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TWO FAMILIES OF CYCLIC INEQUALITIES

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Abstract. In this paper two families of cyclic inequalities in three variables are studied. More precisely, necessary and sufficient conditions in order that the cyclic sums $\sum x^m(x+y)^n$ and $\sum s_\alpha(x)s_\beta(x+y)$ are non-negative are stated and proven. Here m,n are positive integers, α,β are positive real numbers and $s_\alpha(x) = x|x|^{\alpha-1}, x \in \mathbb{R}^*, s_\alpha(0) = 0$.

1. Introduction

Cyclic inequalities that are not symmetric represent a domain that has been less studied in the literature. The reason is that inequalities of this class are usually more difficult to be proven than the classical inequalities that involve symmetric functions. For references on cyclic inequalities see ([1]–[10]). One of the inequalities that have initiated a great interest in the study of cyclic inequalities was published by Nesbitt in 1903 [6]. It is a special case ($n = 3$) of the celebrated inequality conjectured by H. S. Shapiro in 1954 ([1], [4]). It states that for $n \geqslant 3$

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geqslant \frac{n}{2},$$

$x_i \geqslant 0, x_{i+1}+x_{i+2} > 0$ and $x_{i+n} = x_i$ for all $i \in \mathbb{N}$. Shapiro's inequality is true for even $n \leqslant 12$ and for odd $n \leqslant 23$. For other natural values of n the inequality is false. In this paper, our goal is to study conditions under which two families of two parameter cyclic sums in three variables are non-negative or fails to be non-negative on $\mathbb{R}^3$. It will be proven that the cyclic sums are non-negative or fails to be non-negative on $\mathbb{R}^3$ for some strange regions in the spaces of parameters. This behavior is similar somewhat with that of Shapiro's inequality. Analytical proofs will be combined with computer experiments in order to obtain the results.

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2. The study of $g_{\alpha,\beta}$

For every positive numbers α,β we consider the function $g_{\alpha,\beta}:\mathbb{R}^3\to\mathbb{R}$ defined by

$$g_{\alpha,\beta}(x,y,z)=s_\alpha(x)s_\beta(x+y)+s_\alpha(y)s_\beta(y+z)+s_\alpha(z)s_\beta(z+x),\quad (x,y,z)\in\mathbb{R}^3$$

where we denoted $s_\alpha(x)=x|x|^{\alpha-1}, x\in\mathbb{R}^*, s_\alpha(0)=0$.

Note that the functions s_α, $(\alpha>0)$ are increasing odd functions as can be easily checked.

LEMMA 2.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a real odd increasing function such that $f(x+y)\geqslant f(x)+f(y)$ for all $x,y\geqslant 0$. Then, $\sum\limits_{cyclic} f(x^2+xy)\geqslant 0$, for all $x,y,z\in\mathbb{R}$.*

Proof. Taking into account that f is odd and increasing, we have that if $x,y,z\geqslant 0$ or $x,y,z\leqslant 0$, then $\sum\limits_{cyclic} f(x^2+xy)\geqslant 0$. Suppose that $x\geqslant 0$ and $y,z\leqslant 0$. Putting $x=a, y=-b, z=-c$ we have $a,b,c\geqslant 0$ and $\sum\limits_{cyclic} f(x^2+xy)\geqslant 0$ becomes

$$f(a^2-ab)+f(b^2+bc)+f(c^2-ca)\geqslant 0,\ a,b,c\geqslant 0 \tag{2.1}$$

To prove the above inequality we distinguish the following cases:

(1) If $c\geqslant a\geqslant b\geqslant 0$ then $a^2-ab\geqslant 0$, $b^2+bc\geqslant 0$, $c^2-ca\geqslant 0$, and (2.1) holds.

(2) If $b\geqslant a\geqslant c\geqslant 0$ then $f(b^2+bc)\geqslant f(ab-a^2+ac-c^2)\geqslant f(ab-a^2)+f(ac-c^2)=-f(a^2-ab)-f(c^2-ac)$ from which (2.1) follows.

(3) If $a\geqslant c\geqslant b\geqslant 0$ then $a^2-ab+c^2-ca\geqslant 0$ that jointly with the fact that $b^2+bc\geqslant 0$ imply (2.1).

(4) Let $a\geqslant b\geqslant c\geqslant 0$. Suppose that $a^2-ab+c^2-ca<0$ and $b^2+bc+c^2-ca\leqslant 0$. Then, $a^2+b^2+2c^2-ab-2ac+bc<0$ or equivalently, $(a-b)^2+(a-2c)^2+b^2+2bc<0$. Contradiction. Therefore, $a^2-ab+c^2-ca\geqslant 0$ or $b^2+bc+c^2-ca\geqslant 0$. In the first case we get $f(a^2-ab+c^2-ca)\geqslant 0$. Thus, $f(a^2-ab)+f(b^2+bc)+f(c^2-ca)\geqslant 0$. If $b^2+bc+c^2-ca\geqslant 0$, then $f(b^2+bc)+f(c^2-ca)+f(a^2-ab)\geqslant 0$. This proves that (2.1) holds.

(5) Let $c\geqslant b\geqslant a\geqslant 0$ or $b\geqslant c\geqslant a\geqslant 0$. Then $a^2-ab+b^2+bc\geqslant 0$ and $c^2-ca\geqslant 0$. This proves that (2.1) holds. □

THEOREM 2.2. *Let $\alpha\geqslant 1$ be a real number. Then $g_{\alpha,\alpha}$ is non-negative on $\mathbb{R}^3$, that is*

$$s_\alpha(x)s_\alpha(x+y)+s_\alpha(y)s_\alpha(y+z)+s_\alpha(z)s_\alpha(z+x)\geqslant 0,$$

holds for all $x,y,z\in\mathbb{R}$.

Proof. Note that the function s_α is multiplicative and the function $f=s_\alpha$ is supper-additive. Applying the preceding lemma we obtain

$$g_{\alpha,\alpha}(x,y,z)=\sum_{cyclic} s_\alpha(x^2+xy)=\sum_{cyclic} s_\alpha(x)s_\alpha(x+y)\geqslant 0$$

for all $x,y,z \in \mathbb{R}$. This completes the proof. $\square$

THEOREM 2.3. *Let* α,β *be positive real numbers such that* $g_{\alpha,\beta}$ *is non-negative for all* $x,y,z \in \mathbb{R}$ *and let*

$$h(t,\alpha) = \frac{\ln\left(t^\alpha + \left(\frac{t-1}{2}\right)^\alpha\right)}{\ln\left(\frac{2(t-1)}{t+1}\right)},\ t \in (3,\infty)$$

$$\phi(\alpha) = \inf_{t>3} h(t,\alpha)$$

Then $\beta \leqslant \phi(\alpha)$.

Proof. We consider the case when $z = -(x+y)/2$. Since $g_{\alpha,\beta}$ is non-negative, then

$$g_{\alpha,\beta}\left(x,y,-\frac{x+y}{2}\right) = s_\alpha(x)s_\beta(x+y) + s_\alpha(y)s_\beta\left(\frac{y-x}{2}\right) + s_\alpha\left(\frac{x+y}{2}\right)s_\beta\left(\frac{y-x}{2}\right) \geqslant 0$$

for all $x,y,z \in \mathbb{R}$, or equivalently,

$$s_\beta\left(\frac{y-x}{2}\right)\left(s_\alpha(y) + s_\alpha\left(\frac{x+y}{2}\right)\right) \geqslant -s_\alpha(x)s_\beta(x+y)$$

for all $x,y \in \mathbb{R}$. So, for $x \neq 0, y \in \mathbb{R}, x \neq y$, we have

$$s_\alpha\left(-\frac{y}{x}\right) + s_\alpha\left(-\frac{x+y}{2x}\right) \geqslant s_\beta\left(\frac{2(x+y)}{y-x}\right)$$

Putting $t = -y/x$ where $x < 0 < y$ we have $t > 0$. Now, it is easy to obtain that for all $t > 3$, the following inequality holds:

$$\beta \leqslant \frac{\ln\left[s_\alpha(t) + s_\alpha\left(\frac{t-1}{2}\right)\right]}{\ln\left(\frac{2(t-1)}{t+1}\right)} = h(t,\alpha)$$

from which follows that $\beta \leqslant \phi(\alpha)$ as claimed. $\square$

LEMMA 2.4. *Let* x,y *be real numbers such that* $x \geqslant y \geqslant 1$. *Then*

$$x(x^3-y^3) + y(y^3+1) - 2x^3 + 2 \geqslant 0$$

Proof. Let $y \geqslant 1$ be a fixed real number. We consider the function $f : [y,+\infty) \to \mathbb{R}$ defined by $f(x) = x(x^3-y^3) + y(y^3+1) - 2x^3 + 2$. To prove that $f(x) > 0$ we observe that $f'(x) = 4x^3 - 6x^2 - y^3$ and $f''(x) = 12x(x-1) \geqslant 0$ for all $x \in [y,+\infty)$. Since $f'' \geqslant 0$ then f' is increasing in $[y,+\infty)$. That is, $f'(x) \geqslant f'(y) = 3y^2(y-2)$.

If $y \geqslant 2$ then $f'(y) \geqslant 0$ and therefore $f'(x) \geqslant f'(y) \geqslant 0$ for all $x \geqslant y$. So, f is increasing in $[y,+\infty)$ and $f(x) \geqslant f(y) \geqslant 0$ for all $x \in [y,+\infty)$. If $y \in [1,2]$ then

$f'(y) \leqslant 0$. Since f' is increasing in $[y,+\infty)$, then exists $x_0 \geqslant y \geqslant 1$ such that $f'(x_0) = 4x_0^3 - 6x_0^2 - y^3 = 0$. From $1 \leqslant y \leqslant x_0$ follows $1 \leqslant y^3 \leqslant x_0^3$ or $1 \leqslant 4x_0^3 - 6x_0^2 \leqslant x_0^3$. Therefore $3x_0^3 - 6x_0^2 = 3x_0^2(x_0 - 2) \leqslant 0$ and $x_0 \leqslant 2$. But $4x_0^3 - 6x_0^2 = y^3 > 0$ implies $4x_0 - 6 \geqslant 0$. Thus, $x_0 \in [3/2, 2]$.

Therefore, it will be suffice to prove that $f(x) \geqslant 0$, for all $x \in [3/2,2]$ when $4x^3 - 6x^2 = y^3$. That is, we have to prove that, for all $x \in [3/2,2]$, holds $y(4x^3 - 6x^2 + 1) \geqslant 3x^4 - 4x^3 - 2$. By raising to cube, the preceding inequality is equivalent to $y^3(4x^3 - 6x^2 + 1)^3 \geqslant (3x^4 - 4x^3 - 2)^3$ or

$$(4x^3 - 6x^2)(4x^3 - 6x^2 + 1)^3 \geqslant (3x^4 - 4x^3 - 2)^3, \; x \in [3/2,2] \qquad (2.2)$$

In order to prove inequality (2.2) denote $g(x) = (4x^3 - 6x^2)(4x^3 - 6x^2 + 1)^3$, $h(x) = (3x^4 - 4x^3 - 2)^3$, $x \in [3/2,2]$. Note that g and h are increasing functions on $[3/2,2]$. Indeed, $u(x) = 4x^3 - 6x^2, x \in [3/2,2]$ is increasing because $u'(x) = 12x^2 - 12x \geqslant 0, x \in [3/2,2]$. Since $u(x) \geqslant 0$, $x \in [3/2,2]$ then $g = u(u+1)^3$ is increasing. The function $v(x) = 3x^4 - 4x^3 - 2, x \in [3/2,2]$ is increasing because $v'(x) = 12x^3 - 12x^2 \geqslant 0$ for every $x \in [3/2,2]$. Therefore $h = v^3$ is increasing.

i	x_i	$g(x_i)$	$h(x_{i+1})$	$g(x_i) > h(x_{i+1})$
1	1.5	0.000000	$-5.6 \cdot 10^{-7}$	True
2	1.521739	0.349143	0.326456	True
3	1.566478	2.944544	2.924692	True
4	1.608217	10.66051	10.39077	True
5	1.645957	27.21668	26.82787	True
6	1.682696	59.82348	59.36183	True
7	1.719935	121.6644	120.61781	True
8	1.758674	237.1416	235.76013	True
9	1.800413	456.8968	453.03250	True
10	1.846152	882.7773	878.73940	True
11	1.897891	1749.6589	1740.51079	True
12	1.957130	3591.2176	2744.0000	True
13	2.000000			

Table 1. *In the preceding table are displayed the set of non-equidistant nodes* $x_1, x_2, \ldots, x_{12}, x_{13}$ *in the interval* $[3/2,2]$, *the values of* $g(x_i)$ *and* $h(x_{i+1})$ *and the logical variable* $g(x_i) > h(x_{i+1})$.

Consider the set of non-equidistant nodes $x_1, x_2, \ldots, x_{12}, x_{13}$ in the interval $[3/2,2]$ displayed in table 1. Note that $g(x) \geqslant g(x_i) \geqslant h(x_{i+1}) \geqslant h(x)$ for every $x \in [x_i, x_{i+1}]$ and $i \in \{1,2,\ldots,12\}$. Consequently inequality (2.2) is proven. □

LEMMA 2.5. *Let* $\alpha_0 = \inf\{\alpha > 0 : x^\alpha(x-y) + y^\alpha(y+1) - 2x + 2 \geqslant 0$ *for all* $x \geqslant y \geqslant 1\}$. *Then*

(i) $\alpha_0 \in \left(\frac{1}{4}, \frac{1}{3}\right]$

(ii) $s_\alpha(x)(x-y)+s_\alpha(y)(y+z)+2s_\alpha(z)(z-x) \geqslant 0$ *for all* $x \geqslant y \geqslant z \geqslant 0$ *and* $\alpha \geqslant \alpha_0$.

Proof. Let $h(\alpha,x,y,z) = s_\alpha(x)(x-y)+s_\alpha(y)(y+z)+2s_\alpha(z)(z-x)$ for $x,y,z \in \mathbb{R}$ and $\alpha > 0$. Assume that $\alpha_0 \leqslant 1/4$. Note that $h(\alpha,x,y,1) = x^\alpha(x-y)+y^\alpha(y+1)-2x+2$ and $h(\alpha,x,y,1) \geqslant 0$ for all $x \geqslant y \geqslant 1$. One can easily see that $h(\alpha_0,6,1,1) \leqslant h(1/4,6,1,1) < 0$. From the definition of α_0 it follows that $h(1/4,x,y,1) \geqslant 0$ for all $x \geqslant y \geqslant 1$. The contradiction we obtained shows that $\alpha_0 > 1/4$. To prove that $\alpha_0 \leqslant 1/3$ it is sufficient to prove that $h(1/3,x,y,1) = x^{1/3}(x-y)+y^{1/3}(y+1)-2x+2 \geqslant 0$ for $x \geqslant y \geqslant 1$. The preceding inequality is equivalent to $x(x^3-y^3)+y(y^3+1)-2x^3+2 \geqslant 0$ for $x \geqslant y \geqslant 1$ which holds by the previous Lemma. Thus the statement (i) is proved. Let $\alpha \geqslant \alpha_0$. Note that the inequality $h(\alpha,x,y,0) = x^\alpha(x-y)+y^{\alpha+1} \geqslant 0$ is valid for all $x \geqslant y \geqslant 0$. If $x \geqslant y \geqslant z > 0$ we set $a = x/z,\ b = y/z$. Note that $a \geqslant b \geqslant 1$ and $h(\alpha,x,y,z) = z^\alpha h(\alpha,x/z,y/z,1) = a^\alpha(a-b)+b^\alpha(b+1)-2a+2 \geqslant 0$. Thus the statement (ii) is proved. □

LEMMA 2.6. *Let* $f,g:\mathbb{R}\to\mathbb{R}$ *be real functions such that*
(i) Both are odd increasing functions
(ii) $f(x+y) \geqslant f(x)+f(y)$, *for all* $x,y \in \mathbb{R}_+$
(iii) $g(x)(x-y)+g(y)(y+z)+2g(z)(z-x) \geqslant 0$ *if* $x \geqslant y \geqslant z \geqslant 0$
Then, for all $x,y,z \in \mathbb{R}$, *holds*

$$f(g(x)(x+y))+f(g(y)(y+z))+f(g(z)(z+x)) \geqslant 0$$

Proof. First, we note that for $x,y,z \geqslant 0$ or $x,y,z \leqslant 0$ the statement trivially holds. So, it is suffice to prove the inequality in the case when $x \geqslant 0 \geqslant \max\{y,z\}$, where $y,z \leqslant 0$. Putting $a = x, b = -y, c = -z$ we have to prove

$$f(g(a)(a-b))+f(g(b)(b+c))+f(g(c)(c-a)) \geqslant 0,\ a,b,c \geqslant 0 \quad (2.3)$$

We will distinguish the following cases:

(1) If $c \geqslant a \geqslant b \geqslant 0$ then $g(a)(a-b) \geqslant 0, g(b)(b+c) \geqslant 0,\ g(c)(c-a) \geqslant 0$ and (2.3) holds.

(2) Let $b \geqslant a \geqslant c \geqslant 0$. From $ag(a)+bg(b)+cg(c) \geqslant bg(a)+ag(c)-cg(b)$, we get $g(b)(b+c) \geqslant g(a)(b-a)+g(c)(a-c)$. Taking into account (i) and (ii), the preceding relation becomes $f(g(b)(b+c)) \geqslant -f(g(a)(a-b))-f(g(c)(c-a))$ from which (2.3) follows.

(3) Let $a \geqslant c \geqslant b \geqslant 0$. We have $g(b)(b+c) \geqslant 0$ and from $(a-b)g(a)-(a-c)g(c) = ag(a)+cg(c)-bg(a)-ag(c) \geqslant ag(a)+cg(c)-cg(a)-ag(c) = (a-c)(g(a)-g(c)) \geqslant 0$ we get $f(g(a)(a-b))+f(g(c)(c-a)) \geqslant 0$ and the statement follows.

(4) Let $c \geqslant b \geqslant a \geqslant 0$ or $b \geqslant c \geqslant a \geqslant 0$. We have $f(g(c)(c-a)) \geqslant 0$. Furthermore, from $g(a)(a-b)+g(b)(b+c) = ag(a)+bg(b)+bg(a)+cg(b) = (g(a)-g(b))(a-b)+g(b)(a+c) \geqslant 0$, we get $f(g(a)(a-b))+f(g(b)(b+c)) \geqslant 0$ and (2.3) holds.

(5) Let $a \geqslant b \geqslant c \geqslant 0$. Suppose that $g(a)(a-b)+g(c)(c-a)<0$ and $g(b)(b+c)+g(c)(c-a)<0$. Adding the above inequalities, we obtain $g(a)(a-b)+g(b)(b+c)+2g(c)(c-a)<0$ which is impossible by (iii). If $g(a)(a-b)+g(c)(c-a) \geqslant 0$ then $f(g(a)(a-b))+f(g(c)(c-a)) \geqslant 0$. If $g(b)(b+c)+g(c)(c-a) \geqslant 0$ then $f(g(b)(b+c))+f(g(c)(c-a)) \geqslant 0$. In both cases (2.3) holds. □

THEOREM 2.7. *Let $\alpha, \beta \in [1,+\infty)$ be real numbers such that $\beta \leqslant 3\alpha$. Then $g_{\alpha,\beta}$ is non-negative on $\mathbb{R}^3$, that is the following inequality holds*

$$s_\alpha(x)s_\beta(x+y)+s_\alpha(y)s_\beta(y+z)+s_\alpha(z)s_\beta(z+x) \geqslant 0$$

for all $x,y,z \in \mathbb{R}$.

Proof. Apply Lemma 2.5. and Lemma 2.6. □

3. The study of $f_{m,n}$

For every natural numbers m, $n \geqslant 1$ we consider the function $f_{m,n} : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f_{m,n}(x,y,z) = x^m(x+y)^n + y^m(y+z)^n + z^m(z+x)^n, \quad (x,y,z) \in \mathbb{R}^3$$

Note that $f_{m,n} = g_{m,n}$ if m,n are odd numbers. We shall study conditions under which the family of cyclic polynomials $\{f_{m,n}\}$ in three variables is nonnegative.

In the case when m,n are even numbers one can easily see that $f_{m,n}$ is non-negative on $\mathbb{R}^3$. If m and n have distinct parity, then $m+n$ is an odd number and $f_{m,n}$ fails to be non-negative on $\mathbb{R}^3$. This assertion follows at once from the following remark: for every $y,z \in \mathbb{R}$ is

$$\lim_{x \to -\infty} f_{m,n}(x,y,z) = -\infty$$

Therefore, the only case that remains to study is the case when m,n are odd numbers.

LEMMA 3.1. *Let x,y,z be any real numbers. Then the inequality*

$$f_{1,3}(x,y,z) = x(x+y)^3 + y(y+z)^3 + z(z+x)^3 \geqslant 0$$

holds.

Proof. Let $a = \frac{y+z}{2}$, $b = \frac{x+z}{2}$, $c = \frac{x+y}{2}$. Then $x = b+c-a$, $y = a+c-b$, $z = a+b-c$ and

$$\begin{aligned} f_{1,3}(x,y,z) &= \sum_{cyclic} x(x+y)^3 = 8 \sum_{cyclic} (b+c-a)c^3 \\ &= 8\left(\sum_{cyclic} c^4 - \sum_{cyclic} (a-b)c^3\right), \end{aligned}$$

Note that

$$\sum_{cyclic}(a-b)c^3 = \sum_{cyclic}(b-c)a^3 = \sum_{cyclic}(a^3b-a^3c) = \sum_{cyclic}(a^3b-ab^3) = \sum_{cyclic}ab(a^2-b^2)$$

Since for all $a,b\in\mathbb{R}$, holds

$$a^4+b^4 \geqslant |a^4-b^4| = |a^2-b^2|(a^2+b^2) \geqslant 2|ab|\cdot|a^2-b^2|$$

then, for all $a,b,c\in\mathbb{R}$,

$$\sum_{cyclic}a^4 = \frac{1}{2}\sum_{cyclic}(a^4+b^4) \geqslant \sum_{cyclic}|ab|\cdot|a^2-b^2| \geqslant \sum_{cyclic}ab(a^2-b^2)$$

Hence $f_{1,3}(x,y,z)\geqslant 0$, for all $x,y,z\in\mathbb{R}$, and this completes the proof. □

LEMMA 3.2. *Let x,y,z be any real numbers. Then the inequality*

$$f_{1,5}(x,y,z) = x(x+y)^5+y(y+z)^5+z(z+x)^5 \geqslant 0.$$

holds.

Proof. Let $a=\frac{y+z}{2}$, $b=\frac{x+z}{2}$, $c=\frac{x+y}{2}$. Then $x=b+c-a$, $y=a+c-b$, $z=a+b-c$ and

$$f_{1,5}(x,y,z) = \sum_{cyclic}x(x+y)^5 = 32\sum_{cyclic}(b+c-a)c^5 = 32\left(\sum_{cyclic}c^6-\sum_{cyclic}(a-b)c^5\right)$$

Note that

$$\sum_{cyclic}(a-b)c^5 = \sum_{cyclic}(b-c)a^5 = \sum_{cyclic}(a^5b-a^5c) = \sum_{cyclic}(a^5b-ab^5) = \sum_{cyclic}ab(a^4-b^4)$$

Now, we claim that

$$\sum_{cyclic}(a^6+b^6) \geqslant 2\sum_{cyclic}|ab|\cdot|a^4-b^4| \geqslant \sum_{cyclic}ab(a^4-b^4),\ a,b,c\in\mathbb{R}$$

To prove our claim will be suffice to establish that

$$(a^6+b^6)^2 \geqslant 4a^2b^2(a^4-b^4)^2,\ a,b\in\mathbb{R},$$

or equivalently,

$$a^{12}-4a^{10}b^{2}+10a^{6}b^{6}-4a^{2}b^{10}+b^{12}\geqslant 0$$

Setting $c=a/b,(b\neq 0)$ in the preceding we obtain

$$c^{12}-4c^{10}+10c^{6}-4c^{2}+1\geqslant 0$$

Putting $t=c^{2}+\frac{1}{c^{2}}$, we get $c^{4}+\frac{1}{c^{4}}=t^{2}-2$ and $c^{6}+\frac{1}{c^{6}}=t^{3}-3t$, and the above inequality is equivalent to

$$t^{3}-4t^{2}-3t+18\geqslant 0,\ \text{for all}\ t\in[2,+\infty),$$

which trivially holds because the function $f(t)=t^{3}-4t^{2}-3t+18$ is non-negative in $[2,+\infty)$ as can be easily checked using elementary calculus. This proof our claim. Finally, we have

$$\sum_{cyclic}a^{6}=\frac{1}{2}\sum_{cyclic}(a^{6}+b^{6})\geqslant\sum_{cyclic}|ab|\cdot|a^{4}-b^{4}|\geqslant\sum_{cyclic}ab(a^{4}-b^{4})$$

and the Lemma is proven. □

THEOREM 3.3. *Let n be an odd positive integer. Then $f_{1,n}$ is non-negative on $\mathbb{R}^{3}$ if and only if $n\in\{1,3,5\}$.*

Proof. We note that $f_{1,1}$ is non-negative on $\mathbb{R}^{3}$ and from Lemma 3.1. and 3.2. we have that $f_{1,n}$ is non-negative on $\mathbb{R}^{3}$ for $n\in\{1,3,5\}$. On the other hand, if n is an odd positive integer and $f_{1,n}$ is non-negative on $\mathbb{R}^{3}$, then

$$0\leqslant f_{1,n}(-1,5,-2)=-4^{n}+5\cdot 3^{n}+2\cdot 3^{n}=3^{n}\left[7-\left(\frac{4}{3}\right)^{n}\right]$$

Hence $n<7$. It follows that $n\in\{1,3,5\}$. Therefore, $f_{1,n}$ is non-negative on $\mathbb{R}^{3}$ if and only if $n\in\{1,3,5\}$ and this completes the proof. □

Another important result is the following

THEOREM 3.4. *Let m be an odd positive integer. Then $f_{m,1}$ is non-negative on $\mathbb{R}^{3}$.*

Proof. We have $f_{m,1}(x,y,z)=\sum_{cyclic}x^{m}(x+y)=\sum_{cyclic}|x|^{m+1}+\sum_{cyclic}x^{m}y$. By the rearrangement inequality $\sum_{cyclic}|x|^{m+1}\geqslant\sum_{cyclic}|x|^{m}|y|\geqslant-\sum_{cyclic}x^{m}y$, hence $f_{m,1}(x,y,z)\geqslant 0$ and the theorem is proven. □

Finally, we summarize the preceding in the following main result.

THEOREM 3.5. *Let* $\psi : \mathbb{N} \to \mathbb{N}$, $\psi(r) = \ln(4^r + 9^r)/\ln(8/5)$, $r \in \mathbb{N}$. *Suppose that* m, n *are two odd positive integers such that* $f_{m,n}$ *is non-negative on* $\mathbb{R}^3$. *Then* $n \leqslant \psi(m)$. *Also the following inequality holds* $\psi(r) \leqslant 4.74\, r$ *for every* $r \geqslant 3$.

Proof. Note that $f_{m,n}(-1,9,-4) = 5^n(4^m + 9^m) - 8^n$ is negative when $n > \psi(m)$. Consider the function $u(x) = \frac{1}{x}\ln(4^x + 9^x)$, $x \in (0,\infty)$. One can easily check that the derivative of u is negative. Consequently u is a decreasing function. Denote $k_0 = u(3)/\ln(8/5) = 4.734589\ldots$. If $r \geqslant 3$ then $\psi(r) = \frac{r\,u(r)}{\ln(8/5)} \leqslant \frac{r\,u(3)}{\ln(8/5)} = k_0\, r \leqslant 4.74\, r$.

In the table 2 are listed some values of the functions ϕ (from theorem 2.3.) and ψ and some odd values of n from which $f_{m,n}$ fails to be non-negative on $\mathbb{R}^3$. □

m	$\phi(m)$	t	$\psi(m)$	n
3	14.19484862	9.477960556	14.204	15
5	23.39818368	9.445738092	23.411	25
7	32.71330693	9.448561108	32.732	33
9	42.0519171	9.44990273	42.076	43
11	51.39527022	9.450253907	51.424	53
13	60.73957339	9.450334504	60.774	61
15	70.08406654	9.450351976	70.124	71
17	79.42859768	9.450355666	79.473	81
19	88.77313641	9.450356364	88.823	89

Table 2. *In the preceding table are displayed some values of* m, $\phi(m)$, *the values of* t *where the infimum of* $h(t,x)$ *is attained,* $\psi(m)$ *and the odd values of* n *for which* $f_{m,n}$ *fails to be nonnegative on* $\mathbb{R}^3$.

THEOREM 3.6. *Let* m, n *be odd positive integers and consider the sets*

$$A_m = \{s \in \mathbb{N} \,|\, s \text{ is odd and } f_{m,s} \text{ is nonnegative on } \mathbb{R}^3\},$$
$$B_m = \{s \in \mathbb{N} \,|\, s \text{ is odd and } f_{m,s} \text{ fails to be non-negative on } \mathbb{R}^3\},$$
$$C_n = \{r \in \mathbb{N} \,|\, r \text{ is odd and } f_{r,n} \text{ is nonnegative on } \mathbb{R}^3\},$$
$$F_m = \{k \in \mathbb{N} \,|\, k \text{ is odd and } k \leqslant m\},$$
$$L_m = \{k \in \mathbb{N} \,|\, k \text{ is odd and } k \geqslant m\}.$$

Then the following assertions hold:

(i) $A_1 = \{1,3,5\}$

(ii) $m \in A_m$ *for every odd number* m

(iii) $F_{3m} \subseteq A_m$ *for every odd number* m

(iv) $C_1 = L_1$

(v) $L_{15} \subseteq B_3, L_{25} \subseteq B_5, L_{33} \subseteq B_7, L_{43} \subseteq B_9, L_{53} \subseteq B_{11}, L_{61} \subseteq B_{13},\ L_{71} \subseteq B_{15}, L_{81} \subseteq B_{17}, L_{89} \subseteq B_{19}$

Proof. (i) follows from theorem 3.3, (ii) follows from theorem 2.2. From theorem 2.7, we get (iii) and (iv) trivially holds from theorem 3.4. Finally, from table 2, we get (v). □

4. Conclusions

We considered two families of cyclic sums depending on two parameters. Our goal was to determine the values of parameters for which the functions of the two families belong to two classes of functions. The first class is the class of non-negative functions on $\mathbb{R}^3$ while the second class is the class of functions that fails to be non-negative on $\mathbb{R}^3$. We proved that for every $\alpha \geqslant 1$ there exist two positive numbers $\phi_1(\alpha)$ and $\phi_2(\alpha)$ such that

(1) $\phi_1(\alpha) < \phi_2(\alpha)$

(2) for every $1 \leqslant \beta \leqslant \phi_1(\alpha)$ the cyclic sum $g_{\alpha,\beta}$ is non-negative on $\mathbb{R}^3$.

(3) for every $\beta \geqslant \phi_2(\alpha)$ the cyclic sum $g_{\alpha,\beta}$ fails to be non-negative on $\mathbb{R}^3$.

For β in the interval $(\phi_1(\alpha), \phi_2(\alpha))$ we have undecided cases. We proved that:

(4) for every natural numbers m,n such that $m+n$ is odd, the cyclic sum $f_{m,n}$ fails to be non-negative on $\mathbb{R}^3$.

(5) for every even numbers m,n the cyclic sum $f_{m,n}$ is non-negative on $\mathbb{R}^3$.

(6) for every odd number m there exist two positive numbers $\psi_1(m)$ and $\psi_2(m)$ such that

(i) $\psi_1(m) < \psi_2(m)$

(ii) for every odd natural number $n \leqslant \psi_1(m)$ the cyclic sum $f_{m,n}$ is non-negative on $\mathbb{R}^3$.

(iii) for every odd natural number $n \geqslant \psi_2(m)$ the cyclic sum $f_{m,n}$ fails to be non-negative on $\mathbb{R}^3$.

For n odd natural number in the interval $(\psi_1(m), \psi_2(m))$ we have undecided cases. Computer experiments may suggest for such n if $f_{m,n}$ is non-negative or fails to be non-negative on $\mathbb{R}^3$. But the confidence in the results suggested by computer experiments cannot be 100% because of the accumulation of roundoff errors. That is why these results must be accompanied by sound analytical proofs.

In the following we shall make two conjectures.

CONJECTURE 1. For every $\alpha \geqslant 1$ there exist a positive number $\phi(\alpha)$ with the following property: for every $\beta \leqslant \phi(\alpha)$ the cyclic sum $g_{\alpha,\beta}$ is non-negative on $\mathbb{R}^3$ and for every $\beta > \phi(\alpha)$ the cyclic sum $g_{\alpha,\beta}$ fails to be non-negative on $\mathbb{R}^3$.

CONJECTURE 2. For every positive odd number m there exists an odd positive number $\psi(m)$ with the following property: for every odd number $n \leqslant \psi(m)$ the cyclic sum $f_{m,n}$ is non-negative on $\mathbb{R}^3$ and for every odd number $n > \psi(m)$ the cyclic sum $f_{m,n}$ fails to be non-negative on $\mathbb{R}^3$.

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