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ON A FORMULA FOR THE nTH DERIVATIVE AND ITS APPLICATIONS

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Abstract. In this paper we obtain a formula which involves the nth derivative of the first order divided difference and a corresponding inequality for functions whose $(n+1)$-th derivative belongs to a L^p space. These results are a generalization of the results from [1] and [2]. Finally, some examples are given.

1. Introduction

T.H. Gronwall in [1] and [2] gave the following results for the nth derivative of functions $f(x)=\dfrac{\sin x}{x}$ and $f(x)=\dfrac{\cos x}{x}$:

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)=\frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y+\frac{(n+1)\pi}{2}\right)dy, \tag{1}$$

and
$$\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)=\frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y+\frac{n\pi}{2}\right)dy$$

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right|\leqslant\frac{1}{n+1} \quad\text{and}\quad \left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right|\leqslant\frac{1}{n+1}. \tag{2}$$

In this note, the formula for the nth derivative of the first order divided difference $\dfrac{f(x)-f(0)}{x}$ which generalizes (1) is proven. Furthermore, corresponding generalization of the inequalities in (2) for functions f such that $f^{(n+1)}\in L^p$ is obtained. Some examples for certain elementary functions are also given.

Before we proceed to the main results, let us recall one of the most important integral inequality in analysis, the Hölder inequality (cf. [3]), since it will be applied in one of the proofs:

THEOREM 1.1. *Let p and q be such that $1\leqslant p,q\leqslant\infty$ and $\frac{1}{p}+\frac{1}{q}=1$ and let S be a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure. Finally, let $f\in L^p(S)$ and $g\in L^q(S)$. Then*

$$\|fg\|_1\leqslant\|f\|_p\cdot\|g\|_q,$$

where $\|f\|_p=\begin{cases}\left(\int_S|f(x)|^p\,dx\right)^{1/p} & (1\leqslant p<\infty),\\ \sup\limits_{x\in S}|f(x)| & (p=\infty).\end{cases}$

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2. Main results

THEOREM 2.1. *Let $I=[-a,a]\subseteq\mathbb{R}$, $a>0$ and $f:I\to\mathbb{R}$ be such that $f^{(n+1)}$ is integrable on I. Then it holds*

$$\frac{d^n}{dx^n}\left(\frac{f(x)-f(0)}{x}\right)\Big|_{x\neq 0}=\frac{1}{x^{n+1}}\int_0^x y^n f^{(n+1)}(y)dy, \tag{3}$$

$$\frac{d^n}{dx^n}\left(\frac{f(x)-f(0)}{x}\right)\Big|_{x=0}=\frac{f^{(n+1)}(0)}{n+1}. \tag{4}$$

Proof. Let $x\neq 0$, since $\int_0^1 f'(tx)\,dt=\int_0^x f'(y)\,\frac{dy}{x}=\frac{f(x)-f(0)}{x}$, we get

$$\begin{aligned}\frac{d^n}{dx^n}\left(\frac{f(x)-f(0)}{x}\right)&=\frac{d^n}{dx^n}\int_0^1 f'(tx)\,dt=\int_0^1\frac{d^n}{dx^n}f'(tx)\,dt\\&=\int_0^1 t^n f^{(n+1)}(tx)\,dt\\&=\frac{1}{x^{n+1}}\int_0^x y^n f^{(n+1)}(y)\,dy\end{aligned}$$

by the substitution $tx=y$.

For $x=0$ we have

$$\begin{aligned}\frac{d^n}{dx^n}\left(\frac{f(x)-f(0)}{x}\right)\Big|_{x=0}&=\lim_{x\to 0}\frac{\int_0^x y^n f^{(n+1)}(y)\,dy}{x^{n+1}}\qquad\text{(L'Hospital's rule)}\\&=\lim_{x\to 0}\frac{x^n f^{(n+1)}(x)}{(n+1)x^n}=\frac{f^{(n+1)}(0)}{n+1}.\quad\square\end{aligned}$$

THEOREM 2.2. *Let $p,q\in\mathbb{R}$ be such that $1\leqslant p,q\leqslant\infty$ and $1/p+1/q=1$. Let $I=[-a,a]\subseteq\mathbb{R}$, $a>0$, and $f:I\to\mathbb{R}$ be such that $f^{(n+1)}\in L^p([0,|x|])$ for $x\in I$ and $n\geqslant 0$.*

Then it holds

$$\left|\frac{d^n}{dx^n}\left(\frac{f(x)-f(0)}{x}\right)\right|\leqslant\frac{\|f^{(n+1)}\|_p}{(nq+1)^{1/q}\,|x|^{1/p}}. \tag{5}$$

The inequality is sharp for $1<p\leqslant\infty$ and the best possible for $p=1$.

Proof. To prove (5), we start from identity (3). For $x\in I$ and $f^{(n+1)}\in L^p([0,|x|])$

we can estimate this expression by using Hölder inequality as follows

$$\begin{aligned}\left|\frac{d^n}{dx^n}\left(\frac{f(x)-f(0)}{x}\right)\right| &= \frac{1}{|x|^{n+1}}\left|\int_0^{|x|} y^n f^{(n+1)}(y)dy\right| \\ &\leqslant \frac{1}{|x|^{n+1}}\int_0^{|x|} |y^n f^{(n+1)}(y)|dy \\ &\leqslant \frac{1}{|x|^{n+1}}\left(\int_0^{|x|}|y^n|^q dy\right)^{1/q}\left(\int_0^{|x|}|f^{(n+1)}(y)|^p dy\right)^{1/p} \\ &= \frac{1}{|x|^{1/p}}\frac{\|f^{(n+1)}\|_p}{(nq+1)^{1/q}}.\end{aligned}$$

Now, we consider the sharpness of this inequality. For $x\in I$ we have to find a function f for which equality in (5) is attained, i.e. such that

$$\left|\int_0^{|x|} y^n f^{(n+1)}(y)dy\right| = \left(\int_0^{|x|}|y^n|^q dy\right)^{1/q}\left(\int_0^{|x|}|f^{(n+1)}(y)|^p dy\right)^{1/p}.$$

For $1<p<\infty$, take $\widetilde{f}$ such that $\widetilde{f}^{(n+1)}(y)=\operatorname{sgn}(y^n)|y^n|^{\frac{1}{p-1}}$. Then we have

$$\left|\int_0^{|x|} y^n \widetilde{f}^{(n+1)}(y)dy\right| = \frac{|x|^{nq+1}}{nq+1} = \left(\frac{|x|^{nq+1}}{nq+1}\right)^{1/p}\left(\frac{|x|^{nq+1}}{nq+1}\right)^{1/q} = \|\widetilde{f}^{(n+1)}\|_p\,\|y^n\|_q$$

For $p=\infty$, take $\widetilde{f}$ such that $\widetilde{f}^{(n+1)}(y)=\operatorname{sgn}(y^n)$. Then

$$\left|\int_0^{|x|} y^n \widetilde{f}^{(n+1)}(y)dy\right| = \|y^n\|_1 = \|\widetilde{f}^{(n+1)}\|_\infty\,\|y^n\|_1.$$

Finally, for $p=1$ we prove that inequality

$$\left|\frac{d^n}{dx^n}\left(\frac{f(x)-f(0)}{x}\right)\right| \leqslant \frac{1}{|x|}\|f^{(n+1)}\|_1,$$

which is equivalent to

$$\begin{aligned}\frac{1}{|x|^{n+1}}\left|\int_0^{|x|} y^n f^{(n+1)}(y)dy\right| &\leqslant \frac{1}{|x|^{n+1}}\sup_{y\in[0,|x|]}|y^n|\int_0^{|x|}|f^{(n+1)}(y)|dy \\ &= \frac{1}{|x|}\|f^{(n+1)}\|_1,\end{aligned} \tag{6}$$

is the best possible, i.e. that the constant $\frac{1}{|x|}$ cannot be replaced by a smaller one. Note that $\lim\limits_{q\to\infty}(nq+1)^{1/q}=1$.

For a small enough $\delta>0$, define function $\widetilde{f}_\delta$ such that

$$\widetilde{f}_\delta^{(n)}(y)=\begin{cases} 0 & (0\leqslant y\leqslant |x|-\delta), \\ \frac{1}{\delta}(y-|x|+\delta) & (|x|-\delta\leqslant y\leqslant |x|).\end{cases}$$

For this function, the left-hand side of (6) becomes

$$\frac{1}{|x|^{n+1}}\left|\int_0^{|x|} y^n \widetilde{f}_\delta^{(n+1)}(y)\,dy\right| = \frac{1}{|x|^{n+1}}\int_{|x|-\delta}^{|x|} y^n \frac{1}{\delta}\,dy$$
$$= \frac{1}{|x|^{n+1}}\,\frac{1}{\delta}\,\frac{|x|^{n+1}-(|x|-\delta)^{n+1}}{n+1},$$

while the right-hand side becomes

$$\frac{|x|^n}{|x|^{n+1}}\int_0^{|x|} |\widetilde{f}_\delta^{(n+1)}(y)|dy = \frac{1}{|x|}\int_{|x|-\delta}^{|x|}\frac{1}{\delta}\,dy = \frac{1}{|x|}.$$

Now, taking the limit value of the left-hand side of (6) when $\delta \to 0$

$$\frac{1}{(n+1)|x|^{n+1}}\lim_{\delta\to 0}\frac{|x|^{n+1}-(|x|-\delta)^{n+1}}{\delta} = \frac{1}{|x|},$$

the statement follows. □

REMARK 2.1. Applying identity (3) for function $f(x) = \sin x$ and $f(x) = \cos x$, we recapture the formulas (1) from [1].

REMARK 2.2. Applying inequality (5) for $p = \infty$ and function $f(x) = \sin x$ and $f(x) = \cos x$, we obtain the inequalities (2) from [2].

EXAMPLE 1. If in (5) we take $f(x) = e^{\alpha x}$, $f(x) = \operatorname{ch}\alpha x \sin\beta x$, $f(x) = \operatorname{ch}\alpha x\cos\beta x$ and $p = \infty$, we get

$$\left|\frac{d^n}{dx^n}\left(\frac{e^{\alpha x}-1}{x}\right)\right| \leqslant |\alpha|^{n+1}\,\frac{e^{|\alpha x|}}{n+1} \qquad (x\in\mathbb{R}),$$
$$\left|\frac{d^n}{dx^n}\left(\frac{\operatorname{ch}\alpha x\sin\beta x}{x}\right)\right| \leqslant (\alpha^2+\beta^2)^{(n+1)/2}\,\frac{e^{|\alpha x|}}{n+1} \qquad (x\in\mathbb{R}),$$
$$\left|\frac{d^n}{dx^n}\left(\frac{1-\operatorname{ch}\alpha x\cos\beta x}{x}\right)\right| \leqslant (\alpha^2+\beta^2)^{(n+1)/2}\,\frac{e^{|\alpha x|}}{n+1} \qquad (x\in\mathbb{R}).$$

If in (5) we take $f(x) = e^{\alpha x}$ and $p = 1$ we have the following estimation

$$\left|\frac{d^n}{dx^n}\left(\frac{e^{\alpha x}-1}{x}\right)\right| \leqslant \frac{1}{|x|}|\alpha|^n\,(e^{|\alpha x|}-1) \qquad (x\in\mathbb{R}).$$

EXAMPLE 2. If in (3) we take $f(x) = \ln(1+x)$, $(x \neq 0, x > -1)$ we get

$$\left|\frac{d^n}{dx^n}\left(\frac{\ln(x+1)}{x}\right)\right| = \left|\frac{1}{x^{n+1}}\int_0^x y^n\,\frac{(-1)^{n+1}n!}{(1+y)^{n+1}}\,dy\right|$$
$$\leqslant \frac{n!}{|x|^{n+1}}\int_0^{|x|}\frac{y^n}{(1+y)^{n+1}}\,dy$$
$$= \frac{n!}{|x|^{n+1}}\left(\ln(1+|x|)+\sum_{k=1}^{n}(-1)^k\binom{n}{k}\frac{1}{k}\left(1-\frac{1}{(1+|x|)^k}\right)\right).$$

Using (5) for $p=\infty$ we obtain

$$\left|\frac{d^n}{dx^n}\left(\frac{\ln(x+1)}{x}\right)\right| \leqslant \frac{1}{n+1}\sup_{y\in[0,|x|]}\left|\frac{(-1)^{n+1}n!}{(1+y)^{n+1}}\right| = \frac{n!}{n+1}.$$

For $p=1$ it holds

$$\begin{aligned}\left|\frac{d^n}{dx^n}\left(\frac{\ln(x+1)}{x}\right)\right| &\leqslant \frac{1}{|x|}\int_0^{|x|}\left|\frac{(-1)^{n+1}n!}{(1+y)^{n+1}}\right|dy\\ &= \frac{n!}{|x|}\int_0^{|x|}\frac{1}{(1+y)^{n+1}}\,dy\\ &= \frac{(n-1)!}{|x|}\left(1-\frac{1}{(1+|x|)^n}\right).\end{aligned}$$

EXAMPLE 3. Finally, if in (5) we take $f(x)=\arctan x$, for $1\leqslant p<+\infty$ and $x\neq 0$, we have

$$\begin{aligned}\left|\frac{d^n}{dx^n}\left(\frac{\arctan x}{x}\right)\right| &\leqslant \frac{1}{|x|^{1/p}}\frac{1}{(nq+1)^{1/q}}\\ &\qquad\left(\int_0^{|x|}\left|\frac{(-1)^n n!}{(1+y^2)^{(n+1)/2}}\sin\big((n+1)\arctan y\big)\right|^p dy\right)^{1/p}\\ &\leqslant \frac{n!}{|x|^{1/p}}\frac{1}{(nq+1)^{1/q}}\left(\int_0^{|x|}\left(\frac{1}{\sqrt{1+y^2}}\right)^{(n+1)p}dy\right)^{1/p}, (\text{by } y=\tan t)\\ &= \frac{n!}{|x|^{1/p}}\frac{1}{(nq+1)^{1/q}}\left(\int_0^{\arctan|x|}\cos^{(n+1)p-2}t\,dt\right)^{1/p}\\ &< \frac{n!}{|x|^{1/p}}\frac{1}{(nq+1)^{1/q}}\left(\int_0^{\pi/2}\cos^{(n+1)p-2}t\,dt\right)^{1/p}\\ &= \frac{n!}{|x|^{1/p}}\frac{1}{(nq+1)^{1/q}}\left(\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{(n+1)p-1}{2}\right)}{\Gamma\left(\frac{(n+1)p}{2}\right)}\right)^{1/p}.\end{aligned}$$

Using (5) for $p=\infty$ we obtain

$$\left|\frac{d^n}{dx^n}\left(\frac{\arctan x}{x}\right)\right| \leqslant \frac{n!}{n+1}.$$

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