Mathematical
Inequalities
& Applications
Volume 15, Number 2 (2012), 247–269

INEQUALITIES FOR THE NORMS OF FINITE DIFFERENCE OPERATORS OF MULTIPLY MONOTONE SEQUENCES

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(*Communicated by B. Opic*)

Abstract. In this paper we shall present discrete Kolmogorov type inequalities for multiply monotone sequences defined on non-positive integers. Moreover, we will provide a more delicate information by obtaining the description of the following modulus of continuity

$$\omega_{p,q}^{k,j,r}(\delta,\varepsilon)=\sup\{\|\Delta^k x\|_q : x,\Delta x,\dots,\Delta^j x\geqslant 0,\ \|x\|_p=\delta,\ \|\Delta^r x\|_\infty=\varepsilon\}$$

for $\delta\geqslant\varepsilon>0$ and values of $j=r-2$ or $j=r-1$ depending on values of other parameters.

1. Notation, definitions, and history

Let $M:=\mathbf{Z}_-\cup\{0\}=\{\cdots,-2,-1,0\}$. We shall define the (forward) difference operator as follows: for the sequence $x=\{x_m\}_{m\in M}$ set

$$\Delta x:=\{x_m-x_{m-1}\}_{m\in M},$$

$$\Delta^0 x:=x,\ \ \Delta^1 x:=\Delta x,$$

and, recursively,

$$\Delta^j x:=\Delta(\Delta^{j-1})x\quad\text{for}\quad j=2,3,\dots.$$

Let $l_p=l_p(M)$, $p\in[1,\infty]$, be the space of all real-valued sequences defined on M such that the norm

$$\|x\|_p:=\begin{cases}\left(\sum\limits_{m\in M}|x_m|^p\right)^{1/p}, & 1\leqslant p<\infty,\\ \sup\limits_{m\in M}|x_m|, & p=\infty,\end{cases}$$

is finite. Note that in contrast with the derivative operator $D=\frac{d}{dt}$, the difference operator Δ is a bounded linear operator defined on all $l_p(M)$ for any $1\leqslant p\leqslant\infty$.

In this paper we shall consider sharp discrete inequalities of Kolmogorov type, i.e. inequalities of the form

$$\|\Delta^k x\|_q\leqslant C\|x\|_p^\alpha\|\Delta^r x\|_s^{1-\alpha},\qquad x\in l_p(U),\ \ \Delta^r x\in l_s(U)\tag{1}$$

where $U=M$ or $U=\mathbf{Z}$ and $\alpha\in[0,1]$ is defined by relation (9) below. The exact constants $C=C(k,r,p,q,s)$ in inequalities of this form have been found in the following cases:

Mathematics subject classification (2010): 26D10, 47B39, 39A70, 65L12.
Keywords and phrases: Sequences, finite differences, multiply monotone, discrete inequality, rearrangements, comparison theorem.

Paper MIA-15-21

1. $U = \mathbf{Z}$, $q = p = s = \infty$:

   (a) $k = r-1$ by Z. Ditzian in 1983 [3];

   (b) $2 \leqslant r \leqslant 4, k < r$ and $k = 2, r = 5$ by H.G. Kaper and B.E. Spellman in 1987 [4];

   (c) $k = 3, r = 5$ by M. Kwong and A. Zettl in 1988 [9] (see also [10]);

   (d) necessary and sufficient conditions for the existence of inequality (1) were obtained by M. Kwong and A. Zettl in 1988 [9]

   (e) $k = 1, \forall r \in \mathbf{N}$ by J. Velikina in 1999 [13];

   (f) upper estimate for $C(k, r, \infty, \infty, \infty)$ by J. Velikina in 1999 [13].

2. $U = \mathbf{Z}$, $k = 1, r = 2$ and $p = q = s = 1, 2$ (see [8] for references).

3. $U = \mathbf{Z}_+$, $k = 1, r = 2$ and $p = q = s = 1, 2, \infty$ (see [8] for references).

Note that the sharp constants in the discrete and continuous cases do not necessarily coincide (see [9]).

In this paper we shall investigate a more narrow class of sequences, namely, the class $l_p^j(M)$, $j \in \mathbf{N} \cup \{0\}$, $p \in [1, \infty]$, of sequences $x : M \to \mathbf{R}$ with the properties:
1) $x(m) \geqslant 0$, $\forall m \in M$;
2) $x \in l_p(M)$;
3) $\Delta^j x(m) \geqslant 0$, $\forall m \in M$.

Sequences from this class we shall call *multiply monotone* or j*-monotone* sequences.

The inequalities we obtain in this paper are discrete analogues of results by V. F. Babenko and Yu. Babenko [1]. We adopt some of the ideas from this paper for the discrete case and use results by Hardy, Littlewood and Polya [11] from majorization theory to obtain our results. Once again, we would like to emphasize that best constants do not have to coincide in the continuous and discrete cases.

Properties 1)–3) imply a larger set of properties some of which are described below.

PROPOSITION 1. *Let* $j \in \mathbf{N}$ *and* $p \in [1, \infty]$. *For every sequence* $x \in l_p^j(M)$*, there holds*

$$1)\ \ \Delta^k x(m) \geqslant 0,\ \ \forall m \in M,\ \ 0 \leqslant k \leqslant j;$$

$$2)\ \ \|\Delta^k x\|_q < \infty,\ \ k \in \mathbf{N},\ \ q \in [1, \infty].$$

The proof of this proposition is given in Section 3

Considering this class $l_p^j(M)$ (more narrow comparing to $l_p(M)$) will allow us to substantially expand the possibilities for the parameters. More precisely, in this paper we shall solve (a more delicate) problem of finding the modulus of continuity

$$\omega_{p,q}^{k,j,r}(\delta, \varepsilon) := \sup\{\|\Delta^k x\|_q : x \in l_p^j(M),\ \|x\|_p = \delta,\ \|\Delta^r x\|_\infty = \varepsilon\}, \qquad (2)$$

for numbers $p, q \in [1, \infty]$, $\delta \geqslant \varepsilon > 0$, integers $0 < k < r$, and $j = r-2$ or $r-1$ (depending on the values of q and k). Note that whenever $\delta < \varepsilon$, the set over which the supremum is taken in the above definition of the modulus of continuity, is empty if $2j \geqslant r$ (see relation (12) in Proposition 6). In the case $r \geqslant 4$, inequality $\delta \geqslant \varepsilon > 0$ describes the set of all admissible values of δ and ε. In the case $r = 3$ the set of admissible pairs (δ, ε) can be wider (see Remark after Proposition 6).

Note that for the general class of sequences $l_\infty(M)$ inequality without any restrictions on the parameters k, r, q, p, s is not possible (see the discrete analog of Gabushin's existence theorem in [9]).

2. Main results

Before we state the main results of this paper, let us construct the extremal sequence that will play a crucial role in our results.

Construction of the extremal (or comparison) sequence. The r^{th} difference $\Delta^r \varphi_{a,h}$ of the extremal sequence $\varphi_{a,h}$ for $a \in M$ and $h \in [0,1)$ we define to be

$$\Delta^r \varphi_{a,h}(n) := \begin{cases} 0, & n < a; \\ h, & n = a; \\ 1, & n > a. \end{cases} \tag{3}$$

Now define the operator $S : l_1(M) \to l_\infty(M)$ in the following way:

$$S(x;n) := \sum_{i=-\infty}^{n} x(i) \tag{4}$$

and let $S^1 := S$ and

$$S^k := S^1 \circ S^{k-1}, \quad k = 2, 3, \ldots. \tag{5}$$

Define the extremal sequence $\varphi_{a,h}$ to be

$$\varphi_{a,h}(n) := S^r(\Delta^r \varphi_{a,h}; n). \tag{6}$$

For every value of parameters $a \in M \setminus \{0\}$ and $h \in [0,1)$, we have $\|\varphi_{a,h}\|_\infty = \varphi_{a,h}(0) = C^r_{r-a-1} + hC^{r-1}_{r-a-1}$ (see Proposition 7), and $\varphi_{0,h}(0) = h$, $h \in [0,1)$. It is not difficult to see that the value of $\|\varphi_{a,h}\|_p$, $p \in [1,\infty]$, varies continuously from 0 to ∞ as parameter $-a+h$ increases. This allows us to choose the values of parameters a and h to obtain any given non-negative value for $\|\varphi_{a,h}\|_p$.

THEOREM 1. *Let $r, k \in \mathbf{N}$, $k < r$, and $p, q \in [1,\infty]$. If we have either $k \leqslant r-2$ or $q = \infty$, then for every $\delta \geqslant \varepsilon > 0$, we have*

$$\omega^{k,r-2,r}_{p,q}(\delta,\varepsilon) = \varepsilon \|\Delta^k \varphi_{a,h}\|_q, \tag{7}$$

where parameters a and h of sequence $\varphi_{a,h}$ are chosen so that $\|\varphi_{a,h}\|_p = \delta/\varepsilon$.

In the case $k = r-1$ and $1 \leqslant q < \infty$, for every $\delta \geqslant \varepsilon > 0$, there holds

$$\omega^{r-1,r-1,r}_{p,q}(\delta,\varepsilon) = \varepsilon \|\Delta^{r-1} \varphi_{a,h}\|_q, \tag{8}$$

where parameters a and h are chosen so that $\|\varphi_{a,h}\|_p = \delta/\varepsilon$.

REMARK. At least in some cases when $k = r-1$ and $1 \leqslant q < \infty$, relation (7) does not hold. We give a counterexample for the case $r = 3$, $k = 2$, $p = \infty$, and $q = 1$ in Section 4.

As an application of Theorem 1 we obtain the multiplicative inequality for multiply monotone sequences for arbitrary $q, p \in [1, \infty]$ and any $k < r$. The result is provided by the following theorem and the details of the proof are given in the Section 8.

We let

$$\alpha = \alpha(k,r,p,q) := \frac{r-k+1/q}{r+1/p}, \tag{9}$$

where quantities $1/p$ and $1/q$ are set equal to zero if p or q equal ∞.

THEOREM 2. *Let* $1 \leqslant k < r$ *be integers and* $p, q \in [1, \infty]$. *Let* $x \in l_p^{r-1}(M)$ *if* $1 \leqslant q < \infty$ *and* $k = r-1$, *and* $x \in l_p^{r-2}(M)$ *in all other cases. Then the following sharp inequality holds*

$$\|\Delta^k x\|_q \leqslant D_{p,q}^{k,r} \cdot \|x\|_p^{\alpha} \cdot \|\Delta^r x\|_\infty^{1-\alpha}, \tag{10}$$

where $D_{p,q}^{k,r}$ *is a finite constant defined as*

$$D_{p,q}^{k,r} := \sup_{\substack{a \in M,\ h \in [0,1) \\ (a,h) \neq (0,0)}} \frac{\|\Delta^k \varphi_{a,h}\|_q}{\|\varphi_{a,h}\|_p^{\alpha}}.$$

Moreover, for $1 \leqslant k \leqslant r-1$, *we have*

$$D_{\infty,\infty}^{k,r} = \frac{(r!)^{\frac{r-k}{r}}}{(r-k)!},$$

$$D_{1,\infty}^{k,r} = \frac{((r+1)!)^{\frac{r-k}{r+1}}}{(r-k)!},$$

$$D_{\infty,1}^{k,r} = \frac{(r!)^{\frac{r-k+1}{r}}}{(r-k+1)!},$$

$$D_{1,1}^{k,r} = \frac{((r+1)!)^{\frac{r-k+1}{r+1}}}{(r-k+1)!}.$$

REMARK. In the case $p = q = \infty$ inequality (10) turns into equality only for constant sequences x.

REMARK. Note that the sharp constant in inequality (10) for $p = q = \infty$ is the same as in its continuous version obtained by Olovyanishnikov [12].

As a corollary of Theorem 2 one can obtain the known inequality for general bounded sequences on M when $p = q = s = \infty$, $k = 1$, $r = 2$ (see Kwong and Zettl [9]).

3. Some properties of multiply monotone sequences

The proof of Proposition 1 will use the following auxiliary statements.

PROPOSITION 2. *If sequence x is bounded, then any $\Delta^k x$ is bounded as well.*

Proof.

$$\|\Delta x\|_\infty = \sup_{n\in M} |x(n) - x(n-1)| \leqslant 2\|x\|_\infty.$$

For an arbitrary k, the statement is obtained by induction. □

PROPOSITION 3. *If sequence x is non-negative, bounded, and $\Delta^j x(m) \geqslant 0$ for all $m \in M$, then $\Delta^k x(m) \geqslant 0$ for all $m \in M$ and all $k = 0,1,...,j$.*

Proof. The assertion of Proposition 3 holds trivially when $j = 1$. Let now $j \geqslant 2$ be arbitrary integer and assume that the proposition holds for $j-1$. Let $x \in l_\infty^j(M)$ be an arbitrary sequence. Then $\Delta^{j-1}x$ is non-decreasing (since $\Delta^j x \geqslant 0$ by assumption) and is bounded due to Proposition 2. Denote

$$b := \lim_{m\to-\infty} \Delta^{j-1}x(m).$$

If it were that $b < 0$, for every integer m less than or equal to some number $n_0 \in M$, we would have $\Delta^{j-1}x(m) < b/2$. Hence, for each $m < n_0$, we would have

$$\begin{aligned}\Delta^{j-2}x(m) &= \Delta^{j-2}x(n_0) - \sum_{i=m+1}^{n_0} (\Delta^{j-2}x(i) - \Delta^{j-2}x(i-1)) \\ &= \Delta^{j-2}x(n_0) - \sum_{i=m+1}^{n_0} \Delta^{j-1}x(i) > \Delta^{j-2}x(n_0) - b(n_0 - m)/2,\end{aligned}$$

which would imply that $\Delta^{j-2}x$ is unbounded. Since by Proposition 2, sequence $\Delta^{j-2}x$ is bounded, we obtain a contradiction, which shows that $b \geqslant 0$. Since $\Delta^{j-1}x$ is non-decreasing, we have that $\Delta^{j-1}x$ is non-negative and, hence, $x \in l_\infty^{j-1}(M)$. By the induction assumption, we have that $\Delta^k x$ is non-negative for every $k = 0,1,\ldots,j-1$. Since $\Delta^j x$ is also non-negative, we obtain the assertion of the proposition for the considered value of j. Proposition 3 is proved. □

PROPOSITION 4. *Let $q \in [1,\infty]$. If $x \in l_q(M)$, then for every $k \in \mathbf{N}$, we have $\Delta^k x \in l_q(M)$.*

Proof. In the case $q = \infty$, the assertion of this proposition follows from Proposition 2. Let us now take $1 \leqslant q < \infty$. For every $x \in l_q(M)$, in view of convexity of the function $z(t) = t^q$, $q \geqslant 1$, we have

$$\begin{aligned}\|\Delta x\|_q^q &= \sum_{m\in M} |\Delta x(m)|^q = \sum_{m\in M} |x(m) - x(m-1)|^q \\ &\leqslant 2^q \sum_{m\in M} \left(\frac{|x(m)| + |x(m-1)|}{2}\right)^q \\ &\leqslant 2^{q-1} \sum_{m\in M} (|x(m)|^q + |x(m-1)|^q) \leqslant 2^q \|x\|_q^q,\end{aligned}$$

which implies that $\Delta x \in l_q(M)$. For an arbitrary k the statement is obtained by induction. $\square$

Proof of Proposition 1. It is not difficult to see that every sequence in $l_p^j(M)$ belongs to $l_\infty^j(M)$ and in view of Propositions 3, we obtain 1). Statement 2) for $q=\infty$ follows from Proposition 2. It remains to prove 2) for $q<\infty$. In view of Proposition 3, we have $\Delta x(m) \geqslant 0$, $m \in M$. Hence,

$$\|\Delta x\|_1 = \sum_{m\in M} \Delta x(m) = x(0) - \lim_{m\to-\infty} x(m) \leqslant x(0) = \|x\|_\infty < \infty, \tag{11}$$

and we have that $\Delta x \in l_1(M)$. Then $0 \leqslant \Delta x(m) < 1$ for every $m \in M$ sufficiently large negative, and we will have $(\Delta x(m))^q \leqslant \Delta x(m)$ for such m. Hence, $\Delta x \in l_q(M)$. Then by Proposition 4, we have $\Delta^k x \in l_q(M)$ for every $k \in \mathbf{N}$, which completes the proof of statement 2). Proposition 1 is proved. $\square$

Next statement gives a pointwise estimate of differences of order up to $2j$ of a sequence from $l_\infty^j(M)$.

PROPOSITION 5. *Let $j \in \mathbf{N}$ and $x \in l_\infty^j(M)$. Then for every $k = 0,1,\dots,2j$, there holds*

$$\left|\Delta^k x(m)\right| \leqslant x(m), \quad m \in M.$$

Proof. We shall proceed by induction. In the case $x \in l_\infty^1(M)$ we have $\left|\Delta^0 x(m)\right| = x(m)$, $m \in M$,

$$|\Delta x(m)| = \Delta x(m) = x(m) - x(m-1) \leqslant x(m), \ \ m \in M,$$

and for $m \in M$,

$$\begin{aligned} \left|\Delta^2 x(m)\right| &= |\Delta x(m) - \Delta x(m-1)| \\ &\leqslant \Delta x(m) + \Delta x(m-1) = x(m) - x(m-2) \leqslant x(m). \end{aligned}$$

If the assertion of the proposition holds for $j-1$, $j \geqslant 2$, then for every $x \in l_\infty^j(M)$, we have $\Delta x \in l_\infty^{j-1}(M)$ and by the induction assumption, for every $k = 1,2,\dots,2j-1$, there holds

$$\left|\Delta^k x(m)\right| = \left|\Delta^{k-1}\Delta x(m)\right| \leqslant \Delta x(m) = x(m) - x(m-1) \leqslant x(m), \ \ m \in M.$$

Finally, using the induction assumption again, for all $m \in M$, we obtain

$$\begin{aligned} \left|\Delta^{2j} x(m)\right| &= \left|\Delta^{2j-1} x(m) - \Delta^{2j-1} x(m-1)\right| \\ &\leqslant \left|\Delta^{2(j-1)}\Delta x(m)\right| + \left|\Delta^{2(j-1)}\Delta x(m-1)\right| \\ &\leqslant \Delta x(m) + \Delta x(m-1) = x(m) - x(m-2) \leqslant x(m). \end{aligned}$$

Proposition 5 is proved. $\square$

As a corollary of Proposition 5 we obtain the following statement.

PROPOSITION 6. *Let* $j \in \mathbf{N}$ *and* $q \in [1,\infty]$.

1. *For every* $x \in l_q^j(M)$*, there holds*

$$\|\Delta^k x\|_q \leqslant \|x\|_q, \quad k = 0,1,\ldots,2j. \tag{12}$$

In the case $x \in l_\infty^0(M)$*, relation (12) holds for* $q = \infty$ *and* $k = 1$.

2. *For every* $x \in l_\infty^j(M)$*, we have*

$$\|\Delta^k x\|_q \leqslant \|x\|_\infty, \quad k = 1,\ldots,2j-1. \tag{13}$$

REMARK. Even if $x \in l_1^j(M)$, inequalities (12) and (13) do not hold in general for $k = 2j+1$ and $k = 2j$, respectively. A counterexample is given by the sequence $\varphi_{a,0}$ corresponding to $a = -2$ and $r = j-1$. For this sequence we have

$$\|\varphi_{a,0}\|_q = (j^q+1)^{1/q} < \|\Delta^{2j+1}\varphi_{a,0}\|_q = ((j+1)^q+1)^{1/q}$$

and

$$\|\varphi_{a,0}\|_\infty = j < \|\Delta^{2j}\varphi_{a,0}\|_q = (j^q+1)^{1/q}.$$

Proof of Proposition 6. Relation (12) follows immediately from Proposition 5. If $x \in l_\infty^0(M)$, for every $m \in M$, we have

$$|\Delta x(m)| = |x(m) - x(m-1)| \leqslant \max\{x(m), x(m-1)\} \leqslant \|x\|_\infty,$$

which implies (12) with $q = \infty$ and $k = 1$.

For every $x \in l_\infty^j(M)$, $j \in \mathbf{N}$, by Proposition 1, we have $\Delta x \in l_q^{j-1}(M)$. Applying (12) to sequence Δx, in the case $j \geqslant 2$, we have

$$\|\Delta^{k+1} x\|_q \leqslant \|\Delta x\|_q, \quad k = 0,1,\ldots,2j-2$$

(in the case $j = 1$ this relation is trivial). Since for every sequence $y \in l_1(M)$, there holds $\|y\|_q \leqslant \|y\|_1$, using estimate analogous to (11), we obtain

$$\|\Delta^k x\|_q \leqslant \|\Delta x\|_q \leqslant \|\Delta x\|_1 \leqslant \|x\|_\infty, \quad k = 1,\ldots,2j-1.$$

Proposition 6 is proved. □

4. Some auxiliary statements

Formula for the values of the sequence $\varphi_{a,h}$. As usual, by C_r^k we denote the binomial coefficients and we agree that $C_{r-1}^r = 0$.

PROPOSITION 7. *For every* $a \in M$ *and* $h \in [0,1)$*, we have*

$$\varphi_{a,h}(m) = \begin{cases} 0, & m < a, \\ h, & m = a, \\ C_{m-a+r-1}^r + hC_{m-a+r-1}^{r-1}, & m > a. \end{cases}$$

Hence,

$$\|\varphi_{a,h}\|_\infty = C_{r-a-1}^r + hC_{r-a-1}^{r-1}. \tag{14}$$

Proof. We denote by

$$\psi(m) := \begin{cases} 0, & m < 0, \\ h, & m = 0, \\ 1, & m > 0. \end{cases}$$

Note that $\Delta^r \varphi_{a,h}(m) = \psi(m-a)$, $m \in M$. We shall show that

$$S^k(\psi; m) = C^k_{m+k-1} + hC^{k-1}_{m+k-1}, \quad m, k \in \mathbf{N}, \tag{15}$$

using induction (on the sum). It is not difficult to see that $S^1(\psi;m) = m + h = C^1_m + hC^0_m$, $m \in \mathbf{N}$, and $S^k(\psi;1) = 1 + kh = C^k_k + C^{k-1}_k h$. Let $m > 1$ and $k > 1$ be arbitrary integers. Assume that (15) holds for pairs $(m-1,k)$ and $(m,k-1)$. Then

$$\begin{aligned} S^k(\psi;m) &= \sum_{i=-\infty}^{m} S^{k-1}(\psi;i) = \sum_{i=-\infty}^{m-1} S^{k-1}(\psi;i) + S^{k-1}(\psi;m) \\ &= S^k(\psi;m-1) + S^{k-1}(\psi;m) = C^k_{m+k-2} + hC^{k-1}_{m+k-2} \\ &\quad + C^{k-1}_{m+k-2} + hC^{k-2}_{m+k-2} = C^k_{m+k-1} + hC^{k-1}_{m+k-1}. \end{aligned}$$

Then for every $m \in M$, $m > a$, we have

$$\varphi_{a,h}(m) = S^r\left(\Delta^r \varphi_{a,h}; m\right) = S^r(\psi; m-a) = C^r_{m-a+r-1} + hC^{r-1}_{m-a+r-1}.$$

Since $\|\varphi_{a,h}\|_\infty = \varphi_{a,h}(0)$, we obtain (14). Values of $\varphi_{a,h}$ for $m \leqslant a$ are obtained trivially. Proposition 7 is proved. □

Counterexample for relation (7) in case $k = r-1$ and $q < \infty$. Let $k = 2$, $r = 3$, $p = \infty$, and $q = 1$. Define a parametric family of sequences $z_c = S(y;\cdot)$, $c \in \mathbf{N}$, where

$$y(n) = \begin{cases} 1, & 1-2c \leqslant n \leqslant -1, \ n \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases}$$

Then $z_c \in l^1_\infty(M)$ and we also have

$$\Delta^2 z_c(n) = \Delta y(n) = \begin{cases} 1, & 1-2c \leqslant n \leqslant -1, \ n \text{ is odd}, \\ -1, & 2-2c \leqslant n \leqslant 0, \ n \text{ is even}, \\ 0, & n \leqslant -2c, \end{cases}$$

Let $c \geqslant 2$. For every $\varepsilon > 0$, let $\delta = c\varepsilon/2$. Choose a and h so that $\|\varphi_{a,h}\|_\infty = \delta/\varepsilon = c/2$, where $\varphi_{a,h}$ corresponds to $r = 3$. Then $x = \frac{\varepsilon}{2} z_c \in l^1_\infty(M)$ and

$$\|x\|_\infty = \frac{\varepsilon}{2}\|z_c\|_\infty = \frac{\varepsilon}{2}\|y\|_1 = \frac{\varepsilon c}{2} = \delta$$

and

$$\|\Delta^3 x\|_\infty = \frac{\varepsilon}{2}\|\Delta^2 y\|_\infty = \varepsilon.$$

However,

$$\|\Delta^2 x\|_1 = \tfrac{\varepsilon}{2}\|\Delta^2 z_c\|_1 = \tfrac{\varepsilon}{2}\|\Delta y\|_1 = \varepsilon c = 2\varepsilon\|\varphi_{a,h}\|_\infty$$
$$\geqslant 2\varepsilon\|\Delta\varphi_{a,h}\|_\infty = 2\varepsilon\|\Delta^2\varphi_{a,h}\|_1 > \varepsilon\|\Delta^2\varphi_{a,h}\|_1.$$

This implies that

$$\omega_{\infty,1}^{2,1,3}(c\varepsilon/2,\varepsilon) > \varepsilon\|\Delta^2\varphi_{a,h}\|_1,$$

which shows that (7) does not hold in the considered case.

5. Comparison theorem for the class $l_\infty^{r-2}(M)$

Comparison theorems are results which provide an estimate of some characteristic of a sequence (or a function) $x(n)$ from a certain class (for instance, value at a point, value of the norm $\|x\|_p$, value of the norm of the k-th difference $\|\Delta^k x\|_q$ etc.) with the help of the same characteristic of some fixed sequence (function) from the same class. This fixed sequence (function) is called *étalon* or *comparison* sequence (function) for this class.

The first theorem of this type was proved by Kolmogorov. He showed that perfect Euler splines are comparison functions for the class $W_\infty^r(\mathbf{R}) := \{x \in L_\infty^r(\mathbf{R}) : \ \|x^{(r)}\|_\infty \leqslant 1\}$. We shall show that above constructed sequence $\varphi_{a,h}$ is the comparison sequence for the class $l_\infty^{r-2}(M)$ by proving several comparison theorems, which will be needed for completeness of the picture and the proof of the main result.

THEOREM 3. *Let $x \in l_\infty^{r-2}(M)$, $r \in \mathbf{N}$, $r \geqslant 2$. If parameters $a \in M$ and $h \in [0,1)$ of sequence $\varphi_{a,h}$ are such that*

$$\|x\|_\infty \leqslant \|\varphi_{a,h}\|_\infty \tag{16}$$

and

$$\|\Delta^r x\|_\infty \leqslant \|\Delta^r\varphi_{a,h}\|_\infty, \tag{17}$$

then for every integer $0 < k < r$, there holds

$$\|\Delta^k x\|_\infty \leqslant \|\Delta^k\varphi_{a,h}\|_\infty. \tag{18}$$

Proof. First, assume that $a = 0$. For every $k = 0,1,\ldots,r$, we have $\Delta^k\varphi_{a,h}(0) = h$ and $\Delta^k\varphi_{a,h}(n) = 0$, $n < 0$. In view of (16), we have $0 \leqslant x(n) \leqslant h$, $n \in M$. Then $|\Delta x(n)| = |x(n) - x(n-1)| \leqslant h$, $n \in M$, and hence, $\|\Delta x\|_\infty \leqslant h = \|\Delta\varphi_{a,h}\|_\infty$. In the case $r = 2$ this completes the proof for $a = 0$. If $r \geqslant 3$, then Δx is non-negative and we obtain that $0 \leqslant \Delta x(n) \leqslant h$. By induction, we can prove that for every $k = 1,\ldots,r-2$, there holds $0 \leqslant \Delta^k x(n) \leqslant h$, $n \in M$, which immediately implies that $\|\Delta^k x\|_\infty \leqslant h = \|\Delta^k\varphi_{a,h}\|_\infty$, $k = 1,\ldots,r-2$, and we also have $\|\Delta^{r-1}x\|_\infty \leqslant h = \|\Delta^{r-1}\varphi_{a,h}\|_\infty$.

Let now $a \leqslant -1$. The main ingredients of the proof in this case are the following statements.

LEMMA 1. *(The discrete analogue of the Rolle's theorem). Let* $r \in \mathbf{N}$, $r \geqslant 2$, $a \in M \setminus \{0\}$, *and* $w : M \to \mathbf{R}$ *be such a sequence that* $w(m_1) \leqslant 0$, $w(m_0) > 0$ *for some* $a \leqslant m_0 < m_1 \leqslant 0$, *and* $\Delta^k w(a-1) \leqslant 0$ *for every* $0 \leqslant k \leqslant r-2$. *Then there is* $m' \in M$, $a < m' \leqslant 0$, *such that* $\Delta^r w(m') < 0$.

Proof. Since $w(a-1) \leqslant 0$, $w(m_0) > 0$, and $w(m_1) \leqslant 0$, there are $n_1, n_2 \in M$, $a-1 < n_2 \leqslant m_0 < n_1 \leqslant m_1$, such that $\Delta w(n_2) > 0$ and $\Delta w(n_1) < 0$. By assumption, we also have $\Delta w(a-1) \leqslant 0$. Using induction, one can show that there are $n_3, n_4 \in M$ such that $a-1 < n_4 < n_3 \leqslant 0$, $\Delta^{r-2} w(n_4) > 0$, and $\Delta^{r-2} w(n_3) < 0$. Since by assumption, $\Delta^{r-2} w(a-1) \leqslant 0$, there exist $n_5, n_6 \in M$ such that $a-1 < n_6 < n_5 \leqslant 0$, $\Delta^{r-1} w(n_6) > 0$, and $\Delta^{r-1} w(n_5) < 0$. Then there is $m' \in M$ such that $a \leqslant n_6 < m' \leqslant n_5 \leqslant 0$ and $\Delta^r w(m') < 0$. Lemma 1 is proved. □

LEMMA 2. *Let sequence* $y \in l_\infty^{\rho-2}(M)$, $\rho \in \mathbf{N}$, $\rho \geqslant 3$, *be such that*

$$\|y\|_\infty \leqslant \|\varphi_{a,h}\|_\infty \qquad \text{and} \qquad \|\Delta^\rho y\|_\infty \leqslant 1,$$

where the sequence $\varphi_{a,h}$ *corresponds to* $r = \rho$. *Then*

$$\|\Delta y\|_\infty \leqslant \|\Delta \varphi_{a,h}\|_\infty.$$

Proof. Assume to the contrary that under assumptions of the lemma we have $\|\Delta y\|_\infty > \|\Delta \varphi_{a,h}\|_\infty$, i.e. $\Delta y(\widetilde{n}) > \Delta \varphi_{a,h}(0)$ for some $\widetilde{n} \in M$. Let $\widetilde{y}(n) := y(n+\widetilde{n})$. The relation $\Delta \widetilde{y}(n) \geqslant \Delta \varphi_{a,h}(n)$ does not hold for every $n \in M$, since by assumptions of the lemma and definition of the difference operator,

$$\begin{aligned} \sum_{n=-\infty}^{0} \Delta \widetilde{y}(n) &= \widetilde{y}(0) - \lim_{n \to -\infty} \widetilde{y}(n) \leqslant \widetilde{y}(0) \leqslant \|y\|_\infty \\ &\leqslant \|\varphi_{a,h}\|_\infty = \varphi_{a,h}(0) = \sum_{n=-\infty}^{0} \Delta \varphi_{a,h}(n), \end{aligned}$$

and $\Delta \widetilde{y}(0) > \Delta \varphi_{a,h}(0)$. Hence, $\Delta \varphi_{a,h}(n_0) > \Delta \widetilde{y}(n_0)$ for some $n_0 < 0$. Denote $\theta(n) := \Delta \varphi_{a,h}(n) - \Delta \widetilde{y}(n)$. Then $\theta(0) < 0$, $\theta(n_0) > 0$, and $\Delta^k \theta(a-1) \leqslant 0$, $k = 0, 1, \ldots, \rho-3$. Hence, $a \leqslant n_0 < 0$ and by Lemma 1 (with $r = \rho - 1$), there is $m' \in M$ such that $a < m' \leqslant 0$ and $\Delta^{\rho-1} \theta(m') < 0$. Then $\Delta^{\rho-1} \theta(m') = \Delta^\rho \varphi_{a,h}(m') - \Delta^\rho \widetilde{y}(m') = 1 - \Delta^\rho y(m' + \widetilde{n}) < 0$, which contradicts the assumption $\|\Delta^\rho y\|_\infty \leqslant 1$. Lemma 2 is proved. □

For $0 < k \leqslant r-2$, $r \geqslant 3$, we will conduct the proof of Theorem 3 by induction on k. Letting $\rho = r$ and $y = x$, in Lemma 2 we obtain

$$\|\Delta x\|_\infty \leqslant \|\Delta \varphi_{a,h}\|_\infty,$$

which is the assertion of the theorem for $k = 1$. Let $k \in \mathbf{N}$, $1 < k \leqslant r-2$. Let us assume that Theorem 3 holds for $k-1$. Set $\rho = r-k+1$ and $y := \Delta^{k-1} x$. Since x is bounded, by Proposition 2, y is bounded as well. In addition, by assumption of the theorem, y is non-negative and $\Delta^{\rho-2} y = \Delta^{r-2} x \geqslant 0$. Hence, $y \in l_\infty^{\rho-2}(M)$, where $\rho \geqslant 3$. Moreover, $\|\Delta^\rho y\|_\infty = \|\Delta^r x\|_\infty \leqslant 1$, and by the induction assumption,

$$\|y\|_\infty = \|\Delta^{k-1} x\|_\infty \leqslant \|\Delta^{k-1} \varphi_{a,h}\|_\infty = \|S^\rho(\Delta^r \varphi_{a,h}; \cdot)\|_\infty.$$

In view of Lemma 2, we have

$$\|\Delta^k x\|_\infty = \|\Delta y\|_\infty \leqslant \|S^{\rho-1}(\Delta^r \varphi_{a,h};\ \cdot\ )\|_\infty = \|\Delta^k \varphi_{a,h}\|_\infty,$$

which completes the proof of (18) for $0 < k \leqslant r-2$.

To show (18) for $k = r-1$ we assume to the contrary that there exists a point $n_0 \in M$ such that $\left|\Delta^{r-1}x(n_0)\right| > \|\Delta^{r-1}\varphi_{a,h}\|_\infty = -a+h$. For every $n \in M$, $a \leqslant n \leqslant 0$, we have

$$\Delta^{r-1}x(n_0) - \Delta^{r-1}x(n_0+n) = \sum_{k=n+1}^{0} \Delta^r x(n_0+k).$$

If $\Delta^{r-1}x(n_0) > 0$, since $\|\Delta^r x\|_\infty \leqslant 1$, we have

$$\Delta^{r-1}x(n_0) - \Delta^{r-1}x(n_0+n) \leqslant -n,$$

which implies that

$$\Delta^{r-1}x(n_0+n) \geqslant \Delta^{r-1}x(n_0) + n > -a+h+n,\ \ a \leqslant n \leqslant 0.$$

Hence,

$$\begin{aligned}\Delta^{r-2}x(n_0) - \Delta^{r-2}x(n_0+a-1) &= \sum_{k=a}^{0} \Delta^{r-1}x(n_0+k)\\ &> \sum_{k=a}^{0} (-a+h+k) = \sum_{k=a}^{0} \Delta^{r-1}\varphi_{a,h}(k)\\ &= \|\Delta^{r-2}\varphi_{a,h}\|_\infty.\end{aligned}$$

Since $\Delta^{r-2}x$ is non-negative at every point, we obtain that $\Delta^{r-2}x(n_0) > \|\Delta^{r-2}\varphi_{a,h}\|_\infty$.

In the case $\Delta^{r-1}x(n_0) < 0$, using analogous argument we obtain that $\Delta^{r-2}x(n_0 + a-1) > \|\Delta^{r-2}\varphi_{a,h}\|_\infty$. In both cases we have $\|\Delta^{r-2}x\|_\infty > \|\Delta^{r-2}\varphi_{a,h}\|_\infty$, which contradicts inequality (18) proved above for $k = r-2$. This contradiction shows that (18) is true for $k = r-1$. Theorem 3 is proved. $\square$

LEMMA 3. *Let $x \in l_p^{r-2}(M)$, $r \geqslant 2$, $p \in [1,\infty]$, and numbers $a \in M$ and $h \in [0,1)$ be such that*

$$\|x\|_p \leqslant \|\varphi_{a,h}\|_p \qquad \text{and} \qquad \|\Delta^r x\|_\infty \leqslant \|\Delta^r \varphi_{a,h}\|_\infty.$$

Then

$$\|x\|_\infty \leqslant \|\varphi_{a,h}\|_\infty.$$

Proof. For $p = \infty$, the assertion of the lemma is trivial, and we consider the case $p \in [1,\infty)$. First, let us consider the case $a = 0$. Then $\varphi_{a,h}(0) = h$ and $\varphi_{a,h}(n) = 0$, $n < 0$. If $x \in l_p^{r-2}(M)$ is such that $\|x\|_p \leqslant \|\varphi_{a,h}\|_p = h$, then $x(n) \leqslant h$, $n \in M$, and hence, $\|x\|_\infty \leqslant h = \|\varphi_{a,h}\|_\infty$.

Let now $a \leqslant -1$. Assume to the contrary that $\|x\|_\infty > \|\varphi_{a,h}\|_\infty$. Let $\widetilde{n} \in M$ be such that $x(\widetilde{n}) > \|\varphi_{a,h}\|_\infty = \varphi_{a,h}(0)$. Let $\widetilde{x}(n) = x(n+\widetilde{n})$ and $v(n) := \varphi_{a,h}(n) - \widetilde{x}(n)$, $n \in M$.

If it were that $\varphi_{a,h}(n) \leqslant \widetilde{x}(n)$ for every $n \in M$, since $\varphi_{a,h}(0) < \widetilde{x}(0)$, we would have $\|\varphi_{a,h}\|_p < \|\widetilde{x}\|_p \leqslant \|x\|_p$, which contradicts to the assumption of the lemma. Hence, there is a negative integer n_0 such that $v(n_0) > 0$. Since $\Delta^k \varphi_{a,h}(a-1) = 0$, $0 \leqslant k \leqslant r-2$, and $x \in l_p^{r-2}(M)$, we have $\Delta^k v(a-1) \leqslant 0$, $0 \leqslant k \leqslant r-2$. We have $v(0) < 0$ and $a \leqslant n_0 < 0$. By Lemma 1, we obtain that $\Delta^r v(m') < 0$ for some $a < m' \leqslant 0$. Hence, $\Delta^r \widetilde{x}(m') > \Delta^r \varphi_{a,h}(m') = 1$, which contradicts to the fact that $\|\Delta^r \widetilde{x}\|_\infty \leqslant \|\Delta^r x\|_\infty \leqslant 1 = \|\Delta^r \varphi_{a,h}\|_\infty$. This contradiction shows that $\|x\|_\infty \leqslant \|\varphi_{a,h}\|_\infty$. □

The following statement proves relation (7) of Theorem 1 in the case $q = \infty$.

COROLLARY 1. *Given integers $0 < k < r$, real numbers $p \in [1, \infty]$ and $\delta \geqslant \varepsilon > 0$, let $a \in M$ and $h \in [0,1)$ be such that $\|\varphi_{a,h}\|_p = \delta/\varepsilon$. Then*

$$\omega_{p,\infty}^{k,r-2,r}(\delta, \varepsilon) = \varepsilon \left\| \Delta^k \varphi_{a,h} \right\|_\infty . \tag{19}$$

Proof. Let $x \in l_p^{r-2}(M)$ be any sequence such that $\|x\|_p = \delta$ and $\|\Delta^r x\|_\infty = \varepsilon$. Let $a \in M$ and $h \in [0,1)$ be such that $\|\varphi_{a,h}\|_p = \delta/\varepsilon$. Then $a \leqslant -1$ and $\|\Delta^r \frac{1}{\varepsilon} x\|_\infty = 1 = \|\Delta^r \varphi_{a,h}\|_\infty$ and $\|\frac{1}{\varepsilon} x\|_p = \|\varphi_{a,h}\|_p$. In view of Lemma 3, we have $\|\frac{1}{\varepsilon} x\|_\infty \leqslant \|\varphi_{a,h}\|_\infty$. Since $x \in l_p^{r-2}(M)$, we have $\frac{1}{\varepsilon} x \in l_\infty^{r-2}(M)$, and by Theorem 3, we have $\|\Delta^k x\|_\infty \leqslant \varepsilon \|\Delta^k \varphi_{a,h}\|_\infty$, which implies that $\omega_{p,\infty}^{k,r-2,r}(\delta, \varepsilon) \leqslant \varepsilon \|\Delta^k \varphi_{a,h}\|_\infty$. Since the sequence $\varepsilon \varphi_{a,h}$ belongs to the set defining the modulus of continuity, we have equality in (19). Corollary 1 is proved. □

6. Results involving rearrangements of infinite sequences

Following [11], we say that a vector $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbf{R}^n$ is a *majorization* of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and we write $\mathbf{x} \prec \mathbf{y}$ if

$$\sum_{i=1}^k x_i \leqslant \sum_{i=1}^k y_i, \quad k = 1, \ldots, n-1, \text{ and } \sum_{i=1}^n x_i = \sum_{i=1}^n y_i.$$

The result we cite below is a partial case of the theorem proved by Schur (1923) and Hardy, Littlewood, and Polya (1929). Its proof is given, for example, in [11, p. 73] in greater generality. For completeness, we present the proof of the Theorem in necessary for us form in the Appendix.

THEOREM 4. *Let $I \subset \mathbf{R}$ be an interval, $\mathbf{x}, \mathbf{y} \in I^n$ be such that $\mathbf{x} \prec \mathbf{y}$ and $x_1 \geqslant \ldots \geqslant x_n$, and $g : I \to \mathbf{R}$ be a convex function. Then*

$$\sum_{i=1}^n g(x_i) \leqslant \sum_{i=1}^n g(y_i).$$

Let $x = \{x_m\}_{m \in M}$ be an arbitrary non-negative sequence, which has a zero limit as $m \to \infty$. Let $x[M]$ be the set of all non-zero values of x. Denote by z_k the k-th largest element from $x[M]$ and let $\alpha_k = \#\{m \in M : x(m) = z_k\}$. The *(decreasing)* *rearrangement* of the sequence x is the sequence $\{r(x,m)\}_{m\in M}$ defined in the following way: if $\sum\limits_{i=1}^{l-1}\alpha_i < 1-m \leqslant \sum\limits_{i=1}^{l}\alpha_i$, then $r(x,m)=z_l$. If the set $x[M]$ is finite, then $r(x,m)=0$ whenever $1-m > \sum\limits_{i=1}^{\#x[M]}\alpha_i$. Value $r(x,m)$ can be interpreted as the $(1-m)$-th largest element of x counting the multiplicity of every value of x.

The following statement is a discrete analogue of a special case of the theorem by Hardy, Littlewood, and Polya (see for example [7, Theorem 1.3.11]).

THEOREM 5. *Let $q\in[1,\infty)$, f and g be non-negative sequences from $l_q(M)$ and for every $n\in M$,*

$$\sum_{k=n}^{0} r(g,k) \leqslant \sum_{k=n}^{0} r(f,k).$$

Then $\|g\|_q \leqslant \|f\|_q$.

Proof. Let $m\in M$. Denote

$$\mathbf{x}=(r(g,0),r(g,-1),\dots,r(g,m)),\quad \mathbf{y}=(r(f,0),r(f,-1),\dots,r(f,m)),$$

and let

$$b=\sum_{k=m}^{0} r(f,k)-\sum_{k=m}^{0} r(g,k),$$

If $b=0$ we have $\mathbf{x}\prec\mathbf{y}$ and we let $\mathbf{y}':=\mathbf{y}$. If $b>0$ let $l\in M$ be the maximal index such that

$$\sum_{k=l}^{0} r(f,k) \geqslant \sum_{k=m}^{0} r(g,k).$$

Denote $\mathbf{y}':=(r(f,0),\dots,r(f,l+1),c,0\dots,0)\in\mathbf{R}^{-m+1}$, where c is chosen so that

$$\sum_{k=l+1}^{0} r(f,k)+c=\sum_{k=m}^{0} r(g,k).$$

Note that $0\leqslant c\leqslant r(f,l)$. Then $\mathbf{x}\prec\mathbf{y}'$ and the coordinates of $\mathbf{x}$ are in descending order. By Theorem 4, we have

$$\sum_{k=m}^{0} r(g,k)^q=\sum_{k=1}^{-m+1}(\mathbf{x})_k^q\leqslant\sum_{k=1}^{-m+1}(\mathbf{y}')_k^q\leqslant\sum_{k=m}^{0} r(f,k)^q.$$

Letting $m\to-\infty$ we have

$$\|g\|_q^q=\|r(g,\cdot)\|_q^q=\sum_{k=-\infty}^{0} r(g,k)^q\leqslant\sum_{k=-\infty}^{0} r(f,k)^q=\|r(f,\cdot)\|_q^q=\|f\|_q^q$$

and assertion of Theorem 5 follows. □

7. Results involving the l_q-norm of the k-th difference

Denote by

$$i[k] := \begin{cases} r-2, & 1 \leqslant k \leqslant r-2, \\ r-1, & k = r-1. \end{cases}$$

The statement given below is a comparison theorem for the case of l_q-norm.

LEMMA 4. *Let* $k, r \in \mathbf{N}$, $1 \leqslant k \leqslant r-1$, $x \in l_\infty^{i[k]}(M)$, *and* $a \in M$, $h \in [0,1)$ *be such that*

$$\|\Delta^r x\|_\infty \leqslant \|\Delta^r \varphi_{a,h}\|_\infty, \tag{20}$$

and

$$\|x\|_\infty \leqslant \|\varphi_{a,h}\|_\infty. \tag{21}$$

Then for every number $q \in [1, \infty)$,

$$\|\Delta^k x\|_q \leqslant \|\Delta^k \varphi_{a,h}\|_q. \tag{22}$$

Proof. If $a = 0$, then $\Delta^l \varphi_{0,h}(0) = h$ and $\Delta^l \varphi_{0,h}(n) = 0$, $n \leqslant -1$, for all $0 \leqslant l \leqslant r$. In view of relation (13) from Proposition 6, we have $\|\Delta^k x\|_q \leqslant \|x\|_\infty \leqslant \|\varphi_{0,h}\|_\infty = h = \|\Delta^k \varphi_{0,h}\|_q$.

Let $a \leqslant -1$. For every integer $2 \leqslant k \leqslant r-1$, by Theorem 3, and for $k = 1$ in view of assumption (21), we have

$$\begin{aligned} \|\Delta^k \varphi_{a,h}\|_1 &= \Delta^{k-1} \varphi_{a,h}(0) = \|\Delta^{k-1} \varphi_{a,h}\|_\infty \\ &\geqslant \|\Delta^{k-1} x\|_\infty \geqslant \Delta^{k-1} x(0) \geqslant \|\Delta^k x\|_1, \end{aligned} \tag{23}$$

which proves (22) in the case $q = 1$.

To further proceed with the proof for $1 < q < \infty$ we assume that $1 \leqslant k \leqslant r-3$. Let us show that for every $n \in M$, there holds

$$\sum_{j=n}^{0} \Delta^k x(j) \leqslant \sum_{j=n}^{0} \Delta^k \varphi_{a,h}(j). \tag{24}$$

Assume to the contrary that inequality (24) does not hold for some $n \in M$ and let n_0 be the maximal index such that

$$\sum_{j=n_0}^{0} \Delta^k x(j) > \sum_{j=n_0}^{0} \Delta^k \varphi_{a,h}(j). \tag{25}$$

Since relation (24) holds for $n = n_0 + 1$, we conclude that $\Delta^k x(n_0) > \Delta^k \varphi_{a,h}(n_0)$. Note also that for every $n \leqslant a$, in view of (23), we have

$$\sum_{j=n}^{0} \Delta^k x(j) \leqslant \|\Delta^k x\|_1 \leqslant \|\Delta^k \varphi_{a,h}\|_1 = \sum_{j=a}^{0} \Delta^k \varphi_{a,h}(j) = \sum_{j=n}^{0} \Delta^k \varphi_{a,h}(j).$$

Hence, $n_0 > a$. There exists $n_1 \in M$, $a \leqslant n_1 < n_0$, such that $\Delta^k x(n_1) < \Delta^k \varphi_{a,h}(n_1)$. Indeed, if this were not true, for every $a \leqslant j < n_0$, we would have $\Delta^k x(j) \geqslant \Delta^k \varphi_{a,h}(j)$. Then in view of (25), we would obtain

$$\begin{aligned}\|\Delta^k x\|_1 &\geqslant \sum_{j=a}^{n_0-1} \Delta^k x(j) + \sum_{j=n_0}^{0} \Delta^k x(j)\\ &> \sum_{j=a}^{n_0-1} \Delta^k \varphi_{a,h}(j) + \sum_{j=n_0}^{0} \Delta^k \varphi_{a,h}(j) = \|\Delta^k \varphi_{a,h}\|_1,\end{aligned}$$

which contradicts (23). Denote $u(n) = \Delta^k \varphi_{a,h}(n) - \Delta^k x(n)$, $n \in M$. Then $u(n_0) < 0$, $u(n_1) > 0$, where $a \leqslant n_1 < n_0$, and $\Delta^l u(a-1) \leqslant 0$, $l = 0,1,\ldots,r-k-2$. By Lemma 1, there is $a < m' \leqslant 0$ such that

$$0 > \Delta^{r-k} u(m') = \Delta^r \varphi_{a,h}(m') - \Delta^r x(m') = 1 - \Delta^r x(m').$$

Hence, $\|\Delta^r x\|_\infty > 1 = \|\Delta^r \varphi_{a,h}\|_\infty$, which contradicts to (20). Inequality (24) is proved.

Since in the case $1 \leqslant k \leqslant r-3$, both $\Delta^k x$ and $\Delta^k \varphi_{a,h}$ are monotone, from (24) we immediately have

$$\sum_{j=n}^{0} r(\Delta^k x, j) \leqslant \sum_{j=n}^{0} r(\Delta^k \varphi_{a,h}, j), \quad n \in M. \tag{26}$$

Let now $k = r-2$ or $r-1$. In this case sequence $\Delta^k x$ is non-negative but is not monotone and relation (26) must be shown in a different way. Assume the contrary, and let $m \in M$ be the maximal number such that

$$\sum_{j=m}^{0} r(\Delta^k x, j) > \sum_{j=m}^{0} r(\Delta^k \varphi_{a,h}, j). \tag{27}$$

In view of Theorem 3, and definition of rearrangement

$$r(\Delta^k \varphi_{a,h}, 0) = \|\Delta^k \varphi_{a,h}\|_\infty \leqslant \|\Delta^k x\|_\infty = r(\Delta^k x, 0). \tag{28}$$

In view of (23), for every $n \leqslant a$, we have

$$\sum_{j=n}^{0} r(\Delta^k \varphi_{a,h}, j) = \|\Delta^k \varphi_{a,h}\|_1 \geqslant \|\Delta^k x\|_1 \geqslant \sum_{j=n}^{0} r(\Delta^k x, j).$$

Hence, $a < m < 0$. By the choice of m,

$$\sum_{j=m+1}^{0} r(\Delta^k x, j) \leqslant \sum_{j=m+1}^{0} r(\Delta^k \varphi_{a,h}, j)$$

and in view of (27), we have

$$r(\Delta^k x, m) > r(\Delta^k \varphi_{a,h}, m) = \Delta^k \varphi_{a,h}(m) > 0.$$

Let $v\in M$ be the minimal number such that $\Delta^k x(v)\geqslant r(\Delta^k x,m)$. Then

$$\Delta^k x(v)>\Delta^k\varphi_{a,h}(m). \tag{29}$$

We claim that

$$\Delta^k x(j+v)\geqslant \Delta^k\varphi_{a,h}(j+m),\quad j=0,-1,\dots,a-m. \tag{30}$$

Indeed, if it were that

$$\Delta^k x(u_0+v)<\Delta^k\varphi_{a,h}(u_0+m),\quad \text{for some } \ a-m\leqslant u_0\leqslant 0, \tag{31}$$

then for the sequence $\Theta(n):=\Delta^k x(n+v)-\Delta^k\varphi_{a,h}(n+m)$, $n\in M$, we would have $\Theta(u_0)<0$. Relation (29) would imply that $\Theta(0)>0$ and since $\Delta^k\varphi_{a,h}(a-1)=0$, we would have $\Theta(a-m-1)\geqslant 0$. Hence, we would have $\Delta\Theta(n_2)<0$ and $\Delta\Theta(n_1)>0$ for some $a-m\leqslant n_2\leqslant u_0<n_1\leqslant 0$.

In the case $k=r-1$, since $\Delta\Theta(n_1)=\Delta^r x(n_1+v)-\Delta^r\varphi_{a,h}(n_1+m)=\Delta^r x(n_1+v)-1>0$, we have $\|\Delta^r x\|_\infty>1$, which contradicts assumption (20) of the lemma. In the case $k=r-2$, we would have $\Delta^2\Theta(n_3)>0$ for some $a-m\leqslant n_2<n_3\leqslant n_1$, i.e. $\Delta^r x(n_3+v)>\Delta^r\varphi_{a,h}(n_3+m)=1$, which again would contradict assumption (20) of the lemma. This contradiction proves (30).

By (30) and the choice of v, it is not difficult to see that

$$\begin{aligned}\sum_{j=-\infty}^{m-1} r(\Delta^k x,j) &\geqslant \sum_{j=a-m}^{-1}\Delta^k x(j+v)\\ &\geqslant \sum_{j=a-m}^{-1}\Delta^k\varphi_{a,h}(j+m)=\sum_{j=a}^{m-1}\Delta^k\varphi_{a,h}(j).\end{aligned}$$

Then in view of our assumption (27), we will have

$$\begin{aligned}\|\Delta^k x\|_1 &= \sum_{j=m}^{0} r(\Delta^k x,j)+\sum_{j=-\infty}^{m-1} r(\Delta^k x,j)>\sum_{j=m}^{0} r(\Delta^k\varphi_{a,h},j)\\ &\quad+\sum_{j=a}^{m-1}\Delta^k\varphi_{a,h}(j)=\sum_{j=a}^{0}\Delta^k\varphi_{a,h}(j)=\|\Delta^k\varphi_{a,h}\|_1,\end{aligned}$$

which contradicts (23). This contradiction proves (26) in the case $k=r-2$ or $r-1$.

By Proposition 1 non-negative sequences $\Delta^k x$ and $\Delta^k\varphi_{a,h}$ belong to $l_q(M)$. In view of (26) and Theorem 5, we have $\|\Delta^k x\|_q\leqslant\|\Delta^k\varphi_{a,h}\|_q$. Lemma 4 is proved. □

The following statement proves relation (7) of Theorem 1 in the case $1\leqslant q<\infty$ as well as relation (8), which completes the proof of Theorem 1. □

COROLLARY 2. *Given integers $0<k<r$, numbers $p\in[1,\infty]$, $q\in[1,\infty)$, and $\delta\geqslant\varepsilon>0$, let $a\in M$ and $h\in[0,1)$ be such that $\|\varphi_{a,h}\|_p=\delta/\varepsilon$. Then*

$$\omega_{p,q}^{k,i[k],r}(\delta,\varepsilon)=\varepsilon\left\|\Delta^k\varphi_{a,h}\right\|_q.$$

Proof. The argument is analogous to the proof of Corollary 1, where instead of Theorem 3 one uses Lemma 4. □

8. Proof of Theorem 2

Proof of the general inequality (10)*.* If finiteness of the constant $D^{k,r}_{p,q}$ is proved, then inequality (10) is shown in the following way. In the case when $\Delta^r x$ is identically zero, the k-th difference of $x \in l_p^{r-2}(M)$ will be zero as well. In the case $r=2$ and $q=\infty$ this follows from the fact that x has to be a bounded sequence. In all other cases, this follows from the fact that in view of Proposition 1, every difference of the sequence x will have a zero limit as $n \to -\infty$. Hence, the required inequality will hold trivially. Assume now that $\Delta^r x$ has a non-zero value at some point. Denote by $y := x/\|\Delta^r x\|_\infty$. Let $a \in M$ and $h \in [0,1)$ be such that $\|\varphi_{a,h}\|_p = \|y\|_p$.

If $\|\varphi_{a,h}\|_p = \|y\|_p < 1$, there holds $\varphi_{a,h}(0) < 1$, and in view of Proposition 7, we have $a = 0$. In view of Proposition 6, we have

$$\|\Delta^k y\|_q \leqslant \|y\|_\infty \leqslant \|y\|_p \leqslant \|y\|_p^\alpha,$$

where $\alpha = \frac{r-k+1/q}{r+1/p}$, and hence $0 < \alpha \leqslant 1$. Multiplying both sides by $\|\Delta^r x\|_\infty$ we obtain

$$\|\Delta^k x\|_q \leqslant \|x\|_p^\alpha \|\Delta^r x\|_\infty^{1-\alpha}.$$

Since

$$D^{k,r}_{p,q} \geqslant \frac{\|\Delta^k \varphi_{-1,0}\|_q}{\|\varphi_{-1,0}\|_p^\alpha} = 1,$$

we obtain (10) in the case $\|y\|_p < 1$.

Let us assume now that $\|\varphi_{a,h}\|_p = \|y\|_p \geqslant 1$. Let $\delta = \|y\|_p$ and $\varepsilon = \|\Delta^r y\|_\infty = 1$. Then $\|\varphi_{a,h}\|_p = \delta/\varepsilon$ and in view of Theorem 1, we have

$$\|\Delta^k y\|_q \leqslant \|\Delta^k \varphi_{a,h}\|_q = \frac{\|\Delta^k \varphi_{a,h}\|_q}{\|\varphi_{a,h}\|_p^\alpha} \|y\|_p^\alpha \leqslant D^{k,r}_{p,q} \|y\|_p^\alpha.$$

Since $x = \|\Delta^r x\|_\infty \cdot y$, we can write

$$\|\Delta^k x\|_q \leqslant D^{k,r}_{p,q} \|x\|_p^\alpha \|\Delta^r x\|_\infty^{1-\alpha}.$$

To show that we cannot make the constant $D^{k,r}_{p,q}$ smaller, for every $0 < \varepsilon < D^{k,r}_{p,q}$, choose parameters $a \in M$ and $h \in [0,1)$ so that

$$\frac{\|\Delta^k \varphi_{a,h}\|_q}{\|\varphi_{a,h}\|_p^\alpha} > D^{k,r}_{p,q} - \varepsilon.$$

Then

$$\|\Delta^k \varphi_{a,h}\|_q > (D^{k,r}_{p,q} - \varepsilon)\|\varphi_{a,h}\|_p^\alpha \geqslant (D^{k,r}_{p,q} - \varepsilon)\|\varphi_{a,h}\|_p^\alpha \|\Delta^r \varphi_{a,h}\|_\infty^{1-\alpha}.$$

Since $\varphi_{a,h}$ belongs to the class $l_p^{r-1}(M)$ (or $l_p^{r-2}(M)$), we conclude that constant $D^{k,r}_{p,q}$ is the best possible on this class.

Proof of finiteness of the constant $D^{k,r}_{p,q}$. Assume first that $p<\infty$. From (14) in the case $a\leqslant -1$ we have

$$\begin{aligned}\varphi_{a,h}(0)=\|\varphi_{a,h}\|_\infty &= \frac{(r-a-1)!}{r!(-a-1)!}+h\frac{(r-a-1)!}{(r-1)!(-a)!}\\ &= \frac{1}{r!}(-a+1)\cdot\ldots\cdot(-a+r-1)(-a+hr).\end{aligned} \tag{32}$$

Denote

$$T_\gamma(m):=\sum_{n=1}^{m} n^\gamma,\ \ m\in\mathbf{N},\ \ \gamma>0.$$

Using Riemann sums representation it is not difficult to show that

$$T_\gamma(m)=\frac{m^{\gamma+1}}{\gamma+1}(1+o(1)),\ \ m\to\infty. \tag{33}$$

For every $1\leqslant p<\infty$, $a\leqslant -1$, and $h\in[0,1)$, taking into account (32) we have

$$\begin{aligned}\|\varphi_{a,h}\|_p^p &= \sum_{n=a}^{0}\left(\varphi_{a,h}(n)\right)^p=\sum_{n=a}^{0}\left(\varphi_{a-n,h}(0)\right)^p\\ &= \tfrac{1}{(r!)^p}\sum_{n=a}^{0}(-a+n+1)^p\cdot\ldots\cdot(-a+n+r-1)^p(-a+n+hr)^p\\ &\leqslant \tfrac{1}{(r!)^p}\sum_{n=a}^{0}(-a+n+r)^{rp}=\tfrac{1}{(r!)^p}\sum_{n=r}^{-a+r}n^{rp}\leqslant\tfrac{1}{(r!)^p}T_{rp}(-a+r).\end{aligned}$$

On the other hand,

$$\|\varphi_{a,h}\|_p^p\geqslant\frac{1}{(r!)^p}\sum_{n=a}^{0}(-a+n)^{rp}=\frac{1}{(r!)^p}\sum_{n=0}^{-a}n^{rp}=\frac{1}{(r!)^p}T_{rp}(-a).$$

Hence,

$$\frac{1}{r!}\left(T_{rp}(-a)\right)^{1/p}\leqslant\|\varphi_{a,h}\|_p\leqslant\frac{1}{r!}\left(T_{rp}(-a+r)\right)^{1/p}. \tag{34}$$

From (32) we also have

$$\frac{1}{r!}(-a)^r\leqslant\|\varphi_{a,h}\|_\infty\leqslant\frac{1}{r!}(-a+r)^r. \tag{35}$$

From estimates (34) and relation (33), for $1\leqslant p,q<\infty$, we obtain

$$f_a(h):=\frac{\|\Delta^k\varphi_{a,h}\|_q}{\|\varphi_{a,h}\|_p^\alpha}\leqslant\beta_a:=\frac{\left(T_{(r-k)q}(-a+r-k)\right)^{1/q}(r!)^\alpha}{(r-k)!(T_{rp}(-a))^{\alpha/p}},\ \ a\leqslant -1.$$

Using asymptotic estimate (33) it is not difficult to see that

$$\begin{aligned}\lim_{a\to-\infty}\beta_a &= \lim_{a\to-\infty}\frac{(T_{(r-k)q}(-a))^{1/q}(r!)^\alpha}{(r-k)!(T_{rp}(-a))^{\alpha/p}}\\ &= \lim_{a\to-\infty}\frac{(-a)^{r-k+\frac{1}{q}}(r!)^\alpha(rp+1)^{\frac{\alpha}{p}}}{(r-k)!((r-k)q+1)^{\frac{1}{q}}(-a)^{(r+\frac{1}{p})\alpha}}=\frac{(r!)^\alpha(rp+1)^{\frac{\alpha}{p}}}{(r-k)!((r-k)q+1)^{\frac{1}{q}}}.\end{aligned}$$

This implies that β_α, $a \leqslant -1$, is a bounded sequence (whenever $p < \infty$ and $q < \infty$), and hence, $\{f_a\}_{a \leqslant -1}$ is a uniformly bounded family of functions on $[0,1)$. It remains to check whether $f_0(h)$ is bounded. We have

$$f_0(h) = \frac{\|\Delta^k \varphi_{0,h}\|_q}{\|\varphi_{0,h}\|_p^\alpha} = \frac{h}{h^\alpha} = h^{1-\alpha} \leqslant 1, \;\; h \in (0,1).$$

In the case $p = \infty$, $q < \infty$ and $p < \infty$, $q = \infty$ using relations (34), (35), and (33), one can show analogously that family of functions $\{f_a\}_{a \in M}$ is uniformly bounded.

Hence, quantity $D_{p,q}^{k,r}$ is finite in the case when p or q is finite.

In the case $p = q = \infty$ we prove the following statement.

PROPOSITION 8. *For all integers $0 < k < r$, there holds*

$$D_{\infty,\infty}^{k,r} = \sup_{\substack{a \in M,\ h \in [0,1) \\ (a,h) \neq (0,0)}} \frac{\|\Delta^k \varphi_{a,h}\|_\infty}{\|\varphi_{a,h}\|_\infty^{1-k/r}} = \frac{(r!)^{1-k/r}}{(r-k)!}. \tag{36}$$

The supremum in (36) is not attained for any a and h.

Proof. In view of (32), we obtain

$$\begin{aligned} &\frac{\|\Delta^k \varphi_{a,h}\|_\infty}{\|\varphi_{a,h}\|_\infty^{1-k/r}} = \\ &= \frac{(r!)^{1-k/r}}{(r-k)!} \frac{(-a+1)\cdot \ldots \cdot (-a+r-k-1)\cdot(-a+h(r-k))}{((-a+1)\cdot \ldots \cdot (-a+r-1)(-a+hr))^{1-k/r}} \\ &= \frac{(r!)^{1-\frac{k}{r}}}{(r-k)!} \cdot \left(\frac{-a+h(r-k)}{-a+hr}\right)^{1-\frac{k}{r}} \left(\frac{(-a+1)^k \cdot \ldots \cdot (-a+r-k-1)^k (-a+h(r-k))^k}{(-a+r-k)^{r-k} \cdot \ldots \cdot (-a+r-1)^{r-k}}\right)^{\frac{1}{r}} \end{aligned} \tag{37}$$

Since in the third factor both the numerator and the denominator contain equal number of positive factors ($k(r-k)$) and every factor in the numerator is less than or equal to any factor in the denominator, we obtain that the third factor is less than or equal to 1. Moreover, equalities can occur there only if $h = 1$ but in this case the second factor is strictly less than 1. Hence,

$$\frac{\|\Delta^k \varphi_{a,h}\|_\infty}{\|\varphi_{a,h}\|_\infty^{1-k/r}} < \frac{(r!)^{1-k/r}}{(r-k)!}, \quad a \leqslant -1. \tag{38}$$

In the case $a = 0$ we also have

$$\frac{\|\Delta^k \varphi_{0,h}\|_\infty}{\|\varphi_{0,h}\|_\infty^{1-k/r}} = \frac{h}{h^{1-k/r}} = h^{k/r} < 1 < \frac{(r!)^{1-k/r}}{(r-k)!}. \tag{39}$$

Taking into account (37), it is not difficult to see that for every $h \in [0,1)$ fixed, we have

$$\lim_{a \to -\infty} \frac{\|\Delta^k \varphi_{a,h}\|_\infty}{\|\varphi_{a,h}\|_\infty^{1-k/r}} = \frac{(r!)^{1-k/r}}{(r-k)!},$$

which combined with relations (38) and (39) completes the proof of the Proposition 8. □

To compute other explicit values of the constant $D^{k,r}_{p,q}$ we use the following statement.

LEMMA 5. *For $p,q\in[1,\infty]$, we have*

$$D^{k,r}_{p,1}=D^{k-1,r}_{p,\infty},\ \ 2\leqslant k\leqslant r-1,$$

and

$$D^{k,r}_{1,q}=D^{k+1,r+1}_{\infty,q},\quad 1\leqslant k\leqslant r-1.$$

Proof. Note that $\alpha(k,r,p,1)=\frac{r-k+1}{r+1/p}=\alpha(k-1,r,p,\infty)$. Then with $\alpha=\alpha(k,r,p,1)$ $=\frac{r-k+1}{r+1/p}$ for $2\leqslant k\leqslant r-1$, we have

$$D^{k,r}_{p,1}=\sup_{\substack{a\in M,\ h\in[0,1)\\(a,h)\neq(0,0)}}\frac{\|\Delta^k\varphi_{a,h}\|_1}{\|\varphi_{a,h}\|_p^{\alpha}}=\sup_{\substack{a\in M,\ h\in[0,1)\\(a,h)\neq(0,0)}}\frac{\|\Delta^{k-1}\varphi_{a,h}\|_\infty}{\|\varphi_{a,h}\|_p^{\alpha}}=D^{k-1,r}_{p,\infty}.$$

The second equality can be shown analogously. Lemma 5 is proved. □

In view of Lemma 5, we have

$$D^{k,r}_{\infty,1}=D^{k-1,r}_{\infty,\infty}=\frac{(r!)^{\frac{r-k+1}{r}}}{(r-k+1)!},\ \ 2\leqslant k\leqslant r-1.$$

In the case $k=1$ we have $D^{1,r}_{\infty,1}=1$, since $\|\Delta\varphi_{a,h}\|_1=\|\varphi_{a,h}\|_\infty$ and $\alpha(1,r,\infty,1)=1$. The remaining equalities can be proved analogously. Theorem 2 is proved. □

9. Appendix

This section contains the proof of Theorem 4 obtained in the works by Schur and Hardy, Littlewood and Polya (for a citation see [11, p. 73]).

We say that an $n\times n$ matrix P with non-negative entries p_{ij}, $i=1,\dots,n$, $j=1,\dots,n$, is called *doubly stochastic* if for every $j=1,\dots,n$, there holds $\sum_{i=1}^n p_{ij}=1$ and for every $i=1,\dots,n$, we have $\sum_{j=1}^n p_{ij}=1$.

We will use a partial case of the theorem by Hardy, Littlewood, and Polya, obtained in 1929 and cited, for example, in [11, p. 30].

THEOREM 6. *Let $\mathbf{x},\mathbf{y}\in\mathbf{R}^n$ be such that $x_1\geqslant\ldots\geqslant x_n$ and $\mathbf{x}\prec\mathbf{y}$ (relation $\prec$ is defined in Section 6). Then there is a doubly stochastic matrix P such that $\mathbf{x}=\mathbf{y}P$.*

Proof. For vectors $\mathbf{u}=(u_1,\dots,u_n),\mathbf{v}=(v_1,\dots,v_n)\in\mathbf{R}^n$, denote by $d(\mathbf{u},\mathbf{v})$ the number of non-zero differences u_i-v_i, $i=1,\dots,n$. Let I_n be the $n\times n$ identity matrix. Matrix A of size $n\times n$ is called *a matrix of a T-transform* (or a *T-matrix*) if $A=\lambda I_n+(1-\lambda)Q$ for some $\lambda\in[0,1]$ and a permutation matrix Q which can be obtained from I_n by switching only two rows.

LEMMA 6. *If* $\mathbf{x} \neq \mathbf{y} \in \mathbf{R}^n$ *are such that* $x_1 \geqslant \ldots \geqslant x_n$ *and* $\mathbf{x} \prec \mathbf{y}$ *then there is an* $n \times n$ T*-matrix* A *such that* $\mathbf{x} \prec \mathbf{y}A$ *and* $d(\mathbf{x}, \mathbf{y}A) \leqslant d(\mathbf{x}, \mathbf{y}) - 1$.

Proof. Let j be the largest index such that $y_j > x_j$ and let k be the smallest index such that $y_k < x_k$ and $k > j$ (such index k exists since the for the largest index i with $x_i \neq y_i$ we have $y_i < x_i$). Then $y_j > x_j \geqslant x_k > y_k$. Let $\varepsilon := \min\{y_j - x_j, x_k - y_k\}$, $1 - \lambda := \varepsilon/(y_j - y_k)$, and

$$\mathbf{y}^* = (y_1, \ldots, y_{j-1}, y_j - \varepsilon, y_{j+1}, \ldots, y_{k-1}, y_k + \varepsilon, y_{k+1}, \ldots, y_n).$$

Then $0 < \lambda < 1$ and we have

$$\mathbf{y}^* = \lambda \mathbf{y} + (1-\lambda)(y_1, \ldots, y_{j-1}, y_k, y_{j+1}, \ldots, y_{k-1}, y_j, y_{k+1}, \ldots, y_n).$$

Hence,

$$\mathbf{y}^* = \mathbf{y}(\lambda I_n + (1-\lambda)Q_{jk}),$$

where Q_{jk} is the permutation matrix obtained from I_n by switching the j-th and the k-th rows. It is not difficult to see that $\mathbf{y}^* \prec \mathbf{y}$. Let us show that $\mathbf{x} \prec \mathbf{y}^*$. For every $\nu = 1, \ldots, j-1$ and $\nu = k, \ldots, n$, we have

$$\sum_{i=1}^{\nu} y_i^* = \sum_{i=1}^{\nu} y_i \geqslant \sum_{i=1}^{\nu} x_i$$

(with all three quantities being equal for $\nu = n$). By the choice of indices j and k, we have $y_i^* = x_i$, $i = j+1, \ldots, k-1$. Then taking also into account inequality $y_j - \varepsilon \geqslant x_j$, for every $\nu = j, \ldots, k-1$, we have

$$\sum_{i=1}^{\nu} y_i^* = \sum_{i=1}^{j-1} y_i + y_j - \varepsilon + \sum_{i=j+1}^{\nu} y_i \geqslant \sum_{i=1}^{j-1} x_i + x_j + \sum_{i=j+1}^{\nu} x_i = \sum_{i=1}^{\nu} x_i,$$

which yields the necessary relation between $\mathbf{x}$ and $\mathbf{y}^*$.

Let $A = \lambda I_n + (1-\lambda)Q_{jk}$. Then A is a T-matrix. Since $y_j^* = x_j$ if $\varepsilon = y_j - x_j$ and $y_k^* = x_k$ if $\varepsilon = x_k - y_k$, we have $d(\mathbf{x}, \mathbf{y}A) = d(\mathbf{x}, \mathbf{y}^*) \leqslant d(\mathbf{x}, \mathbf{y}) - 1$. Lemma 6 is proved. □

Under assumptions of Theorem 6, applying Lemma 6 appropriate number of times, we find a finite sequence of T-matrices $A_1, \ldots, A_l$ such that $d(\mathbf{x}, \mathbf{y}A_1 \cdot \ldots \cdot A_l) = 0$, i.e.

$$\mathbf{x} = \mathbf{y}A_1 \cdot \ldots \cdot A_l. \tag{40}$$

LEMMA 7. *The product of any* $n \times n$ *doubly stochastic matrices is a doubly stochastic matrix.*

Proof. Denote $\mathbf{e} = (1, \ldots, 1) \in \mathbf{R}^n$. It is not difficult to see that double stochastic property of matrix P is equivalent to the fact that P has non-negative entries and $\mathbf{e}P = \mathbf{e}$ and $P\mathbf{e}^T = \mathbf{e}^T$, where $\mathbf{e}^T$ is the transpose of $1 \times n$ matrix $\mathbf{e}$.

Let P_1 and P_2 be $n\times n$ doubly stochastic matrices. Then P_1P_2 also has nonnegative entries and for vector $\mathbf{e}=(1,\dots,1)\in\mathbf{R}^n$, we will have

$$P_1P_2\mathbf{e}^T=P_1\mathbf{e}^T=\mathbf{e}^T \quad \text{and} \quad \mathbf{e}P_1P_2=\mathbf{e}P_2=\mathbf{e}.$$

Hence, P_1P_2 is also doubly stochastic. Lemma 7 is proved. □

Every matrix A_i, $i=1,\dots,l$, in (40) is doubly stochastic. Hence, their product is also doubly stochastic and the assertion of Theorem 6 follows. □

Proof of Theorem 4. By Theorem 6, there is a doubly stochastic matrix P with entries p_{ij}, $i=1,\dots,n$, $j=1,\dots,n$, such that $\mathbf{x}=\mathbf{y}P$. Hence,

$$x_j=y_1p_{1j}+\dots+y_np_{nj}, \quad j=1,\dots,n,$$

and using convexity of g we obtain

$$\begin{aligned}\sum_{j=1}^n g(x_j) &= \sum_{j=1}^n g\left(\sum_{i=1}^n p_{ij}y_i\right) \leqslant \sum_{j=1}^n\sum_{i=1}^n p_{ij}g(y_i)\\ &= \sum_{i=1}^n\sum_{j=1}^n p_{ij}g(y_i)=\sum_{i=1}^n g(y_i)\sum_{j=1}^n p_{ij}=\sum_{i=1}^n g(y_i).\end{aligned}$$

Theorem 4 is proved. □

REFERENCES

[1] V. F. BABENKO, YU. V. BABENKO, *Kolmogorov inequalities for multiply monotone functions defined on a half-line*, East J. Approx. **11**, 2 (2005), 169–186.

[2] V. F. BABENKO, N. P. KORNEICHUK, V. A. KOFANOV, S. A. PICHUGOV, *Inequalities for derivatives and their applications*, Naukova Dumka, Kiev, 2003 (Russian).

[3] Z. DITZIAN, *Discrete and shift Kolmogorov type inequalities*, Proc. Roy. Soc. Edinburgh, Sect. A **93**, 3–4 (1982/83), 307–317.

[4] H. G. KAPER, B. E. SPELLMAN, *Best constants in norm inequalities for the difference operator*, Trans. Amer. Math. Soc. **299**, 1 (1987), 351–372.

[5] A. N. KOLMOGOROV, *On inequalities between upper bounds of consecutive derivatives of arbitrary function on the infinite interval*, Scientific Notes MGU **30** (1939), 3–16 (in Russian).

[6] A. KOLMOGOROFF, *Une generalisation de J.Hadamard entre les bornes superier des derivees seccesives d'une fonction*, C. R. Acad. Sci. **207** (1938), 764–765.

[7] N. P. KORNEICHUK, V. F. BABENKO, A. A. LIGUN, *Extremal properties of polynomials and splines*, Kiev, Naukova Dumka, 1992.

[8] M. K. KWONG, A. ZETTL, *Ramifications of Landau's inequality*, Proc. Roy. Soc. Edinburgh **86** (1980), 175–212.

[9] M. K. KWONG, A. ZETTL, *Landau's inequality for the difference operator*, Proc. Amer. Math. Soc. **104**, 1 (1988), 201–206.

[10] M. K. KWONG, A. ZETTL, *Norm inequalities for derivatives and differences*, Lecture Notes in Mathematics, 1536. Springer-Verlag, Berlin, 1992. viii+150 pp.

[11] A. W. MARSHALL, I. OLKIN, *Inequalities: theory of majorization and its applications*, Mathematics in Science and Engineering, Vol. 143. Academic Press, 1979.

[12] V. M. OLOVYANISHNIKOV, *To the question on inequalities between upper bounds of consecutive derivatives on a half-line*, Uspekhi Mat. Nauk **6**, 2(42) (1951), 167–170.
[13] J. VELIKINA, *Sharp constants in discrete Kolmogorov type inequalities*, East J. Approx. **5**, 1 (1999), 89–100.
[14] J. V. VELIKINA, *Some sharp inequalities and estimates of approximation of functions of discrete argument by splines*, Ph. D. thesis, Dnepropetrovsk National University, Ukraine, 1999.

(Received January 24, 2010)

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