Mathematical
Inequalities
& Applications
Volume 15, Number 2 (2012), 271–279

ON WILKER AND HUYGENS TYPE INEQUALITIES

EDWARD NEUMAN

Abstract. Trigonometric inequalities, which have been obtained by J. Wilker [11] and C. Huygens [4], have attracted attention of several researchers (see, e.g., [1], [3], [6], [7], [8], [9], [10], [12], [13], [14], [15], [16]). In this paper we offer several refinements and generalizations of these results.

1. Introduction

In this section we give a brief overview of known results which pertain to the main results of this paper.

In what follows let $0 < |x| < \frac{\pi}{2}$ unless otherwise stated. The following inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1}$$

is due to Wilker [11]. It has attracted attention of several researchers (see, e.g., [3], [6], [8], [10], [13], [14]). A hyperbolic counterpart of Wilker's inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 \tag{2}$$

$(x \neq 0)$ has been established by L. Zhu [15]. A. Baricz and J. Sándor [1] have pointed out that inequalities (1) and (2) follow from the first inequalities in

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2+\cos x}{3} \tag{3}$$

$(0 < |x| < \frac{\pi}{2})$ and

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2+\cosh x}{3} \tag{4}$$

$(x \neq 0)$ where the first inequality in (3) is due to D.D. Adamović and D.S. Mitrinović (see, e.g., [5, p. 238]) while the corresponding inequality in (4) was established by Lazarević (see e.g., [2, p. 131]). The second inequalities in (3) and (4) appear respectively in [9], and [7]. Inequalities (1) and (2) have been refined and generalizated as follows

$$\left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^{p} > \left(\frac{x}{\sin x}\right)^{2p} + \left(\frac{x}{\tan x}\right)^{p} > 2 \tag{5}$$

Mathematics subject classification (2010): 26D05, 33B10.

Keywords and phrases: inequalities, trigonometric functions, hyperbolic functions, Wilker's inequality, Huygen's inequality..

Paper MIA-15-22

$(0 < |x| < \frac{\pi}{2})$ and

$$\left(\frac{\sinh x}{x}\right)^{2p} + \left(\frac{\tanh x}{x}\right)^{p} > \left(\frac{x}{\sinh x}\right)^{2p} + \left(\frac{x}{\tanh x}\right)^{p} > 2 \tag{6}$$

$(x \neq 0)$. These results or particular cases of those appear in [12], [16], [6], [7], and [13].

A. Baricz and J. Sándor have pointed out in [1] that the first inequalities in (3) and (4) also imply the following one

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3 \tag{7}$$

$(0 < |x| < \frac{\pi}{2})$ and

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3 \tag{8}$$

$(x \neq 0)$. Inequality (7) is due to C. Huygens [4]. Refinements of the last two inequalities read as follows

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3 \tag{9}$$

$(0 < |x| < \frac{\pi}{2})$ and

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 2\frac{x}{\sinh x} + \frac{x}{\tanh x} > 3 \tag{10}$$

$(x \neq 0)$ (see [7, Thm. 2.6]). The generalizations of (9) and (10), similar to (5) and (6), are obtained immediately by applying Lemma 2 (see Section 2) to (9) and (10). We omit further details.

This paper is a continuation of our earlier work [7] and is organized as follows. In Section 2 we give some lemmas and preliminary results. Refinements and generalizations of Wilker-type inequalities (5) and (6) are presented in Section 3. In Section 4 we prove generalizations and refinements of the Huygens-type inequalities (9) and (10).

2. Lemmas and Preliminaries

The following lemmas will be used in the subsequent sections.

LEMMA 1. ([7]) *Let u and v be different numbers which satisfy the inequality $uu > 1$. Then*

$$u + v > \frac{1}{u} + \frac{1}{v}.$$

LEMMA 2. *Let $u > 0$, $v > 0$ be different numbers and let the positive numbers α and β be such that $\alpha + \beta = 1$. If*

$$\alpha u + \beta v > \alpha\frac{1}{u} + \beta\frac{1}{v} > 1, \tag{11}$$

then the first inequality in

$$\alpha u^p + \beta v^p > \alpha \frac{1}{u^p} + \beta \frac{1}{v^p} > 1 \tag{12}$$

holds true if $p > 0$ *while the second one is valid for* $p \geqslant 1$.

Proof. The first inequality in (12) follows from the first inequality in (11) by letting $u := u^p$ and $v = v^p$. For the proof of the second inequality in (12) we utilize monotonicity of the power mean in its parameter together with (11) to obtain

$$\left(\alpha \frac{1}{u^p} + \beta \frac{1}{v^p}\right)^{1/p} > \alpha \frac{1}{u} + \beta \frac{1}{v} > 1$$

($p \geqslant 1$). Hence the desired inequality follows.

LEMMA 3. *Under the same assumptions about* u, v, α *and* β *as in Lemma* 2 *if* $u^\alpha v^\beta > 1$, *then*

$$\alpha u + \beta v > \alpha \left(\frac{u}{v}\right)^\beta + \beta \left(\frac{v}{u}\right)^\alpha.$$

Proof. We multiply both sides of $u^\alpha v^\beta > 1$ by $\alpha u + \beta v$ to obtain $u^\alpha v^\beta(\alpha u + \beta v) > \alpha u + \beta v$. Dividing both sides of the last inequality by $u^\alpha v^\beta$ we obtain the desired result.

For later use let us record two simultaneous inequalities

$$\left(\cos \frac{x}{2}\right)^{4/3} < \frac{\sin x}{x} < \frac{\cos^2 \frac{x}{2} + 2\cos \frac{x}{2}}{3} \tag{13}$$

($0 < |x| < \frac{\pi}{2}$) and

$$\left(\cosh \frac{x}{2}\right)^{4/3} < \frac{\sinh x}{x} < \frac{\cosh^2 \frac{x}{2} + 2\cosh \frac{x}{2}}{3} \tag{14}$$

($x \neq 0$). (See [7, (2.5) and (2.7)]). They are tighter than the corresponding inequalities (3) and (4).

3. Wilker's Type Inequalities

The goal of this section is to obtain some inequalities similar to those listed in (5) and (6).

A refinement of the second inequality in (5), when $p = 1$, is contained in the following.

THEOREM 1. *Let* $0 < |x| < \frac{\pi}{2}$. *Then*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > \frac{\sin x}{x} + \left(\frac{\tan(x/2)}{x/2}\right)^2$$
$$> \frac{x}{\sin x} + \left(\frac{x/2}{\tan(x/2)}\right)^2 > 2. \quad (15)$$

Proof. First inequality in (15) is a special case of (5) when $p = 1$. For the proof of the second inequality in (15) we let $a = \frac{\sin x}{x}$ and $c = \frac{2}{1+\cos x}$. Using the half-angle formula for the tangent function $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$ the inequality in question can be written as

$$1 + a\cos x > a^3 + a^4c^2.$$

For the proof of the last inequality let

$$f(x) = 1 + a\cos x - (a^3 + a^4c^2) =: g(x) + h(x), \quad (16)$$

where $g(x) = 1 + a\cos x$ and $h(x) = -(a^3 + a^4c^2)$. It is easy to verify that $1 < g(x) < 2$ and also that $g(x)$ is strictly decreasing and strictly concave for $0 < x < \frac{\pi}{2}$. We shall establish now some properties of the function $h(x)$. It follows from the left inequality in (13) that $c^2 > a^{-3}$. This in turn implies that $a^3 + a^4c^2 > a^3 + a$. Since $a = a(x)$ is strictly decreasing and strictly concave on $(0, \frac{\pi}{2})$, $h(x) < -(a^3 + a)$, where the right side of the last inequality is a strictly increasing and strictly convex function. Moreover, $-2 < h(x) < -\left[(\frac{\pi}{2})^3 + \frac{\pi}{2}\right] = -0.680\ldots$ Making use of all properties of functions we conclude using (16), that $f(x) > 0$ for $0 < x < \frac{\pi}{2}$. Since $f(x)$ is an even function we conclude that $f(x)$ is positive for all x which satisfy $0 < |x| < \frac{\pi}{2}$. For the proof of third inequality in (15) we write the first inequality in (13), using the half-angle formula for tangents function, as

$$uv > 1,$$

where $u = \frac{\sin x}{x}$ and $v = \left(\frac{\tan(x/2)}{x/2}\right)^2$. Making use of Lemma 1 gives the desired result. We shall establish now the fourth inequality in (15). With $t = x/2$, the inequality in question can be written as

$$\frac{\sin x}{x} < \frac{1}{2}\left(1 + \cos^3 t \frac{t}{\sin t}\right). \quad (17)$$

Utilizing the second inequality in (13)

$$\frac{\sin x}{x} < \frac{\cos^2 t + 2\cos t}{3}$$

we see that in order to establish (17) it suffices to prove that

$$\frac{\cos^2 t + 2\cos t}{3} < \frac{1}{2}\left(1 + \cos^3 t \frac{t}{\sin t}\right) \quad (18)$$

holds true for $0<|t|<\pi/4$. Since both sides of (18) are even functions, we will assume that $0<t<\pi/4$. Inequality (18) can be written as

$$2\cos^2 t+4\cos t<3+3\cos^3 t\frac{t}{\sin t}. \tag{19}$$

Application of

$$\frac{t}{\sin t}<(\cos t)^{-1/3}$$

(see the left inequality in (3)) to the last member of (19) gives

$$2\cos^2 t+4\cos t-3(\cos t)^{8/3}-3<0. \tag{20}$$

Let $f(x)$ denote the left side of (20). In order to complete the proof of the theorem we have to demonstrate that $f(x)<0$ for $0<x<\frac{\pi}{2}$. Differentiating $f(x)$ we obtain

$$f'(x)=-4\sin t\left(\frac{d+1}{2}-d^{5/3}\right)<0$$

($d=\cos t$) where the last inequality is a consequence of

$$\frac{d+1}{2}>d^{1/2}>d^{5/3}$$

which follows from the inequality of arithmetic and geometric means of d and 1 and also from the monotonicity of the exponential function d^z. Since $f(0)=0$ and $f(x)$ is strictly decreasing we conclude that $f(x)<0$ on $(0,\frac{\pi}{2})$. The proof is complete.

COROLLARY 1. *If* $0<|x|<\frac{\pi}{2}$, *then*

$$\left(\frac{\sin x}{x}\right)^p+\left(\frac{\tan(x/2)}{x/2}\right)^{2p}>\left(\frac{x}{\sin x}\right)^p+\left(\frac{x/2}{\tan(x/2)}\right)^{2p}>2. \tag{21}$$

The first inequality in (21) *holds for all* $p>0$ *while the second one is valid if* $p\geqslant 1$.

Proof. Apply Lemma 2 to last two inequalities in (15) with $u=\frac{\sin x}{x}$, $v=\left(\frac{\tan(x/2)}{x/2}\right)^2$ and $\alpha=\beta=\frac{1}{2}$.

There is no full counterpart of (15) for hyperbolic functions. There is numerical evidence that

$$\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}>\frac{\sinh x}{x}+\left(\frac{\tanh(x/2)}{x/2}\right)^2$$

holds for $0<|x|\leqslant 1.88$ with the inequality reversed if $|x|\geqslant 1.89$. The hyperbolic counterparts of the last two inequalities in (15) are contained in the following.

THEOREM 2. *Let* $x\neq 0$. *Then*

$$\frac{\sinh x}{x}+\left(\frac{\tanh(x/2)}{x/2}\right)^2>\frac{x}{\sinh x}+\left(\frac{x/2}{\tanh(x/2)}\right)^2>2. \tag{22}$$

For the proof of (22) *one might follow the lines of proof of the corresponding inequalities in* (15). *We omit further details.*

COROLLARY 2. *Let* $x \neq 0$. *Then*

$$\left(\frac{\sinh x}{x}\right)^p + \left(\frac{\tanh(x/2)}{x/2}\right)^{2p} > \left(\frac{x}{\sinh x}\right)^p + \left(\frac{x/2}{\tanh(x/2)}\right)^{2p} > 2. \tag{23}$$

The first inequality in (23) *is valid for* $p > 0$ *while the second one holds true for all* $p \geqslant 1$.

Proof. Inequalities (23) follow from (22) by use of Lemma 2 with $u = \frac{\sinh x}{x}$, $v = \left(\frac{\tanh(x/2)}{x/2}\right)^2$ and $\alpha = \beta = \frac{1}{2}$.

4. Huygens Type Inequalities

In this section we shall establish refinements of first inequalities in (9) and (10). Generalizations of the last two inequalities

$$2\left(\frac{\sin x}{x}\right)^p + \left(\frac{\tan x}{x}\right)^p > 2\left(\frac{x}{\sin x}\right)^p + \left(\frac{x}{\tan x}\right)^p > 3 \tag{24}$$

$(0 < |x| < \frac{\pi}{2})$ and

$$2\left(\frac{\sinh x}{x}\right)^p + \left(\frac{\tanh x}{x}\right)^p > 2\left(\frac{x}{\sinh x}\right)^p + \left(\frac{x}{\tanh x}\right)^p > 3 \tag{25}$$

$(x \neq 0)$ are obtained applying Lemma 2 with $\alpha = 2/3$ and $\beta = 1/3$ to (9) and (10). First inequalities in (24) and (25) hold true if $p > 0$ while the second inequalities are valid provided $p \geqslant 1$.

We are in a position to prove the following.

THEOREM 3. *Let* $0 < |x| < \frac{\pi}{2}$. *Then*

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x} + 2\frac{\tan(x/2)}{x/2} > 2\frac{x}{\sin x} + \frac{x}{\tan x} > 3. \tag{26}$$

Proof. Without a loss of generality, we may assume that $0 < x < \frac{\pi}{2}$. For the proof of the first inequality in (26) we use the following statement

$$1 + \frac{1}{\cos x} > \frac{4}{1 + \cos x} \tag{27}$$

which can be written as $(1 - \cos x)^2 > 0$. Multiplying both sides of (27) by $\frac{\sin x}{x}$ and next adding to both sides $\frac{\sin x}{x}$, we obtain

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > \frac{\sin x}{x} + 2\frac{\sin x}{1 + \cos x}\frac{1}{x/2}.$$

Application of the half-angle formula for the tangent function $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$ gives the assertion. We shall establish now the second inequality in (26). It suffices to show that

$$\frac{\sin x}{x} + 2\frac{\tan(x/2)}{x/2} > \frac{x}{\sin x} + 2\frac{x/2}{\tan(x/2)} \tag{28}$$

because

$$\frac{x}{\sin x} + \frac{x/2}{\tan(x/2)} = 2\frac{x}{\sin x} + \frac{x}{\tan x}. \tag{29}$$

For the proof of (28) we let $a = \dfrac{\sin t}{t}$, $c = \cos t$ and $t = \frac{x}{2}$. It is easy to see that (28) can be written as

$$a^2 > \frac{2c^2+1}{c^2+2}. \tag{30}$$

The first inequality in (13) can be written as

$$a^2 > \left(\frac{1+c}{2}\right)^{4/3}.$$

We shall prove (30) showing that

$$\left(\frac{1+c}{2}\right)^{4/3} > \frac{2c^2+1}{c^2+2}. \tag{31}$$

For the proof of (31) we introduce a function

$$f(c) = \left(\frac{1+c}{2}\right)^4 - \left(\frac{2c^2+1}{c^2+2}\right)^3.$$

Function $f(c)$ can be written, after a little algebra, as

$$f(c) = \frac{(1-c)^2}{16(c^2+2)^3}(c^8+6c^7+23c^6+68c^5+34c^4+72c^3+4c^2+16c-8).$$

Since $0 < x < \frac{\pi}{2}$, $0 < t < \frac{\pi}{4}$. Thus $\frac{1}{\sqrt{2}} < c < 1$. It is elementary to verify that $f(c) > 0$ in the stated domain for c. This in turn implies that the inequalities (31), (30), and (28) hold true. Making use of (29) and the second inequality in (9) we obtain the asserted result. The proof is complete.

Combining the last three terms in (26) with (28) and next using Lemma 2 we obtain the following.

COROLLARY 3. *If* $0 < |x| < \frac{\pi}{2}$, *then*

$$\left(\frac{\sin x}{x}\right)^p + 2\left(\frac{\tan(x/2)}{x/2}\right)^p > \left(\frac{x}{\sin x}\right)^p + 2\left(\frac{x/2}{\tan(x/2)}\right)^p > 3, \tag{32}$$

where the first inequality is valid if $p > 0$ *while the second one holds true provided* $p \geqslant 1$.

A refinement of the first inequality in (10) is obtained in the following.

THEOREM 4. *If $x \neq 0$, then*

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x} > \frac{\sinh x}{x}+2\frac{\tanh(x/2)}{x/2} > 2\frac{x}{\sinh x}+\frac{x}{\tanh x} > 3. \tag{33}$$

Proof of the last result is very similar to that of Theorem 3 and for these reasons is not included here. The hyperbolic counterpart of (29)

$$\frac{x}{\sinh x}+2\frac{x/2}{\tanh(x/2)} = 2\frac{x}{\sinh x}+\frac{x}{\tanh x} \tag{34}$$

and the first inequality in (14) are utilized in the proof of (33).

The following result is obtained with the aid of the last three inequalities in (33), identity (34), and Lemma 2 used with $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$, $u = \frac{\sinh x}{x}$, and $v = \frac{\tanh(x/2)}{x/2}$. We have

COROLLARY 4. *Let $x \neq 0$. Then*

$$\left(\frac{\sinh x}{x}\right)^p + 2\left(\frac{\tanh(x/2)}{x/2}\right)^p > \left(\frac{x}{\sinh x}\right)^p + 2\left(\frac{x/2}{\tanh(x/2)}\right)^p > 3,$$

where the first inequality holds true if $p > 0$ while the second one is valid for all $p \geqslant 1$.

We close this section with proofs of some refinements of inequalities (7) and (8). We have

THEOREM 5. *The following inequalities*

$$2\frac{\sin x}{x}+\frac{\tan x}{x} > \frac{2\cos x+1}{(\cos x)^{2/3}} > 3 \tag{35}$$

($0 < |x| < \frac{\pi}{2}$) and

$$2\frac{\sinh x}{x}+\frac{\tanh x}{x} > \frac{2\cosh x+1}{(\cosh x)^{2/3}} > 3 \tag{36}$$

($x \neq 0$) are valid.

Proof. We shall establish the inequality (35) only. Its first part follows immediately from Lemma 3 applied with $u = \frac{\sin x}{x}$, $v = \frac{\tan x}{x}$, $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$. Let $c = \cos x$. Then the second inequality in (35) is equivalent to $2c+1 > 3c^{2/3}$ or to $f(c) > 0$, where $f(c) = (2c+1)^3 - 27c^2$. Clearly $f(0) = 1$, $f(1) = 0$, and $f'(c) = 6(4c-1)(c-1)$. Thus $f(c)$ is strictly increasing on the interval $\left(0, \frac{1}{4}\right)$ and strictly decreasing on $\left(\frac{1}{4}, 1\right)$. Hence the assertion $f(c) > 0$ for $c \in (0,1)$ follows and the proof is complete.

REFERENCES

[1] A. BARICZ, J. SÁNDOR, *Extensions of generalized Wilker inequality to Bessel functions*, J. Math. Inequal. **2**, 3 (2008), 397–406.

[2] P. S. BULLEN, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 97, Addison Wesley Longman Limited, Longman, Harlow, 1998.

[3] B. -N. GUO, B. -M. QIAO, F. QI, W. LI, *On new proofs of Wilker inequalities involving trigonometric functions*, Math. Inequal. Appl. **6**, 1 (2003), 19–22.

[4] C. HUYGENS, *Oeuvres Completes 1888–1940*, Société Hollandaise des Science, Haga.

[5] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[6] E. NEUMAN, *One- and two-sided ineualities for Jacobian elliptic functions and related results*, Integral Transform. Spec. Funct. **21**, 6 (2010), 399–407.

[7] E. NEUMAN, J. SÁNDOR, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Huygens inaequalities*, Math. Inequal. Appl. **13**, 4 (2010), 715–723.

[8] I. PINELIS, *L'Hospital rules of monotonicity and Wilker-Anglesio inequality*, Amer. Math. Monthly **111**(2004), 905–909.

[9] J. SÁNDOR, M. BENCZE, *On Huygen's trigonometric inequality*, RGMIA Res. Rep. Collection, **8**, 8 (2005), Article 14.

[10] J. S. SUMNER, A. A. JAGERS, M. VOWE, J. ANGLESIO, *Inequalities involving trigonometric functions*, Amer. Math. Monthly **98** (1991), 264–267.

[11] J. B. WILKER, *Problem E 3306*, Amer. Math. Monthly **96** (1989), 55.

[12] S. WU, A. BARICZ, *Generlizations of Mitrinović, Adamović and Lazarevic's inequalities and their applications*, Publ. Math. Debrecen **75** 3–4 (2009), 447–458.

[13] S. -H. WU, H. M. SRIVASTAVA, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transform. Spec. Funct. **18**, 8 (2007), 525–535.

[14] L. ZHU, *A new simple proof of Wilker's inequality*, Math. Inequal. Appl., **8** (2005), No. 4, 749–750.

[15] L. ZHU, *On Wilker-type inequalities*, Math. Inequal. Appl., **10**, 4 (2007), 727–731.

[16] L. ZHU, *Some new Wilker type inequalities for circular and hyperbolic functions*, Abstract Appl. Analysis, Vol. 2009, Article ID 485842, 9 pages.

(Received June 16, 2010)

Edward Neuman
Department of Mathematics
Mailcode 4408
Southern Illinois University
1245 Lincoln Drive, Carbondale
IL 62901, USA.
e-mail: edneuman@math.siu.edu