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# A NOTE ON THE ESTIMATE OF GAMMA DISTRIBUTION

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*Abstract.* The lower and upper estimates with explicit coefficients for Gamma distribution are given. Furthermore, using these results, the estimates of spherically symmetric distribution and ellipsoidal distribution are obtained.

## 1. Introduction

It is well known that Mills' ratio $R(x)$ is defined to be the normal probability beyond a certain point divided by the normal density at that point; that is,

$$R(x) = \int_x^\infty f(y)dy \Big/ f(x),$$

where $f(x) = (2\pi)^{-1/2}e^{-x^2/2}$ is a standard normal density. It has been studied for a long history in view of the purpose of computation. Lots of lower and upper estimates were obtained by many mathematicians, such as Birnbaum [4], Steck [10], Hashorva and Hüsler [7], Lu and Li [9] et. They obtained very sharp lower and upper bounds for Mills' ratio and Multivariate Mills' ratio. In addition, it has been recognized for some time that Gamma distribution plays an important role both in theory and application. The nature of Gamma distribution has been studied by many mathematicians, such as Bar-Lev and Reiser [1], Keating, Glaser and Ketchum [8], Basawa [2], Diaconis and Pealman [6], Batir [3] et. However, we find that there are very few estimates with explicit coefficients for Gamma distribution in the literature. This is the motivation of our work in this paper.

Returning to our problem, in this paper, the Gamma ratio is defined as

$$H(x) = \int_x^\infty g(y,\lambda,\alpha)dy \Big/ g(x,\lambda,\alpha), \tag{1.1}$$

where $g(x,\lambda,\alpha) = \lambda^\alpha x^{\alpha-1}e^{-\lambda x}/\Gamma(\alpha)$ with $\lambda > 0$ and $\alpha > 0$, $\Gamma$ is the usual Gamma function. Based on the method in Lu and Li [9], a pair of lower and upper estimates for the Gamma ratio are obtained.

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The rest of this paper is arranged as follow etc. In Section 2, we give the asymptotic estimates for the Gamma ratio. Furthermore, some lower and upper estimates with explicit coefficients for the Gamma ratio are obtained. In Section 3, applying the result of the Gamma ratio, the lower and upper bounds of spherically symmetric distribution and ellipsoidal distribution are given. The whole proof of Theorem 2.1 is given in Section 4.

## 2. Asymptotic estimates for the Gamma ratio

Throughout the paper, the set of natural numbers is denoted by $N$, and the set of positive integers is denoted by $Z^+$.

From (1.1), the Gamma ratio is denoted again by

$$H(x)=\frac{\int_x^\infty y^{\alpha-1}e^{-\lambda y}dy}{x^{\alpha-1}e^{-\lambda x}}. \tag{2.1}$$

For the brevity, let

$$g(x)=\int_x^\infty y^{\alpha-1}e^{-\lambda y}dy, \quad G(x)=e^{\lambda x}g(x), \quad \prod_{i=1}^{0}\overset{def}{=}1. \tag{2.2}$$

If $\alpha\in Z^+$, it is easy to see

$$g(x)=\int_x^\infty y^{\alpha-1}e^{-\lambda y}dy=\sum_{1\leqslant i\leqslant\alpha}\frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-\alpha}e^{\lambda x}}.$$

Plugging $g(x)$ into (2.1) yields

$$H(x)=\sum_{1\leqslant i\leqslant\alpha}\frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}}. \tag{2.3}$$

Then we give an asymptotic estimate for $H(x)$ .

PROPOSITION 2.1. *$H(x)$ is defined as above in (1.1), for any $\alpha>0$, we have*

$$\lim_{x\to\infty}H(x)=1/\lambda. \tag{2.4}$$

*Proof.* Since $H(x)$ is an increasing function for $\alpha$, if $k<\alpha\leqslant k+1, k\in N$, we have

$$\frac{\int_x^\infty y^{k-1}e^{-\lambda y}dy}{x^{k-1}e^{-\lambda x}}\leqslant\frac{\int_x^\infty y^{\alpha-1}e^{-\lambda y}dy}{x^{\alpha-1}e^{-\lambda x}}\leqslant\frac{\int_x^\infty y^{k}e^{-\lambda y}dy}{x^{k}e^{-\lambda x}}.$$

If $k\geqslant 1$, from (2.3), we have

$$\sum_{1\leqslant i\leqslant k}\frac{\prod_{j=1}^{i-1}(k-j)}{\lambda^i x^{i-1}}\leqslant\frac{\int_x^\infty y^{\alpha-1}e^{-\lambda y}dy}{x^{\alpha-1}e^{-\lambda x}}\leqslant\sum_{1\leqslant i\leqslant k+1}\frac{\prod_{j=1}^{i-1}(k+1-j)}{\lambda^i x^{i-1}}.$$

For $x \to \infty$, obviously

$$\sum_{1 \leqslant i \leqslant k} \frac{\prod_{j=1}^{i-1}(k-j)}{\lambda^i x^{i-1}} \to \frac{1}{\lambda} \quad \text{and} \quad \sum_{1 \leqslant i \leqslant k+1} \frac{\prod_{j=1}^{i-1}(k+1-j)}{\lambda^i x^{i-1}} \to \frac{1}{\lambda}.$$

By the sandwich theorem, we obtain (2.4).

If $k = 0$, we have

$$\frac{\int_x^\infty y^{-1} e^{-\lambda y} dy}{x^{-1} e^{-\lambda x}} \leqslant \frac{\int_x^\infty y^{\alpha-1} e^{-\lambda y} dy}{x^{\alpha-1} e^{-\lambda x}} \leqslant \frac{\int_x^\infty e^{-\lambda y} dy}{e^{-\lambda x}} = \frac{1}{\lambda}.$$

For $x \to \infty$, it is easy to verify

$$\frac{\int_x^\infty y^{-1} e^{-\lambda y} dy}{x^{-1} e^{-\lambda x}} \to \frac{1}{\lambda}.$$

By the sandwich theorem again, we also obtain (2.4).

Next we give some estimates with explicit coefficients for $H(x)$.

THEOREM 2.1. *For $2k+1 < \alpha < 2k+2, k \in N$, $n \in Z^+$ and $n > k$. If $x \in [\frac{2n-\alpha}{\lambda}, \frac{2n+1-\alpha}{\lambda})$, we have*

$$\sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}} < H(x) < \sum_{1 \leqslant i \leqslant 2n} \frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}}. \tag{2.5}$$

*If $x \in [\frac{2n+1-\alpha}{\lambda}, \frac{2n+2-\alpha}{\lambda})$, we have*

$$\sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}} < H(x) < \sum_{1 \leqslant i \leqslant 2n+2} \frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}}. \tag{2.6}$$

*For $2k < \alpha < 2k+1$, if $x \in [\frac{2n-\alpha}{\lambda}, \frac{2n+1-\alpha}{\lambda})$, we have*

$$\sum_{1 \leqslant i \leqslant 2n} \frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}} < H(x) < \sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}}. \tag{2.7}$$

*If $x \in [\frac{2n+1-\alpha}{\lambda}, \frac{2n+2-\alpha}{\lambda})$, we have*

$$\sum_{1 \leqslant i \leqslant 2n+2} \frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}} < H(x) < \sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha-j)}{\lambda^i x^{i-1}}. \tag{2.8}$$

For the brevity, the detail of proof is given in Section 4.

In addition, for $\alpha < 0$, using the similar argument of $2k < \alpha < 2k+1$, we have the following conclusion.

COROLLARY 2.1. *For $\alpha < 0$, $n \in Z^+$. If $x \in [\frac{2n-\alpha}{\lambda}, \frac{2n+1-\alpha}{\lambda})$, we have*

$$\sum_{1 \leqslant i \leqslant 2n} \frac{\prod_{j=1}^{i-1}(\alpha - j)}{\lambda^i x^{i-1}} < H(x) < \sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha - j)}{\lambda^i x^{i-1}}. \tag{2.9}$$

*If $x \in [\frac{2n+1-\alpha}{\lambda}, \frac{2n+2-\alpha}{\lambda})$, we have*

$$\sum_{1 \leqslant i \leqslant 2n+2} \frac{\prod_{j=1}^{i-1}(\alpha - j)}{\lambda^i x^{i-1}} < H(x) < \sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha - j)}{\lambda^i x^{i-1}}. \tag{2.10}$$

## 3. Application in spherically symmetric distribution and ellipsoidal distribution

In this section, applying the results of the Gamma ratio into spherically symmetric distribution and ellipsoidal distribution, their exact estimates are obtained, respectively. Comparing with the case of 2-dimensional, for the one of d-dimensional, we only need consider more parameters. Thus, for the clarity, in this section, we only provide the result of 2-dimensional. First, we introduce the definitions of elliptically contoured distribution and spherically symmetric distribution as follows:

DEFINITION 3.1. *Let $\boldsymbol{x}$ be a random vector in $\boldsymbol{R}^n$ with eigenfunction*

$$exp(i\boldsymbol{t}'\mu)\phi(\boldsymbol{t}'\Sigma\boldsymbol{t}),$$

*where $\mu : n \times 1$, $\Sigma : n \times n$ and $\Sigma > 0$, $\phi$ is an arbitrary function. Then we say that $\boldsymbol{x}$ obey elliptically contoured distribution with parameter $\mu, \Sigma$, $\phi$ and write $\boldsymbol{x} \sim EC_n(\mu, \Sigma, \phi)$. Especially, if $\mu = 0$, $\Sigma = I_n$, $EC_n(\boldsymbol{0}, I_n, \phi)$ is called spherically symmetric distribution, write $S_n(\phi)$.*

In addition, we introduce an useful Lemma.

LEMMA 3.1. *Assuming $\boldsymbol{x} \stackrel{d}{=} Ru^{(n)} \sim S_n(\phi)$, the density function of $\boldsymbol{x}$ exists if and only if the density function $g(\cdot)$ of $R$ exists, and the relation between $f(\cdot)$ and $g(\cdot)$ is*

$$g(x) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} x^{n-1} f(x^2).$$

The result can be easily found in Cambanis, Huang and Simons [5].

Let $g(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$, i.e.: the distribution of $R$ is Gamma distribution. From Lemma (3.1), we have $f(\mathbf{x}'\mathbf{x}) = \frac{\lambda^r}{2\pi\Gamma(r)} (\mathbf{x}'\mathbf{x})^{\frac{r-2}{2}} e^{-\lambda(\mathbf{x}'\mathbf{x})^{\frac{1}{2}}}$. For 2-dimensional spherically symmetric distribution, we have

$$f(x_1, x_2) = \frac{\lambda^r}{2\pi\Gamma(r)} (x_1^2 + x_2^2)^{(r-2)/2} e^{-\lambda\sqrt{x_1^2 + x_2^2}}.$$

where $r > 0$. Furthermore,

$$P\{\mathbf{X} > \mathbf{t}\} = \frac{\lambda^r}{2\pi\Gamma(r)} \int_{t_2}^{\infty} \int_{t_1}^{\infty} (x_1^2 + x_2^2)^{(r-2)/2} e^{-\lambda\sqrt{x_1^2+x_2^2}} dx_1 dx_2. \tag{3.1}$$

Next, using the result in Section 2, we give the estimates of (3.1). By

$$(x_1 + x_2)/\sqrt{2} \leqslant \sqrt{x_1^2 + x_2^2} \leqslant x_1 + x_2,$$

$$\max(x_1^{r-2}, x_2^{r-2}) \leqslant (x_1^2 + x_2^2)^{(r-2)/2} \leqslant C_r(x_1^{r-2} + x_2^{r-2}),$$

where $C_r$ is a constant. We have

$$\begin{aligned}
&\max\left\{ \int_{t_2}^{\infty} \int_{t_1}^{\infty} \frac{x_1^{r-2}}{e^{\lambda(x_1+x_2)}} dx_1 dx_2, \int_{t_2}^{\infty} \int_{t_1}^{\infty} \frac{x_2^{r-2}}{e^{\lambda(x_1+x_2)}} dx_1 dx_2 \right\} \\
&\leqslant \int_{t_2}^{\infty} \int_{t_1}^{\infty} \frac{(x_1^2 + x_2^2)^{(r-2)/2}}{e^{\lambda\sqrt{x_1^2+x_2^2}}} dx_1 dx_2 \\
&\leqslant \int_{t_2}^{\infty} \int_{t_1}^{\infty} \frac{C_r(x_1^{r-2} + x_2^{r-2})}{e^{\lambda(x_1+x_2)/\sqrt{2}}} dx_1 dx_2.
\end{aligned} \tag{3.2}$$

If we want to estimate (3.1), we should estimate the upper and lower bounds in (3.2), respectively. On the one hand, for the upper bounds in (3.2), we have

$$\begin{aligned}
&\int_{t_2}^{\infty} \int_{t_1}^{\infty} \frac{C_r(x_1^{r-2} + x_2^{r-2})}{e^{\lambda(x_1+x_2)/\sqrt{2}}} dx_1 dx_2 \\
&\quad = \frac{\sqrt{2}C_r}{\lambda} \left( e^{\frac{-\lambda t_2}{\sqrt{2}}} \int_{t_1}^{\infty} \frac{x_1^{r-2}}{e^{\lambda x_1/\sqrt{2}}} dx_1 + e^{\frac{-\lambda t_1}{\sqrt{2}}} \int_{t_2}^{\infty} \frac{x_2^{r-2}}{e^{\lambda x_2/\sqrt{2}}} dx_2 \right).
\end{aligned} \tag{3.3}$$

In the following, we only consider the case of $2k < r-1 < 2k+1$, $k \in N$ and others are similar.

PROPOSITION 3.1. *For $2k+1 < r < 2k+2, k \in N, n_j > k, j = 1,2$. If $t_1 \in [\ \frac{2n_1+1-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_1+2-r}{\lambda/\sqrt{2}})$ , $t_2 \in [\ \frac{2n_2+1-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_2+2-r}{\lambda/\sqrt{2}})$,*

$$\begin{aligned}
&\int_{t_2}^{\infty} \int_{t_1}^{\infty} \frac{(x_1^2 + x_2^2)^{(r-2)/2}}{e^{\lambda\sqrt{x_1^2+x_2^2}}} dx_1 dx_2 \\
&\leqslant \frac{C_r}{e^{\lambda t_1/\sqrt{2}} e^{\lambda t_2/\sqrt{2}}} \left( \sum_{i=1}^{2n_2+1} \frac{2^{(i+1)/2} \prod_{j=1}^{i-1} (r-1-j)}{\lambda^{i+1} t_2^{i+1-r}} + \sum_{i=1}^{2n_1+1} \frac{2^{(i+1)/2} \prod_{j=1}^{i-1} (r-1-j)}{\lambda^{i+1} t_1^{i+1-r}} \right).
\end{aligned} \tag{3.4}$$

*Proof.* Let $r-1 = \alpha$ and $\lambda/\sqrt{2} = \lambda_1$. Plugging $r-1$ and $\lambda/\sqrt{2}$ into (2.7) of Theorem (2.1) yields, if $t_1 \in [\ \frac{2n_1+1-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_1+2-r}{\lambda/\sqrt{2}})$ , $t_2 \in [\ \frac{2n_2+1-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_2+2-r}{\lambda/\sqrt{2}})$,

$$\frac{\sqrt{2}C_r e^{\frac{-\lambda t_1}{\sqrt{2}}}}{\lambda} \int_{t_2}^{\infty} \frac{x_2^{r-2}}{e^{\frac{\lambda x_2}{\sqrt{2}}}} dx_2 < C_r \sum_{1 \leqslant i \leqslant 2n_2+1} \frac{2^{(i+1)/2} \prod_{j=1}^{i-1} (r-1-j)}{\lambda^{i+1} t_2^{i+1-r} e^{\lambda t_1/\sqrt{2}} e^{\lambda t_2/\sqrt{2}}}. \tag{3.5}$$

$$\frac{\sqrt{2}C_r e^{\frac{-\lambda t_2}{\sqrt{2}}}}{\lambda}\int_{t_1}^{\infty}\frac{x_1^{r-2}}{e^{\frac{\lambda x_1}{\sqrt{2}}}}dx_1 < C_r \sum_{1\leqslant i\leqslant 2n_1+1}\frac{2^{(i+1)/2}\prod_{j=1}^{i-1}(r-1-j)}{\lambda^{i+1}t_1^{i+1-r}e^{\lambda t_1/\sqrt{2}}e^{\lambda t_2/\sqrt{2}}}. \tag{3.6}$$

Combining (3.2), (3.3) (3.5) and (3.6), we obtain (3.4).

Using the similar method, we can obtain results in other case as:

1. $t_1 \in [\ \frac{2n_1+1-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_1+2-r}{\lambda/\sqrt{2}})\ ,\ t_2 \in [\ \frac{2n_2+2-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_2+3-r}{\lambda/\sqrt{2}});$
2. $t_1 \in [\ \frac{2n_1+2-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_1+3-r}{\lambda/\sqrt{2}})\ ,\ t_2 \in [\ \frac{2n_2+1-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_2+2-r}{\lambda/\sqrt{2}});$
3. $t_1 \in [\ \frac{2n_1+2-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_1+3-r}{\lambda/\sqrt{2}})\ ,\ t_2 \in [\ \frac{2n_2+2-r}{\lambda/\sqrt{2}}\ ,\ \frac{2n_2+3-r}{\lambda/\sqrt{2}}).$

On the other hand, for the lower bounds in (3.2), we only consider the case of $2k < r-1 < 2k+1$, $k \in N$ and others are similar.

PROPOSITION 3.2. *For $2k+1 < r < 2k+2, k \in N, n_j > k, j = 1,2.$ If $t_1 \in \left[\ \frac{2n_1+1-r}{\lambda}\ ,\ \frac{2n_1+2-r}{\lambda}\right)$, $t_2 \in \left[\ \frac{2n_2+1-r}{\lambda}\ ,\ \frac{2n_2+2-r}{\lambda}\right)$,*

$$\int_{t_2}^{\infty}\int_{t_1}^{\infty}\frac{(x_1^2+x_2^2)^{(r-2)/2}}{e^{\lambda\sqrt{x_1^2+x_2^2}}}dx_1dx_2 \geqslant \max\left\{\sum_{i=1}^{2n_2}\frac{\prod_{j=1}^{i-1}(r-1-j)}{\lambda^{i+1}t_2^{i+1-r}e^{\lambda t_1}e^{\lambda t_2}},\ \sum_{i=1}^{2n_1}\frac{\prod_{j=1}^{i-1}(r-1-j)}{\lambda^{i+1}t_1^{i+1-r}e^{\lambda t_1}e^{\lambda t_2}}\right\}. \tag{3.7}$$

*Proof.* Let $r-1=\alpha$ and note that the relation between $H(x)$ and $g(x)$. Plugging $r-1$ into (2.7) of Theorem (2.1) yields, if $t_1 \in [\ \frac{2n_1+1-r}{\lambda}\ ,\ \frac{2n_1+2-r}{\lambda})\ ,\ t_2 \in [\ \frac{2n_2+1-r}{\lambda}\ ,\ \frac{2n_2+2-r}{\lambda})$,

$$\int_{t_2}^{\infty}\int_{t_1}^{\infty}\frac{x_2^{r-2}}{e^{\lambda(x_1+x_2)}}dx_1dx_2 = \frac{e^{-\lambda t_1}}{\lambda}\int_{t_2}^{\infty}\frac{x_2^{r-2}}{e^{\lambda x_2}}dx_2 > \sum_{1\leqslant i\leqslant 2n_2}\frac{\prod_{j=1}^{i-1}(r-1-j)}{\lambda^{i+1}t_2^{i+1-r}e^{\lambda t_1}e^{\lambda t_2}}. \tag{3.8}$$

$$\int_{t_2}^{\infty}\int_{t_1}^{\infty}\frac{x_1^{r-2}}{e^{\lambda(x_1+x_2)}}dx_1dx_2 = \frac{e^{-\lambda t_2}}{\lambda}\int_{t_1}^{\infty}\frac{x_1^{r-2}}{e^{\lambda x_1}}dx_1 > \sum_{1\leqslant i\leqslant 2n_1}\frac{\prod_{j=1}^{i-1}(r-1-j)}{\lambda^{i+1}t_1^{i+1-r}e^{\lambda t_1}e^{\lambda t_2}}. \tag{3.9}$$

Combining (3.2), (3.8) and (3.9), we obtain (3.7).

Using the similar method, we can obtain results in other case as:

1. $t_1 \in [\ \frac{2n_1+1-r}{\lambda}\ ,\ \frac{2n_1+2-r}{\lambda})\ ,\ t_2 \in [\ \frac{2n_2+2-r}{\lambda}\ ,\ \frac{2n_2+3-r}{\lambda});$
2. $t_1 \in [\ \frac{2n_1+2-r}{\lambda}\ ,\ \frac{2n_1+3-r}{\lambda})\ ,\ t_2 \in [\ \frac{2n_2+1-r}{\lambda}\ ,\ \frac{2n_2+2-r}{\lambda});$
3. $t_1 \in [\ \frac{2n_1+2-r}{\lambda}\ ,\ \frac{2n_1+3-r}{\lambda})\ ,\ t_2 \in [\ \frac{2n_2+2-r}{\lambda}\ ,\ \frac{2n_2+3-r}{\lambda}).$

Next, we consider the ellipsoidal distribution. In this case, the density function of $\mathbf{x}$ is $|\Sigma|^{-\frac{1}{2}} f(\mathbf{x}'\Sigma^{-1}\mathbf{x})$, $\Sigma$ is positive definite. There exists an orthogonal matrix $D$ such that $\Sigma = D\Lambda D'$ , where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ is the diagonal matrix. Let $B = \Sigma^{-1}$, so

$$B = D\Lambda^{-1}D' = D\,\mathrm{diag}(\lambda_1^{-1}, \lambda_2^{-1})D'.$$

Let $\lambda_{\max}$, $\lambda_{\min}$ be the largest and the smallest eigenvalue of $\Sigma$ respectively, clearly $\lambda_{\max} \geqslant \lambda_{\min} > 0$, and moreover for all $\mathbf{x} \in \mathbf{R}^d$, we have

$$\frac{1}{\lambda_{\max}}\langle x, x\rangle \leqslant \langle x, Bx\rangle \leqslant \frac{1}{\lambda_{\min}}\langle x, x\rangle.$$

Since the density function of ellipsoidal distribution is

$$f(x_1, x_2) = \frac{|\Sigma|^{-1/2}\lambda^r}{2\pi\Gamma(r)}(\mathbf{x}'B\mathbf{x})^{(r-2)/2} e^{-\lambda\sqrt{\mathbf{x}'B\mathbf{x}}},$$

we need to estimate

$$\int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{|\Sigma|^{-\frac{1}{2}}\lambda^r}{2\pi\Gamma(r)}(\mathbf{x}'B\mathbf{x})^{(r-2)/2} e^{-\lambda\sqrt{\mathbf{x}'B\mathbf{x}}} dx_1 dx_2. \tag{3.10}$$

By

$$\frac{(\mathbf{x}'\mathbf{x})^{(r-2)/2}}{\lambda_{\max}^{(r-2)/2} e^{\lambda\sqrt{\frac{\mathbf{x}'\mathbf{x}}{\lambda_{\min}}}}} \leqslant \frac{(\mathbf{x}'B\mathbf{x})^{(r-2)/2}}{e^{\lambda\sqrt{\mathbf{x}'B\mathbf{x}}}} \leqslant \frac{(\mathbf{x}'\mathbf{x})^{(r-2)/2}}{\lambda_{\min}^{(r-2)/2} e^{\lambda\sqrt{\frac{\mathbf{x}'\mathbf{x}}{\lambda_{\max}}}}}.$$

We have

$$\begin{aligned}\int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{(\mathbf{x}'\mathbf{x})^{(r-2)/2}}{\lambda_{\max}^{(r-2)/2} e^{\lambda\sqrt{\frac{\mathbf{x}'\mathbf{x}}{\lambda_{\min}}}}} dx_1 dx_2 &\leqslant \int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{(\mathbf{x}'B\mathbf{x})^{(r-2)/2}}{e^{\lambda\sqrt{\mathbf{x}'B\mathbf{x}}}} dx_1 dx_2 \\ &\leqslant \int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{(\mathbf{x}'\mathbf{x})^{(r-2)/2}}{\lambda_{\min}^{(r-2)/2} e^{\lambda\sqrt{\frac{\mathbf{x}'\mathbf{x}}{\lambda_{\max}}}}} dx_1 dx_2. \end{aligned} \tag{3.11}$$

Let $\lambda_2 = \frac{\lambda}{\sqrt{\lambda_{\min}}}$, $\lambda_3 = \frac{\lambda}{\sqrt{\lambda_{\max}}}$, using Proposition 3.1 and 3.2, we can get the estimate of lower and upper bounds in (3.11). Combining all above formulas, we get the final proposition in this paper.

PROPOSITION 3.3. *For* $2k+1 < r < 2k+2, k \in N, n_j > k, j = 1, 2$. *If* $t_1 \in [\frac{2n_1+1-r}{\lambda/\sqrt{2\lambda_{\max}}}, \frac{2n_1+2-r}{\lambda/\sqrt{2\lambda_{\max}}})$, $t_2 \in [\frac{2n_2+1-r}{\lambda/\sqrt{2\lambda_{\max}}}, \frac{2n_2+2-r}{\lambda/\sqrt{2\lambda_{\max}}})$, *we have*

$$\begin{aligned}&\int_{t_2}^{\infty}\int_{t_1}^{\infty} \frac{|\Sigma|^{-\frac{1}{2}}\lambda^r}{2\pi\Gamma(r)}(\boldsymbol{x}'B\boldsymbol{x})^{(r-2)/2} e^{-\lambda\sqrt{\boldsymbol{x}'B\boldsymbol{x}}} dx_1 dx_2 \\ &\leqslant \frac{|\Sigma|^{-\frac{1}{2}}\lambda^r C_r}{2\pi\Gamma(r)\lambda_{\min}^{(r-2)/2} e^{\lambda t_1/\sqrt{2\lambda_{\max}}} e^{\lambda t_2/\sqrt{2\lambda_{\max}}}} \times \\ &\quad \times \left( \sum_{i=1}^{2n_2+1} \frac{2^{(i+1)/2}\prod_{j=1}^{i-1}(r-1-j)}{(\lambda/\sqrt{\lambda_{\max}})^{i+1} t_2^{i+1-r}} + \sum_{i=1}^{2n_1+1} \frac{2^{(i+1)/2}\prod_{j=1}^{i-1}(r-1-j)}{(\lambda/\sqrt{\lambda_{\max}})^{i+1} t_1^{i+1-r}} \right).\end{aligned}$$

*If* $t_1 \in [\ \frac{2n_1+1-r}{\lambda/\sqrt{\lambda_{\min}}}\ ,\ \frac{2n_1+2-r}{\lambda/\sqrt{\lambda_{\min}}})$ , $t_2 \in [\ \frac{2n_2+1-r}{\lambda/\sqrt{\lambda_{\min}}}\ ,\ \frac{2n_2+2-r}{\lambda/\sqrt{\lambda_{\min}}})$, *we have*

$$\begin{aligned}
&\int_{t_2}^{\infty}\int_{t_1}^{\infty}\frac{|\Sigma|^{-\frac{1}{2}}\lambda^r}{2\pi\Gamma(r)}(\boldsymbol{x}'B\boldsymbol{x})^{(r-2)/2}e^{-\lambda\sqrt{\boldsymbol{x}'B\boldsymbol{x}}}dx_1dx_2\\
\geqslant\ &\frac{|\Sigma|^{-\frac{1}{2}}\lambda^r}{2\pi\Gamma(r)\lambda_{\max}^{(r-2)/2}e^{\lambda t_1/\sqrt{\lambda_{\min}}}e^{\lambda t_2/\sqrt{\lambda_{\min}}}}\\
&\quad\times\max\left\{\sum_{i=1}^{2n_2}\frac{\prod_{j=1}^{i-1}(r-1-j)}{(\lambda/\sqrt{\lambda_{\min}})^{i+1}t_2^{i+1-r}},\ \sum_{i=1}^{2n_1}\frac{\prod_{j=1}^{i-1}(r-1-j)}{(\lambda/\sqrt{\lambda_{\min}})^{i+1}t_1^{i+1-r}}\right\}.
\end{aligned}$$

## 4. Proof of Theorem 2.1

First, in the case of $0<\alpha-1<1$, assuming $\alpha_1=\alpha$. On the one hand, from (2.2), by variable substitution, we have

$$G(x)=\int_0^{\infty}(x+t)^{\alpha_1-1}e^{-\lambda t}dt. \tag{4.1}$$

From the derivative of (4.1) with respect to $x$, we have, for $l\in N$,

$$G^{(2l+1)}(x)>0\ ,\ G^{(2l+2)}(x)<0. \tag{4.2}$$

On the other hand, differentiating $G(x)=e^{\lambda x}g(x)$ directly yields

$$G^{(l+1)}(x)=\lambda G^{(l)}(x)-\left[\prod_{1\leqslant i\leqslant l}(\alpha_1-i)\right]x^{\alpha_1-l-1}, \tag{4.3}$$

where $l\in N$. Combining (4.2) with (4.3), we have

$$g_1(x)>\sum_{1\leqslant i\leqslant 2l+1}\frac{\prod_{j=1}^{i-1}(\alpha_1-j)}{\lambda^i x^{i-\alpha_1}e^{\lambda x}}=R_{2l+1}, \tag{4.4}$$

$$g_1(x)<\sum_{1\leqslant i\leqslant 2l+2}\frac{\prod_{j=1}^{i-1}(\alpha_1-j)}{\lambda^i x^{i-\alpha_1}e^{\lambda x}}=R_{2l+2}. \tag{4.5}$$

where $g_1(x)=\int_x^{\infty}y^{\alpha_1-1}e^{-\lambda y}dy$. It is easy to see $R_{2l+1}<R_{2l+2}$.

If $x\in[\ 0,\frac{2-\alpha_1}{\lambda}\ )$, we have

$$\cdots<R_{2l+1}<R_{2l-1}<\cdots<R_1<g_1(x)<R_2<\cdots<R_{2l}<R_{2l+2}<\cdots,$$

thus $R_1$ and $R_2$ are the best lower and upper bounds.

If $x \in [\ \frac{2-\alpha_1}{\lambda}, \frac{3-\alpha_1}{\lambda}\ )$,

$$\cdots < R_{2l+1} < R_{2l-1} < \cdots < R_3 < g_1(x) < R_2 < \cdots < R_{2l} < R_{2l+2} < \cdots,$$

since $R_3 > R_1$, thus $R_3$ and $R_2$ are the best lower and upper bounds.

By induction, if $x \in [\ \frac{2n-\alpha_1}{\lambda}, \frac{2n+1-\alpha_1}{\lambda}\ )$, $n \in Z^+$,

$$\cdots < R_{2n+3} < R_{2n+1} < g_1(x) < R_{2n} < R_{2n+2} < \cdots,$$

since $R_{2n+1} > R_{2m+1}$, $R_{2n} < R_{2m}$, when $n > m$. Thus $R_{2n+1}$ and $R_{2n}$ are the best lower and upper bounds.

If $x \in [\ \frac{2n+1-\alpha_1}{\lambda}, \frac{2n+2-\alpha_1}{\lambda}\ )$,

$$\cdots < R_{2n+3} < R_{2n+1} < g_1(x) < R_{2n+2} < R_{2n+4} < \cdots,$$

since $R_{2n+1} > R_{2m+1}$, $R_{2n+2} < R_{2m+2}$, when $n > m$. Thus $R_{2n+1}$ and $R_{2n+2}$ are the best lower and upper bounds. From (4.4) and (4.5), it is easy to see that $\lim\limits_{x\to\infty} R_{2n+1} = R_{2n+2}$.

Combining (2.1), (4.4) and (4.5), we have, if $x \in [\ \frac{2n-\alpha_1}{\lambda}\ ,\ \frac{2n+1-\alpha_1}{\lambda}\ )$,

$$\sum_{1\leqslant i\leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha_1 - j)}{\lambda^i x^{i-1}} < H_1(x) < \sum_{1\leqslant i\leqslant 2n} \frac{\prod_{j=1}^{i-1}(\alpha_1 - j)}{\lambda^i x^{i-1}}, \tag{4.6}$$

where $H_1(x) = e^{\lambda x} g_1(x)/x^{\alpha_1 - 1}$.

If $x \in [\ \frac{2n+1-\alpha_1}{\lambda}\ ,\ \frac{2n+2-\alpha_1}{\lambda}\ )$,

$$\sum_{1\leqslant i\leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha_1 - j)}{\lambda^i x^{i-1}} < H_1(x) < \sum_{1\leqslant i\leqslant 2n+2} \frac{\prod_{j=1}^{i-1}(\alpha_1 - j)}{\lambda^i x^{i-1}}. \tag{4.7}$$

Next, if $2k < \alpha - 1 < 2k+1$, that is $2k+1 < \alpha < 2k+2$, we write $\alpha_{2k+1} = \alpha$. For every $\alpha_{2k+1} \in (2k+1, 2k+2)$, there exists a $\alpha_1 \in (1,2)$ such that $\alpha_{2k+1} = \alpha_1 + 2k$. Assuming

$$g_{2k+1}(x) = \int_x^\infty y^{\alpha_{2k+1}-1} e^{-\lambda y} dy.$$

Then

$$g_{2k+1}(x) = \sum_{1\leqslant i\leqslant 2k} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+1} - j)}{\lambda^i x^{i-\alpha_{2k+1}} e^{\lambda x}} + \frac{\prod_{i=1}^{2k}(\alpha_{2k+1} - i)}{\lambda^{2k}} \int_x^\infty \frac{y^{\alpha_{2k+1}-2k-1}}{e^{\lambda y}} dy.$$

$$\begin{aligned} H_{2k+1}(x) &= \frac{g_{2k+1}(x)}{x^{\alpha_{2k+1}-1} e^{-\lambda x}} \qquad (4.8)\\ &= \sum_{1\leqslant i\leqslant 2k} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+1} - j)}{\lambda^i x^{i-1}} + \frac{\prod_{i=1}^{2k}(\alpha_{2k+1} - i)}{\lambda^{2k} x^{2k}} \frac{\int_x^\infty y^{\alpha_{2k+1}-2k-1} e^{-\lambda y} dy}{x^{\alpha_{2k+1}-2k-1} e^{-\lambda x}}. \end{aligned}$$

If $x \in [\frac{2n-\alpha_{2k+1}}{\lambda}, \frac{2n+1-\alpha_{2k+1}}{\lambda})$, $n > k$, that is $x \in [\frac{2n-2k-\alpha_1}{\lambda}, \frac{2n-2k+1-\alpha_1}{\lambda})$, combining (4.6) and (4.8), we have

$$\sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+1}-j)}{\lambda^i x^{i-1}} < H_{2k+1}(x) < \sum_{1 \leqslant i \leqslant 2n} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+1}-j)}{\lambda^i x^{i-1}}.$$

Similarly, if $x \in [\frac{2n+1-\alpha_{2k+1}}{\lambda}, \frac{2n+2-\alpha_{2k+1}}{\lambda})$, that is $x \in [\frac{2n-2k+1-\alpha_1}{\lambda}, \frac{2n-2k+2-\alpha_1}{\lambda})$, combining (4.7) and (4.8), we have

$$\sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+1}-j)}{\lambda^i x^{i-1}} < H_{2k+1}(x) < \sum_{1 \leqslant i \leqslant 2n+2} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+1}-j)}{\lambda^i x^{i-1}}.$$

Using the similar method as above for $2k+1 < \alpha - 1 < 2k+2$, we have the following. If $x \in [\frac{2n-\alpha_{2k+2}}{\lambda}, \frac{2n+1-\alpha_{2k+2}}{\lambda})$,

$$\sum_{1 \leqslant i \leqslant 2n} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+2}-j)}{\lambda^i x^{i-1}} < H_{2k+2}(x) < \sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+2}-j)}{\lambda^i x^{i-1}}.$$

If $x \in [\frac{2n+1-\alpha_{2k+2}}{\lambda}, \frac{2n+2-\alpha_{2k+2}}{\lambda})$,

$$\sum_{1 \leqslant i \leqslant 2n+2} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+2}-j)}{\lambda^i x^{i-1}} < H_{2k+2}(x) < \sum_{1 \leqslant i \leqslant 2n+1} \frac{\prod_{j=1}^{i-1}(\alpha_{2k+2}-j)}{\lambda^i x^{i-1}}.$$

Combining similar results in any other case, we obtain the uniform expansions in (2.5), (2.6), (2.7) and (2.8), respectively. Using the similar argument above, we obtain that the result for $0 < \alpha < 1$ is the special case of (2.7) and (2.8).

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