi(Y_\tau(t,y)) \\
&\leqslant C\left[1 + \exp\left(y + \left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)T + \frac{1}{2}\int_0^1 (T-t)\,\sigma^2(t+v(T-t))dv\right)\right] + g(1) \\
&\leqslant C\left[1 + \exp\left(y + \left(\bar{\mu} + \bar{\sigma}^2\right)T\right)\right] + g(1).
\end{aligned}
\tag{2.17}
$$

Next, for any $y \in \mathbb{R},\ 0 \leqslant \tau \leqslant 1,\ 0 \leqslant s,t \leqslant T,$ we have

$$
\begin{aligned}
&\exp\left(2|y| + \left|\int_t^{t+\tau(T-t)}\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right| + \left|\int_s^{s+\tau(T-s)}\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right|\right) \\
&\leqslant \exp\left(2|y| + \left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)\tau(T-t) + \left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)\tau(T-s)\right) \\
&\leqslant \exp\left(2|y| + 2T\left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)\right).
\end{aligned}
\tag{2.18}
$$

Moreover, using inequality (2.13) and that

$$
E\exp\left(2\int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v - \frac{1}{2}\int_0^\tau 4(T-t)\sigma^2(t+v(T-t))dv\right) = 1,
$$

we obtain

$$
E\exp\left(2\left|\int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v\right|\right) \leqslant 2\exp(2\bar{\sigma}^2 T), \tag{2.19}
$$

where $\tau \in [0,1]$.

To obtain the next estimate, we denote $R(u)=\int_0^u\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv,\ 0\leqslant u\leqslant T$. By mean value theorem

$$\begin{aligned}
&\left|\int_t^{t+\tau(T-t)}\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv-\int_s^{s+\tau(T-s)}\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv\right|\\
&=|(R(t+\tau(T-t))-R(t))-(R(s+\tau(T-s))-R(s))|\\
&\leqslant 2\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)|t-s|, \qquad (2.20)
\end{aligned}$$

where $\tau\in\mathscr{T}_{0,1}$.

Moreover, using the Lipschitz condition (1.4), for any $\tau,\ 0\leqslant\tau\leqslant 1,$ we can write

$$\begin{aligned}
&E\left|\int_0^\tau\left(\sqrt{T-t}\,\sigma(t+v(T-t))-\sqrt{T-s}\,\sigma(s+v(T-s))\right)dW_v\right|^2\\
&=E\int_0^\tau\left(\sqrt{T-t}\,\sigma(t+v(T-t))-\sqrt{T-s}\,\sigma(s+v(T-s))\right)^2dv\\
&\leqslant\int_0^1\left(\sqrt{T-t}\,\sigma(t+v(T-t))-\sqrt{T-s}\,\sigma(s+v(T-s))\right)^2dv\\
&\leqslant 2\int_0^1(T-t)(\sigma(t+v(T-t))-\sigma(s+v(T-s)))^2dv\\
&\quad+2\int_0^1\left(\sqrt{T-t}-\sqrt{T-s}\right)^2\sigma^2(s+v(T-s))dv\\
&\leqslant\frac{2K^2T^2+\bar{\sigma}^2}{T-t}\,(t-s)^2, \qquad (2.21)
\end{aligned}$$

where $0\leqslant s\leqslant t<T$.

Now we are ready to prove the following regularity result:

PROPOSITION 5. *The value function $u(t,y),\ 0\leqslant t\leqslant T,\ -\infty<y<\infty,$ of the optimal stopping problem* (2.15) *is Locally Lipschitz continuous in time argument t, i.e.,*

$$|u(t,y)-u(s,y)|\leqslant\frac{B\,e^{2|y|}}{\sqrt{T-t}}\,(t-s),\ 0\leqslant s\leqslant t<T,\ -\infty<y<\infty, \qquad (2.22)$$

where the nonnegative constant B depends only on the parameters $\bar{r},\ \bar{\delta},\ \bar{\sigma},\ K,\ C,\ T$ and $g(1)$.

Proof. Fix $s,t,\ 0\leqslant s\leqslant t<T$, and τ in $\mathscr{T}_{0,1}$. Application of the estimate (2.20) and Lemma 1 gives

$$\left|E\exp\left(-\int_t^{t+\tau(T-t)}r(v)dv\right)\psi(Y_\tau(t,y))-E\exp\left(-\int_s^{s+\tau(T-s)}r(v)dv\right)\psi(Y_\tau(s,y))\right|$$

$$
\begin{aligned}
&\leqslant E\Bigg[\bigg|\exp\left(-\int_t^{t+\tau(T-t)} r(v)dv\right) - \exp\left(-\int_s^{s+\tau(T-s)} r(v)dv\right)\bigg|\,\psi(Y_\tau(t,y)) \\
&\quad + \exp\left(-\int_s^{s+\tau(T-s)} r(v)dv\right)|\psi(Y_\tau(t,y)) - \psi(Y_\tau(s,y))|\Bigg] \\
&\leqslant 2\ \bar{r}\ |t-s|\ E\psi(Y_\tau(t,y)) \\
&\quad + CE\Bigg[\exp\Bigg(2|y| + \left|\int_t^{t+\tau(T-t)}\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right| + \left|\int_0^\tau \sqrt{T-t}\ \sigma(t+v(T-t))dW_v\right| \\
&\quad + \left|\int_s^{s+\tau(T-s)}\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right| + \left|\int_0^\tau \sqrt{T-s}\ \sigma(s+v(T-s))dW_v\right|\Bigg) \\
&\quad \times\Bigg\{\left|\int_t^{t+\tau(T-t)}\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv - \int_s^{s+\tau(T-s)}\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right| \\
&\quad + \left|\int_0^\tau \left(\sqrt{T-t}\ \sigma(t+v(T-t)) - \sqrt{T-s}\ \sigma(s+v(T-s))\right)dW_v\right|\Bigg\}\Bigg].
\end{aligned}
$$

Next, we use estimates (2.17) and (2.20) and find

$$
\begin{aligned}
&\left|E\exp\left(-\int_t^{t+\tau(T-t)} r(v)dv\right)\psi(Y_\tau(t,y)) - E\exp\left(-\int_s^{s+\tau(T-s)} r(v)dv\right)\psi(Y_\tau(s,y))\right| \\
&\leqslant 2\ \bar{r}\left[C\left\{1+\exp\left(y+(\bar{\mu}+\bar{\sigma}^2)T\right)\right\} + g(1)\right]|t-s| + C\exp\left(2|y| + 2\left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)T\right) \\
&\quad \times E\Bigg[\exp\left(\left|\int_0^\tau \sqrt{T-t}\ \sigma(t+v(T-t))dW_v\right| + \left|\int_0^\tau \sqrt{T-s}\ \sigma(s+v(T-s))dW_v\right|\right) \\
&\quad \times\left\{2\left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)|t-s| + \left|\int_0^\tau\left(\sqrt{T-t}\sigma(t+v(T-t)) - \sqrt{T-s}\sigma(s+v(T-s))\right)dW_v\right|\right\}\Bigg] \\
&\leqslant 2\bar{r}\left[C\left\{1+\exp\left(y+(\bar{\mu}+\bar{\sigma}^2)T\right)\right\} + g(1)\right]\ |t-s| \\
&\quad + 2C\left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)\ \exp\left(2|y| + 2\left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)T\right) \\
&\quad \times E\exp\left(\left|\int_0^\tau \sqrt{T-t}\sigma(t+v(T-t))dW_v\right| + \left|\int_0^\tau \sqrt{T-s}\sigma(s+v(T-s))dW_v\right|\right)|t-s| \\
&\quad + C\exp\left(2|y| + 2\left(\bar{\mu} + \frac{\bar{\sigma}^2}{2}\right)T\right) \\
&\quad \times E\Bigg[\exp\left(\left|\int_0^\tau \sqrt{T-t}\ \sigma(t+v(T-t))dW_v\right| + \left|\int_0^\tau \sqrt{T-s}\ \sigma(s+v(T-s))dW_v\right|\right) \\
&\quad \times\left|\int_0^\tau\left(\sqrt{T-t}\ \sigma(t+v(T-t)) - \sqrt{T-s}\ \sigma(s+v(T-s))\right)dW_v\right|\Bigg].
\end{aligned}
$$

Application of Schwartz inequality implies

$$\begin{aligned}
&\left| E\exp\left(-\int_t^{t+\tau(T-t)} r(v)dv\right)\psi(Y_\tau(t,y)) - E\exp\left(-\int_s^{s+\tau(T-s)} r(v)dv\right)\psi(Y_\tau(s,y))\right| \\
&\leqslant 2\bar{r}\left[C\left\{1+\exp\left(y+(\bar{\mu}+\bar{\sigma}^2)T\right)\right\}+g(1)\right]\,|t-s| \\
&\quad +2C\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)\exp\left(2|y|+2\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)T\right) \\
&\quad \times\sqrt{E\exp\left(2\left|\int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v\right|\right)} \\
&\quad \times\sqrt{E\exp\left(2\left|\int_0^\tau \sqrt{T-s}\,\sigma(s+v(T-s))dW_v\right|\right)}\,|t-s| \\
&\quad +C\,\exp\left(2|y|+2\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)T\right)\left[E\exp\left(4\left|\int_0^\tau \sqrt{T-t}\,\sigma(t+v(T-t))dW_v\right|\right)\right]^{\frac{1}{4}} \\
&\quad \times\left[E\exp\left(4\left|\int_0^\tau \sqrt{T-s}\,\sigma(s+v(T-s))dW_v\right|\right)\right]^{\frac{1}{4}} \\
&\quad \times\sqrt{E\left|\int_0^\tau\left(\sqrt{T-t}\,\sigma(t+v(T-t))-\sqrt{T-s}\,\sigma(s+v(T-s))\right)dW_v\right|^2}.
\end{aligned}$$

Finally, we use estimates (2.19) and (2.21) and get

$$\begin{aligned}
&\left| E\exp\left(-\int_t^{t+\tau(T-t)} r(v)dv\right)\psi(Y_\tau(t,y)) - E\exp\left(-\int_s^{s+\tau(T-s)} r(v)dv\right)\psi(Y_\tau(s,y))\right| \\
&\leqslant 2\bar{r}\left[C\left\{1+\exp\left(y+(\bar{\mu}+\bar{\sigma}^2)T\right)\right\}+g(1)\right](t-s) \\
&\quad +4C\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)\,\exp\left(2|y|+(2\bar{\mu}+3\bar{\sigma}^2)T\right)(t-s) \\
&\quad +\sqrt{2}\,C\,\exp\left(2|y|+(2\bar{\mu}+5\bar{\sigma}^2)T\right)\sqrt{\frac{2T^2K^2+\bar{\sigma}^2}{T-t}\,(t-s)^2} \\
&\leqslant \frac{Be^{2|y|}}{\sqrt{T-t}}\,(t-s),
\end{aligned}$$

where the constant B depends on parameters $\bar{r}$, $\bar{\delta}$, $\bar{\sigma}$, K, C, $g(1)$ and T.

Again using the fact that the difference between supremums is less or equal to the supremum of the differences, we complete the proof. □

Theorem 3.6 and its Corollary 3.7 from Jaillet, Lamberton and Lapeyre [6] state that the value function $u(t,y)$ of the optimal stopping problem (2.15) admits partial derivatives $\frac{\partial u(t,y)}{\partial t}$, $\frac{\partial u(t,y)}{\partial y}$ and $\frac{\partial^2 u(t,y)}{\partial y^2}$, which are locally bounded on $[0,T)\times\mathbb{R}$, the partial derivative $\frac{\partial u(t,y)}{\partial y}$ is continuous on $[0,T)\times\mathbb{R}$.

Furthermore, in the sense of measure, the following inequalities hold

$$\begin{aligned}&\frac{\partial u(t,y)}{\partial t}+\frac{\sigma^2(t)}{2}\frac{\partial^2 u(t,y)}{\partial y^2}+\left(\mu(t)-\frac{\sigma^2(t)}{2}\right)\frac{\partial u(t,y)}{\partial y}-r(t)u(t,y)\leqslant 0,\\&\frac{\partial^2 u(t,y)}{\partial y^2}-\frac{\partial u(t,y)}{\partial y}\geqslant 0\end{aligned}\tag{2.23}$$

almost everywhere in $[0,T)\times\mathbb{R}$.

The above system of inequalities implies the two-sided bound

$$\frac{\partial u(t,y)}{\partial y}\leqslant\frac{\partial^2 u(t,y)}{\partial y^2}\leqslant\frac{2}{\underline{\sigma}^2}\left[-\frac{\partial u(t,y)}{\partial t}-\left(\mu(t)-\frac{\sigma^2(t)}{2}\right)\frac{\partial u(t,y)}{\partial y}+r(t)u(t,y)\right]\tag{2.24}$$

a.e. in $[0,T)\times\mathbb{R}$.

PROPOSITION 6. *For the second ordered weak partial derivative* $\frac{\partial^2 u(t,y)}{\partial y^2}$, $0\leqslant t<T$, $-\infty<y<\infty$, *of the value function* $u(t,y)$, *the following estimate does hold*

$$\left|\frac{\partial^2 u(t,y)}{\partial y^2}\right|\leqslant\frac{A\,e^{2|y|}}{\sqrt{T-t}}\tag{2.25}$$

a.e. in $[0,T)\times\mathbb{R}$ *(with respect to the product measure* $dt\times dy$*), where the constant* A *depends only on the parameters* $\bar r$, $\bar\delta$, $\underline{\sigma}$, $\bar\sigma$, K, C, $g(1)$ *and* T.

Proof. From Propositions 4 and 5, we observe that

$$\left|\frac{\partial u(t,y)}{\partial y}\right|\leqslant D\,e^{2|y|}\tag{2.26}$$

for arbitrary (t,y), $0\leqslant t<T$, $-\infty<y<\infty$ and

$$\left|\frac{\partial u(t,y)}{\partial t}\right|\leqslant\frac{B\,e^{2|y|}}{\sqrt{T-t}}\quad\text{a.e. in } [0,T),\ y\in\mathbb{R}.\tag{2.27}$$

Moreover, the two-sided inequality (2.24) implies

$$\left|\frac{\partial^2 u(t,y)}{\partial y^2}\right|\leqslant\left|\frac{\partial u(t,y)}{\partial y}\right|+\frac{2}{\underline{\sigma}^2}\left[\left|\frac{\partial u(t,y)}{\partial t}\right|+\left(\bar r+\frac{\bar\sigma^2}{2}\right)\left|\frac{\partial u(t,y)}{\partial y}\right|+\bar r u(t,y)\right].\tag{2.28}$$

From the expression (2.6) of the value function $u(t,y)$ we can write

$$\begin{aligned}u(t,y)&\leqslant\sup_{\tau\in\mathscr{T}_{t,T}}E\psi(X_\tau(t,y))\\&\leqslant C\sup_{\tau\in\mathscr{T}_{t,T}}E\left[\exp(X_\tau(t,y))+1\right]+g(1)\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\left(e^{y+\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)T}\sup_{\tau\in\mathscr{T}_{t,T}}E\exp\left(\int_t^\tau\sigma(v)dW_v\right)+1\right)+g(1)\\
&\leqslant\left(e^{y+\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)T}\sup_{\tau\in\mathscr{T}_{t,T}}E\exp\left(\int_t^\tau\sigma(v)dW_v-\frac{1}{2}\int_t^\tau\sigma^2(v)dv+\frac{1}{2}\int_t^\tau\sigma^2(v)dv\right)+1\right)\\
&\quad+g(1)\\
&\leqslant\left(e^{y+(\bar{\mu}+\bar{\sigma}^2)T}\sup_{\tau\in\mathscr{T}_{t,T}}E\exp\left(\int_t^\tau\sigma(v)dW_v-\frac{1}{2}\int_t^\tau\sigma^2(v)dv\right)+1\right)+g(1)\\
&=C\left(e^{y+(\bar{\mu}+\bar{\sigma}^2)T}+1\right)+g(1),
\end{aligned}
\tag{2.29}
$$

where we have used (2.16).

Required result follows by using the bounds (2.26), (2.27) and the latter estimate in the inequality (2.28). □

Here is the application of Propositions 5 and 6.

PROPOSITION 7. *The partial derivative* $\gamma(t,y)\equiv\frac{\partial u(t,y)}{\partial y}$, $0\leqslant t<T$, $-\infty<y<\infty$ *of the value function* $u(t,y)$ *of the optimal stopping problem* (2.6) *satisfies, with respect to time argument, the following local Hölder estimate with exponent* $\frac{1}{2}$

$$
|\gamma(t_2,y)-\gamma(t_1,y)|\leqslant\frac{F\ e^{2|y|}}{\sqrt{T-t_2}}\ |t_2-t_1|^{\frac{1}{2}},
\tag{2.30}
$$

where $0\leqslant t_1\leqslant t_2<T$, $-\infty<y<\infty$, *and* F *is a positive constant depending on the parameters* $\bar{r}$, $\bar{\delta}$, $\underline{\sigma}$, $\bar{\sigma}$, K, $g(1)$ *and* T.

Proof. According to Proposition 6

$$
\begin{aligned}
|\gamma(t_1,y)-\gamma(t_1,z)|&\leqslant\frac{A\ e^{|y|+|z|}}{\sqrt{T-t_1}}\ |y-z|,\\
|\gamma(t_2,y)-\gamma(t_2,z)|&\leqslant\frac{A\ e^{|y|+|z|}}{\sqrt{T-t_2}}\ |y-z|,
\end{aligned}
\tag{2.31}
$$

valid for arbitrary t_1,t_2, $0\leqslant t_1\leqslant t_2<T$ and $y,z\in\mathbb{R}$, by the continuity of the function $\gamma(t,y)$.

Lemma 2.4 in Hussain and Shashiashvili [4] states that for arbitrary pairs (t_1,t_2), $0\leqslant t_1\leqslant t_2<T$ and $(y,y+h)$, $-\infty<y<\infty$, $h>0$, the following bound does hold

$$
\begin{aligned}
&|\gamma(t_2,y)-\gamma(t_1,y)|\\
&\leqslant\frac{1}{h}\left[\int_y^{y+h}|\gamma(t_2,y)-\gamma(t_2,z)|dz+\int_y^{y+h}|\gamma(t_1,z)-\gamma(t_1,y)|dz\right.\\
&\quad\left.+|u(t_2,y+h)-u(t_1,y+h)|+|u(t_2,y)-u(t_1,y)|\right].
\end{aligned}
$$

The use of inequalities (2.31) and Proposition 5 in the latter estimate implies

$$
\begin{aligned}
|\gamma(t_2,y)-\gamma(t_1,y)| &\leqslant \frac{1}{h}\Bigg[\int_y^{y+h}\frac{A\,e^{|y|+|z|}}{\sqrt{T-t_2}}\,(z-y)dz+\int_y^{y+h}\frac{A\,e^{|y|+|z|}}{\sqrt{T-t_1}}\,(z-y)dz\\
&\quad+\frac{B\,e^{2|y+h|}}{\sqrt{T-t_2}}\,(t_2-t_1)+\frac{B\,e^{2|y|}}{\sqrt{T-t_2}}\,(t_2-t_1)\Bigg]\\
&\leqslant \frac{2}{h}\left[\frac{A\,e^{2|y|+h}}{\sqrt{T-t_2}}\,\frac{h^2}{2}+\frac{B\,e^{2|y|+2h}}{\sqrt{T-t_2}}\,(t_2-t_1)\right], \qquad (2.32)
\end{aligned}
$$

where $0\leqslant t_1\leqslant t_2<T,\ -\infty<y<\infty$, and $h>0$.

Choosing $h=C^*(t_2-t_1)^{\frac{1}{2}}$, where C^* is an arbitrary positive constant, then the last inequality (2.32) takes the form

$$
\begin{aligned}
|\gamma(t_2,y)-\gamma(t_1,y)| &\leqslant \frac{2e^{2|y|}}{\sqrt{T-t_2}}\left[\frac{A\,C^*}{2}e^{C^*\sqrt{t_2-t_1}}+\frac{B}{C^*}\,e^{2C^*\sqrt{t_2-t_1}}\right](t_2-t_1)^{\frac{1}{2}}\\
&\leqslant \frac{2e^{2|y|}}{\sqrt{T-t_2}}\left[\frac{A\,C^*}{2}e^{C^*\sqrt{T}}+\frac{B}{C^*}\,e^{2C^*\sqrt{T}}\right](t_2-t_1)^{\frac{1}{2}},
\end{aligned}
$$

the minimum of the above bound is attained at the point $C^*=\sqrt{\frac{2B}{A}}$. Thus we arrive to our required bound with constant $F=2\sqrt{2AB}\cdot e^{2\sqrt{\frac{2BT}{A}}}$. □

3. Proof of the main results

This section deals with the proof of our main results stated in Section 1.

From the relation (2.7) between the value functions $v(t,x)$ and $u(t,y)$ we can write

$$
\frac{\partial v(t,x)}{\partial x}=\frac{1}{x}\frac{\partial u(t,\ln x)}{\partial y},\ x>0,\ 0\leqslant t\leqslant T. \qquad (3.1)
$$

As the partial derivative $\frac{\partial u(t,y)}{\partial y}$, $0\leqslant t<T$, $-\infty<y<\infty$ is continuous with respect to the pair of arguments (t,y), the relation (3.1) implies that the partial derivative $\varphi(t,x)=\frac{\partial v(t,x)}{\partial x}$ is also continuous with respect to the pair of arguments (t,x), $0\leqslant t<T$, $x>0$.

Moreover, using the bound (2.26), equality (3.1) yields

$$
|x\,\varphi(t,x)|\leqslant D\,e^{2|\ln x|},\ 0\leqslant t<T,\ x>0. \qquad (3.2)
$$

Proof of Proposition 1. Using the relation (3.1) we can express

$$
\begin{aligned}
&E\int_0^T (D(t)-D_\Delta(t))^2S_t^2dt \qquad (3.3)\\
&=E\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\left(\varphi(t,S_t)-\varphi(t_{k-1},S_{t_{k-1}})\right)^2S_t^2dt
\end{aligned}
$$

$$= E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\frac{1}{S_t}\frac{\partial u(t,\ln S_t)}{\partial y}-\frac{1}{S_{t_{k-1}}}\frac{\partial u(t_{k-1},\ln S_{t_{k-1}})}{\partial y}\right)^2 S_t^2 dt$$

$$= E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\gamma(t,\ln S_t)-\gamma(t_{k-1},\ln S_{t_{k-1}})\frac{S_t}{S_{t_{k-1}}}\right)^2 dt. \tag{3.4}$$

Square of the difference in the last integral can be bounded as

$$\left(\gamma(t,\ln S_t)-\gamma(t_{k-1},\ln S_{t_{k-1}})\ \frac{S_t}{S_{t_{k-1}}}\right)^2$$
$$\leqslant 3\left(\gamma(t,\ln S_t)-\gamma(t_{k-1},\ln S_t)\right)^2+3\left(\gamma(t_{k-1},\ln S_t)-\gamma(t_{k-1},\ln S_{t_{k-1}})\right)^2$$
$$+3\gamma^2(t_{k-1},\ln S_{t_{k-1}})\ \frac{\left(S_{t_{k-1}}-S_t\right)^2}{S_{t_{k-1}}^2}.$$

Thus the expectation on the left-hand side of equality (3.3) is bounded above as

$$E\int_0^T (D(t)-D_\Delta(t))^2 S_t^2 dt \tag{3.5}$$
$$\leqslant 3E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\gamma(t,\ln S_t)-\gamma(t_{k-1},\ln S_t)\right)^2 dt$$
$$+3E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\gamma(t_{t_{k-1}},\ln S_t)-\gamma(t_{k-1},\ln S_{t_{k-1}})\right)^2 dt$$
$$+3E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\gamma^2(t_{k-1},\ln S_{t_{k-1}})\frac{\left(S_{t_{k-1}}-S_t\right)^2}{S_{t_{k-1}}^2}dt. \tag{3.6}$$

To bound the first expectation on the right-hand side of the above inequality we apply Proposition 7 and use the bounds (2.13), (2.26) and obtain

$$E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\gamma(t,\ln S_t)-\gamma(t_{k-1},\ln S_t)\right)^2 dt$$
$$\leqslant E\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{F^2e^{4|\ln S_t|}}{T-t}\ (t-t_{k-1})dt+4D^2E\int_{t_{n-1}}^{T}e^{2|\ln S_t|}dt$$
$$\leqslant F^2\Delta\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{E\left(S_t^4+S_t^{-4}\right)}{T-t}dt+4D^2\int_{t_{n-1}}^{T}E\left(S_t^2+S_t^{-2}\right)dt$$
$$\leqslant F^2\Delta\int_0^{t_{n-1}}\frac{E\left(S_t^4+S_t^{-4}\right)}{T-t}dt+4D^2\int_{t_{n-1}}^{T}E\left(S_t^2+S_t^{-2}\right)dt. \tag{3.7}$$

Using the solution of stochastic differential equation (1.10) we obtain

$$E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\gamma(t,\ln S_t)-\gamma(t_{k-1},\ln S_t)\right)^2 dt$$
$$\leqslant F^2\Delta\, e^{2(2\bar{\mu}+5\bar{\sigma}^2)T}\left(S_0^4+S_0^{-4}\right)\int_0^{t_{n-1}}\frac{1}{T-t}dt+4D^2\Delta\, e^{2(2\bar{\mu}+3\bar{\sigma}^2)T}\left(S_0^2+S_0^{-2}\right)$$

$$\leqslant \left(F^2\left(S_0^4+S_0^{-4}\right)+4D^2\left(S_0^2+S_0^{-2}\right)\right)e^{2(2\bar{\mu}+5\bar{\sigma}^2)T}\Delta\cdot\ln\frac{T}{\Delta}, \tag{3.8}$$

where we have used the obvious inequality $1 \leqslant 2\ln 2 \leqslant 2\ln\frac{T}{\Delta}$.

To estimate the second expectation on the right-hand side of inequality (3.5) we use the bound (2.31) and get

$$\begin{aligned}
& E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\gamma(t_{t_{k-1}},\ln S_t)-\gamma(t_{k-1},\ln S_{t_{k-1}})\right)^2 dt \\
\leqslant\ & A^2 E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{1}{T-t_{k-1}}\, e^{2|\ln S_t|+2|\ln S_{t_{k-1}}|}\left(\ln S_t-\ln S_{t_{k-1}}\right)^2 dt \\
=\ & A^2 E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{1}{T-t_{k-1}}\,\left(S_t^2+S_t^{-2}\right)\left(S_{t_{k-1}}^2+S_{t_{k-1}}^{-2}\right) \\
& \times\left(\int_{t_{k-1}}^{t}\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+\int_{t_{k-1}}^{t}\sigma(v)dW_v\right)^2 dt \\
\leqslant\ & 4A^2\Delta\left(\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)^2+\bar{\sigma}^2\right)e^{2\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)\Delta}E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{e^{2\left|\int_{t_{k-1}}^{t}\sigma(v)dW_v\right|}}{T-t_{k-1}}\left(S_{t_{k-1}}^4+S_{t_{k-1}}^{-4}\right)dt.
\end{aligned}$$

Let us estimate explicitly the expectation in the above inequality

$$\begin{aligned}
& E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{e^{2\left|\int_{t_{k-1}}^{t}\sigma(v)dW_v\right|}}{T-t_{k-1}}\left(S_{t_{k-1}}^4+S_{t_{k-1}}^{-4}\right)dt \\
=\ & \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{1}{T-t_{k-1}}E\left(S_{t_{k-1}}^4+S_{t_{k-1}}^{-4}\right)Ee^{2\left|\int_{t_{k-1}}^{t}\sigma(v)dW_v\right|}dt \\
\leqslant\ & 2e^{2\bar{\sigma}^2\Delta}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{1}{T-t_{k-1}}E\left(S_{t_{k-1}}^4+S_{t_{k-1}}^{-4}\right)dt \\
& +2e^{2\bar{\sigma}^2\Delta}\int_{t_{n-1}}^{T}\frac{1}{T-t_{n-1}}\,E\left(S_{t_{n-1}}^4+S_{t_{n-1}}^{-4}\right)dt \\
=\ & 2e^{2\bar{\sigma}^2\Delta}\int_0^{t_{n-1}}\frac{1}{T-t}\,E\left(S_t^4+S_t^{-4}\right)dt+4e^{6(\bar{\mu}+2\bar{\sigma}^2)T}\left(S_0^4+S_0^{-4}\right) \\
\leqslant\ & 8e^{6(\bar{\mu}+2\bar{\sigma}^2)T}\left(S_0^4+S_0^{-4}\right)\,\ln\frac{T}{\Delta},
\end{aligned}$$

this bound leads to

$$\begin{aligned}
& E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\gamma(t_{t_{k-1}},\ln S_t)-\gamma(t_{k-1},\ln S_{t_{k-1}})\right)^2 dt \\
\leqslant\ & 32A^2(S_0^4+S_0^{-4})\left(\left(\bar{\mu}+\frac{\bar{\sigma}^2}{2}\right)^2+\bar{\sigma}^2\right)e^{(8\bar{\mu}+13\bar{\sigma}^2)T}\,\Delta\,\ln\frac{T}{\Delta}.
\end{aligned} \tag{3.9}$$

To complete the proof it remains to bound the third expectation in (3.5). For this purpose we use the bound (2.26) on $\gamma(t,y)$ and write

$$
\begin{aligned}
& E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\gamma^2(t_{k-1},\ln S_{t_{k-1}})\ \frac{\left(S_{t_{k-1}}-S_t\right)^2}{S_{t_{k-1}}^2}dt \\
\leqslant\ & D^2 E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} e^{4|\ln S_{t_{k-1}}|}\frac{\left(S_{t_{k-1}}-S_t\right)^2}{S_{t_{k-1}}^2}dt \\
\leqslant\ & D^2\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} E\left(S_{t_{k-1}}^4+S_{t_{k-1}}^{-4}\right)E\left(1-e^{\int_{t_{k-1}}^t\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+\int_{t_{k-1}}^t\sigma(v)dW_v}\right)^2 dt \\
\leqslant\ & 2D^2\left(S_0^4+S_0^{-4}\right)e^{(4\bar{\mu}+10\bar{\sigma}^2)T}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} E\Bigg(e^{2\int_{t_{k-1}}^t\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+2\int_{t_{k-1}}^t\sigma(v)dW_v} \\
& -2e^{\int_{t_{k-1}}^t\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+\int_{t_{k-1}}^t\sigma(v)dW_v}+1\Bigg)dt \\
=\ & 2D^2\left(S_0^4+S_0^{-4}\right)e^{(4\bar{\mu}+10\bar{\sigma}^2)T}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(e^{\int_{t_{k-1}}^t\left(2\mu(v)+\sigma^2(v)\right)dv}-2e^{\int_{t_{k-1}}^t\mu(v)dv}+1\right)dt \\
=\ & 2D^2\left(S_0^4+S_0^{-4}\right)e^{(4\bar{\mu}+10\bar{\sigma}^2)T}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(e^{\int_{t_{k-1}}^t\left(2\mu(v)+\sigma^2(v)\right)dv}-1-2e^{\int_{t_{k-1}}^t\mu(v)dv}+2\right)dt.
\end{aligned}
$$

Mean value theorem leads us

$$
\begin{aligned}
& E\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\gamma^2(t_{k-1},\ln S_{t_{k-1}})\ \frac{\left(S_{t_{k-1}}-S_t\right)^2}{S_{t_{k-1}}^2}dt \\
\leqslant\ & 2D^2\left(S_0^4+S_0^{-4}\right)e^{(4\bar{\mu}+10\bar{\sigma}^2)T}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\Bigg(\int_{t_{k-1}}^{t}\left(2\mu(v)+\sigma^2(v)\right)dv \\
& \times e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)\Delta}+2\int_{t_{k-1}}^{t}\delta(v)dv\Bigg)dt \\
\leqslant\ & 2D^2\left(S_0^4+S_0^{-4}\right)e^{(4\bar{\mu}+10\bar{\sigma}^2)T}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\left(2\bar{\mu}+\bar{\sigma}^2\right)e^{(2\bar{\mu}+\bar{\sigma}^2)\Delta}+2\bar{\delta}\right)(t-t_{k-1})dt \\
\leqslant\ & 2D^2T\left(4\bar{\mu}+\bar{\sigma}^2\right)\left(S_0^4+S_0^{-4}\right)e^{(6\bar{\mu}+11\bar{\sigma}^2)T}\ \Delta. \qquad (3.10)
\end{aligned}
$$

Proposition 1 follows by inserting the bounds (3.8), (3.9) and (3.10) in the inequality (3.5). □

Let us explicitly calculate the conditional expectation

$$
\begin{aligned}
E\left[S_t^2\middle|S_{t_{k-1}}\right] &= S_{t_{k-1}}^2\ E\left[\frac{S_t^2}{S_{t_{k-1}}^2}\middle|S_{t_{k-1}}\right] \\
&= S_{t_{k-1}}^2\ E\left[\exp\left(2\int_{t_{k-1}}^t\left(\mu(v)-\frac{\sigma^2(v)}{2}\right)dv+2\int_{t_{k-1}}^t\sigma(v)dW_v\right)\middle|S_{t_{k-1}}\right]
\end{aligned}
$$

$$= S_{t_{k-1}}^2\ E\left[\exp\left(2\int_{t_{k-1}}^t \sigma(v)dW_v - \frac{1}{2}\int_{t_{k-1}}^t (2\sigma(v))^2 dv\right)\right.$$
$$\left.\times \exp\left(\int_{t_{k-1}}^t (2\mu(v)+\sigma^2(v))dv\right)\right]$$
$$= S_{t_{k-1}}^2\ \exp\left(\int_{t_{k-1}}^t (2\mu(v)+\sigma^2(v))dv\right), \tag{3.11}$$

where $t_{k-1} \leqslant t \leqslant t_k, k = 1,2,...,n$.

Proof of Proposition 2. From the obvious identity

$$D_{\Delta,h}(u) - D(u) = D_{\Delta,h}(u) - D_\Delta(u) + D_\Delta(u) - D(u),\ 0 \leqslant u \leqslant T,$$

we can write

$$E\int_0^T (D_{\Delta,h}(u) - D(u))^2 S_u^2 du$$
$$\leqslant 2E\int_0^T (D_{\Delta,h}(u) - D_\Delta(u))^2 S_u^2 du + 2E\int_0^T (D_\Delta(u) - D(u))^2 S_u^2 du,\ 0 \leqslant u \leqslant T.$$

We have to bound the first expectation on the right-hand side of the above inequality. Let us express

$$E\int_0^T (D_{\Delta,h}(u) - D_\Delta(u))^2 S_u^2 du$$
$$= E\sum_{k=1}^n \int_{t_{k-1}}^{t_k} (D_{\Delta,h}(u) - D_\Delta(u))^2 S_u^2 du$$
$$= E\int_0^{t_1} (0 - \varphi(0,S_0))^2 S_u^2 du$$
$$+ \sum_{k=2}^n \int_{t_{k-1}}^{t_k} E\left[\left(\varphi_h(t_{k-1},S_{t_{k-1}}) - \varphi(t_{k-1},S_{t_{k-1}})\right)^2 E\left\{S_u^2 | S_{t_{k-1}}\right\}\right] du$$
$$= \varphi^2(0,S_0)\int_0^{t_1} E\left(S_0^2\ e^{2\int_0^u\left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv + 2\int_0^u \sigma(v)dW_v}\right) du$$
$$+ \sum_{k=2}^n \int_{t_{k-1}}^{t_k} E\left[\left(\varphi_h(t_{k-1},S_{t_{k-1}}) - \varphi(t_{k-1},S_{t_{k-1}})\right)^2 E\left\{S_u^2 | S_{t_{k-1}}\right\}\right] du$$
$$\leqslant D^2\ e^{4|\ln S_0| + \left(2\bar{\mu} + \bar{\sigma}^2\right)\Delta} \cdot \Delta$$
$$+ \sum_{k=2}^n \int_{t_{k-1}}^{t_k} \exp\left(\int_{t_{k-1}}^u (2\mu(v)+\sigma^2(v))dv\right)$$
$$\times E\left[\left(\varphi_h(t_{k-1},S_{t_{k-1}}) - \varphi(t_{k-1},S_{t_{k-1}})\right)^2 S_{t_{k-1}}^2\right] du,$$

where we have used the bound (3.2) and expression (3.11).

Note that explicit form of the density of probability distribution of the random variable S_t can be expressed as

$$f(S_0,t;x) = \frac{1}{\sqrt{2\pi}\left(\int_0^t \sigma^2(v)dv\right)^{\frac{1}{2}} x} \, e^{\frac{-1}{2\int_0^t \sigma^2(v)dv}\left[\ln \frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right]^2},$$

where $S_0 > 0,\ 0 < x < \infty,\ 0 < t \leqslant T$.

Thus we can write

$$\begin{aligned}
&E\int_0^T (D_{\Delta,h}(u) - D_\Delta(u))^2 S_u^2 du \\
&\leqslant D^2\, e^{4|\ln S_0| + \left(2\bar{\mu}+\bar{\sigma}^2\right)\Delta} \cdot \Delta \\
&\quad + e^{\left(2\bar{\mu}+\bar{\delta}^2\right)\Delta} \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} \int_0^\infty (\varphi_h(t_{k-1},x) - \varphi(t_{k-1},x))^2 H(S_0,t_{k-1};x)dxdt, \qquad (3.12)
\end{aligned}$$

where explicit form of the weight function $H(S_0,t;x)$ is given as

$$H(S_0,t;x) = \frac{x}{\sqrt{2\pi}\left(\int_0^t \sigma^2(v)dv\right)^{\frac{1}{2}}} \, e^{\frac{-1}{2\int_0^t \sigma^2(v)dv}\left[\ln \frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right]^2},\ S_0 > 0,\ 0 < x < \infty.$$

Its first and second derivatives are given by

$$\begin{aligned}
\frac{\partial H(S_0,t;x)}{\partial x} &= \frac{1}{\sqrt{2\pi}\left(\int_0^t \sigma^2(v)dv\right)^{\frac{1}{2}}}\left\{1 - \frac{1}{\int_0^t \sigma^2(v)dv}\left[\ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right]\right\} \\
&\quad \times e^{\frac{-1}{2\int_0^t \sigma^2(v)dv}\left[\ln \frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right]^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 H(S_0,t;x)}{\partial x^2} &= \frac{1}{\sqrt{2\pi}\left(\int_0^t \sigma^2(v)dv\right)^{\frac{3}{2}} x}\left\{ - \left[\ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right]\right. \\
&\quad \left. + \frac{1}{\int_0^t \sigma^2(v)dv}\left[\ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right]^2 - 1\right\} \\
&\quad \times e^{\frac{-1}{2\int_0^t \sigma^2(v)dv}\left[\ln \frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv\right]^2}.
\end{aligned}$$

Here we note that

$$\begin{aligned}
&\lim_{x\to 0+} H(S_0,t;x) = 0,\quad \lim_{x\to\infty} H(S_0,t;x) = 0, \\
&\lim_{x\to 0+} \frac{\partial H(S_0,t;x)}{\partial x} = 0,\quad \lim_{x\to\infty} x\,\frac{\partial H(S_0,t;x)}{\partial x} = 0,
\end{aligned} \qquad (3.13)$$

for $S_0 > 0,\ 0 < t \leqslant T$. And

$$
\begin{aligned}
&\int_0^\infty (|v(t,x)| + |\breve{v}_h(t,x)|) \left| \frac{\partial^2 H(S_0,t;x)}{\partial x^2} \right| dx \\
&\leqslant 2\int_0^\infty \frac{v(t,x)+C_1 h}{\sqrt{2\pi}\left(\int_0^t \sigma^2(v)dv\right)^{\frac{3}{2}} x} \left\{ \left| \ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right) dv \right| \right. \\
&\quad \left. + \frac{1}{\int_0^t \sigma^2(v)dv} \left| \ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right) dv \right|^2 + 1 \right\} \\
&\quad \times e^{\frac{-1}{2\int_0^t \sigma^2(v)dv}\left[\ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right) dv\right]^2} dx \\
&\leqslant 2\int_0^\infty \frac{C\left(x\, e^{T(\bar{\mu}+\bar{\sigma}^2)}+1\right)+g(1)+C_1 h}{\sqrt{2\pi}\left(\int_0^t \sigma^2(v)dv\right)^{\frac{3}{2}} x} \left\{ \left| \ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right) dv \right| \right. \\
&\quad \left. + \frac{1}{\int_0^t \sigma^2(v)dv} \left| \ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right) dv \right|^2 + 1 \right\} \\
&\quad \times e^{\frac{-1}{2\int_0^t \sigma^2(v)dv}\left[\ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right) dv\right]^2} dx,
\end{aligned}
$$

where we have used relation (2.7) and the bound (2.29).

By the change of variable $y = \frac{1}{\left(\int_0^t \sigma^2(v)dv\right)^{\frac{1}{2}}} \left[\ln\frac{x}{S_0} - \int_0^t \left(\mu(v) - \frac{\sigma^2(v)}{2}\right) dv\right]$, we obtain

$$
\begin{aligned}
&\int_0^\infty (|v(t,x)| + |\breve{v}_h(t,x)|) \left| \frac{\partial^2 H(S_0,t;x)}{\partial x^2} \right| dx \\
&\leqslant 4CS_0\, e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T} \left\{ 1 + \frac{1}{\int_0^t \sigma^2(v)dv} \right\} \\
&\quad +2(C+g(1)+C_1 h) \left\{ \frac{1}{\left(\int_0^t \sigma^2(v)dv\right)^{\frac{1}{2}}} + \frac{2}{\int_0^t \sigma^2(v)dv} \right\},
\end{aligned}
\tag{3.14}
$$

for $S_0 > 0,\ 0 < t \leqslant T$.

Fix $(t,x) \in [0,T] \times [0,\infty)$ and let $S_\tau(t,x)$ be given by (2.3). We have

$$
\begin{aligned}
|v(t,x) - v(t,y)| &\leqslant \sup_{\tau \in \mathscr{T}_{t,T}} E\left[|g(S_\tau(t,x) - g(S_\tau(t,y))|\right] \\
&\leqslant C\,|x-y| \sup_{\tau \in \mathscr{T}_{t,T}} E e^{\int_t^\tau \left(\mu(v) - \frac{\sigma^2(v)}{2}\right)dv + \int_t^\tau \sigma(v)dW_v} \\
&\leqslant C\,|x-y|\, e^{\bar{\mu}T},
\end{aligned}
$$

which shows that the left-hand derivative $\frac{\partial v(t,x-)}{\partial x}$ is finite.

The function $\varphi_h(t,x)$ is nondecreasing with respect to x, hence

$$\varphi_h(t,1)-\varphi_h(t,x) \text{ if } x\geqslant 1.$$

Denote $\tilde{c}=e^{\bar{\mu}T}$, then

$$\left|\frac{\partial v(t,x-)}{\partial x}\right|\leqslant \tilde{c}.$$

From here we have

$$v(t,x)-v(t,1)\leqslant \tilde{c}(x-1),\ x\geqslant 1.$$

Since $\breve{v}_h(t,x)$ is the lower convex envelope of $v_h(t,x)$, from the last inequality we get the estimate

$$\breve{v}_h(t,x)\leqslant \tilde{c}x+\tilde{d},\ \text{ if } x\geqslant 1.$$

Fix x_0, $x_0\geqslant 1$. The tangent line $\breve{v}_h(t,x_0)+\varphi_h(t,x_0)(x-x_0)$ is below the convex function $\breve{v}_h(t,x)$ and hence below the straight line $\tilde{c}x+d$, if $x\geqslant x_0$. But this is possible only if

$$\varphi_h(t,x_0)\leqslant \tilde{c}.$$

Hence we have

$$\varphi_h(t,1)\leqslant \varphi_h(t,x)\leqslant \tilde{c},\text{ if } x\geqslant 1.$$

Thus applying the weighted square integral inequality Proposition 3, we write

$$\begin{aligned}
&\int_0^\infty (\varphi_h(t_{k-1},x)-\varphi(t_{k-1},x))^2 H(S_0,t_{k-1};x)dx\\
&\leqslant \frac{3}{2}\sup_{x\geqslant 0}|v_h(t_{k-1},x)-v(t_{k-1},x)|\Bigg[4CS_0\, e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T}\left\{1+\frac{1}{\int_0^{t_{k-1}}\sigma^2(v)dv}\right\}\\
&\quad +2(C+g(1)+C_1h)\left\{\frac{1}{\left(\int_0^{t_{k-1}}\sigma^2(v)dv\right)^{\frac{1}{2}}}+\frac{2}{\int_0^{t_{k-1}}\sigma^2(v)dv}\right\}\Bigg]\\
&\leqslant \frac{3}{2}\,C_1h\left[4CS_0\, e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T}\left\{1+\frac{1}{\underline{\sigma}^2 t_{k-1}}\right\}+2\left(C+g(1)+C_1h\right)\left\{\frac{1}{\underline{\sigma}\sqrt{t_{k-1}}}+\frac{2}{\underline{\sigma}^2 t_{k-1}}\right\}\right].
\end{aligned}$$

Thus inequality (3.12) takes the form

$$\begin{aligned}
&E\int_0^T (D_{\Delta,h}(u)-D_\Delta(u))^2 S_u^2 du\\
&\leqslant D^2\, e^{4|\ln S_0|+\left(2\bar{\mu}+\bar{\sigma}^2\right)}\cdot\Delta+\frac{3}{2}\,C_1h\, e^{\left(2\bar{\mu}+\bar{\delta}^2\right)}\Bigg[4CS_0\, e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T}\left\{1+\frac{1}{\underline{\sigma}^2}\right\}\\
&\quad +2(C+g(1)+C_1h)\left\{\frac{\sqrt{\Delta}}{\underline{\sigma}}+\frac{2}{\underline{\sigma}^2}\right\}+\sum_{k=3}^n\int_{t_{k-2}}^{t_{k-1}}\Bigg\{4CS_0\, e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T}\left(1+\frac{1}{\underline{\sigma}^2 t_{k-1}}\right)\\
&\quad +2(C+g(1)+C_1h)\left(\frac{1}{\underline{\sigma}\sqrt{t_{k-1}}}+\frac{2}{\underline{\sigma}^2 t_{k-1}}\right)\Bigg\}\Bigg]dt.
\end{aligned}$$

It is easy to find

$$\sum_{k=3}^{n}\int_{t_{k-2}}^{t_{k-1}}\left\{4CS_0\,e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T}\left(1+\frac{1}{\underline{\sigma}^2 t_{k-1}}\right)+2(C+g(1)+C_1h)\left(\frac{1}{\underline{\sigma}\sqrt{t_{k-1}}}+\frac{2}{\underline{\sigma}^2 t_{k-1}}\right)\right\}dt$$
$$\leqslant \int_{t_1}^{T}\left\{4CS_0\,e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T}\left(1+\frac{1}{\underline{\sigma}^2 t}\right)+2(C+g(1)+C_1h)\left(\frac{1}{\underline{\sigma}\sqrt{t}}+\frac{2}{\underline{\sigma}^2 t}\right)\right\}dt$$
$$=\left\{4CS_0e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T}\left(1+\frac{1}{\underline{\sigma}^2}\ln\frac{T}{\Delta}\right)+2(C+g(1)+C_1h)\left(\frac{2\left(\sqrt{T}-\sqrt{\Delta}\right)}{\underline{\sigma}}+\frac{2}{\underline{\sigma}^2}\ln\frac{T}{\Delta}\right)\right\}.$$

Thus we get

$$\begin{aligned}
&E\int_0^T (D_{\Delta,h}(u)-D_\Delta(u))^2 S_u^2 du \\
&\leqslant D^2\,e^{4|\ln S_0|+\left(2\bar{\mu}+\bar{\sigma}^2\right)}\cdot\Delta \\
&\quad+\frac{3}{2}\,C_1h\,e^{\left(2\bar{\mu}+\bar{\delta}^2\right)}\Bigg[4CS_0\,e^{\left(2\bar{\mu}+\bar{\sigma}^2\right)T}\left\{2+\frac{1}{\underline{\sigma}^2}\left(\ln\frac{T}{\Delta}+1\right)\right\} \\
&\quad+2(C+g(1)+C_1h)\left\{\frac{2\sqrt{T}}{\underline{\sigma}}+\frac{2}{\underline{\sigma}^2}\left(\ln\frac{T}{\Delta}+1\right)\right\}\Bigg] \\
&\leqslant d\cdot\ln\frac{T}{\Delta}\,(\Delta+h),
\end{aligned} \tag{3.15}$$

where d is some positive constant depending on parameters $\bar{r}$, $\bar{\delta}$, $\bar{\sigma}$, $\underline{\sigma}$, C_1, T, $g(1)$, K, C and S_0.

Use of Proposition 1 and the bound (3.15) in the estimate (1.14) completes the proof. □

Proof of Proposition 3. Introduce the notation

$$\sup_{x\geqslant 0}|F_h(x)-F(x)|=\alpha_h,$$

it follows that

$$F(x)-\alpha_h\leqslant F_h(x),\ F_h(x)-\alpha_h\leqslant F(x),\ \text{if } x\geqslant 0.$$

Since $\breve{F}_h(x)$ is the lower convex envelope of $F_h(x)$, this implies

$$F(x)-\alpha_h\leqslant \breve{F}_h(x), x\geqslant 0.$$

On the other side

$$\breve{F}_h(x)-\alpha_h\leqslant F_h(x)-\alpha_h\leqslant F(x), x\geqslant 0,$$

therefore we can write

$$|\breve{F}_h(x)-F(x)|\leqslant \alpha_h, x\geqslant 0.$$

Thus we find

$$\sup_{x\geqslant 0}|\breve{F}_h(x)-F(x)| \leqslant \sup_{x\geqslant 0}|F_h(x)-F(x)|. \tag{3.16}$$

We have from condition (1.5)

$$|F'(x-)| \leqslant C,$$

that is

$$-C \leqslant F'(x-) \leqslant C.$$

After integraton we get

$$F(0)-Cx \leqslant F(x) \leqslant F(0)+Cx,\ x \geqslant 0,$$

hence

$$|F(x)| \leqslant |F(0)|+C.$$

On the other hand

$$F(x)-\alpha_h \leqslant F_h(x) \leqslant F(x)+\alpha_h,$$

therefore, we obtain

$$F(0)-\alpha_h-Cx \leqslant F_h(x) \leqslant F(0)+\alpha_h+Cx.$$

Similarly

$$F(0)-\alpha_h-Cx \leqslant \breve{F}_h(x) \leqslant F(0)+\alpha_h+Cx.$$

Thus

$$|\breve{F}_h(x)| \leqslant |F(0)|+\alpha_h+Cx.$$

From the condition (1.20) we get

$$\int_0^\infty \left(|\breve{F}_h(x)|+|F(x)|\right)|H''(x)|dx < \infty.$$

The left-derivative $\breve{F}_h'(x-)$ is nondecreasing, hence

$$\breve{F}_h'(1-) \leqslant \breve{F}_h'(x-) \ \text{ if } \ x \geqslant 1.$$

Fix $x_0,\ x_0 \geqslant 1$. The tangent line $\breve{F}_h'(x_0)+\breve{F}_h'(x_0-)(x-x_0)$ is below the convex function $\breve{F}_h(x),\ x \geqslant x_0$ and hence below the line

$$F(0)+\alpha_h+Cx,\ x \geqslant x_0.$$

But this is possible only if

$$\breve{F}_h'(x_0-) \leqslant C.$$

Hence we have

$$\breve{F}_h'(1-) \leqslant \breve{F}_h'(x_0-) \text{ if } x \geqslant 1.$$

Taking now $\breve{F}_h(x)$ instead of the convex function $f(x)$ in Theorem 3.1 in Hussain, Pečarić and Shashiashvili [5] and using the bound (3.16), we obtain

$$\int_0^\infty \left(\breve{F}_h'(x-)-F'(x-)\right)^2 H(x)dx$$
$$\leqslant \frac{3}{2}\sup_{x\geqslant 0}|F_h(x)-F(x)| \int_0^\infty \left(|\breve{F}_h(x)|+|F(x)|\right)\left|H''(x)\right|dx.$$

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