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STABILITY PROPERTIES OF THE GENERALIZED CHERNOFF INEQUALITY

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Abstract. In this short note we will present two stability properties of the Chernoff-Ou-Pan inequality, newly obtained in [6], which states that if K is a convex domain in the plane $\mathbb{R}^2$ with area $a(K)$, then one gets

$$a(K) \leqslant \frac{1}{k}\int_0^{\frac{\pi}{k}} \omega_k(\theta)w_k(\theta+\frac{\pi}{k})d\theta,$$

where $w_k(\theta)$ is defined in [6] (see also §3 below), and the equality holds if and only if K is a circular disc.

1. Introduction

The study of stability properties of geometric inequalities has received a lot of attention in convex geometry, see, for example, Fuglede [2], Groemer [3], [4], and Pan-Xu [7], etc.. Let $\mathscr{C}^n$ denote the class of all n-dimensional convex bodies in the Euclidean space $\mathbb{R}^n$. If $n=2$, a 2-dimensional convex body is usually called a *convex domain* in the Euclidean plane $\mathbb{R}^2$. An inequality in convex geometry can be written

$$\Phi(K) \geqslant 0, \tag{1.1}$$

where $\Phi : \mathscr{C}^n \to \mathbb{R}$ is a real valued function and (1.1) is supposed to hold for all $K \in \mathscr{C}^n$. Let $\mathscr{C}^n_\Phi$ denote those elements $K \in \mathscr{C}^n$ for which the equality sign in (1.1) holds, i.e., $\Phi(K)=0$ for all $K \in \mathscr{C}^n_\Phi$. For example, if $n=2$, $p(K)$ denotes the perimeter of K and $a(K)$ its area, $\Phi(K)=p(K)^2-4\pi a(K)$, (1.1) is then the classical isoperimetric inequality in the Euclidean plane $\mathbb{R}^2$, and in this case $\mathscr{C}^2_\Phi$ is known to consist of all circular discs.

We are interested in the stability problem associated with geometric inequalities of type (1.1). That means, we ask if K must be close to a member of $\mathscr{C}^n_\Phi$ whenever $\Phi(K)$ is close to zero. In order to give a precise formulation of this problem, it is necessary for us to be given a measurement function $g : \mathscr{C}^n \times \mathscr{C}^n \to \mathbb{R}$ that describes in some sense the deviation between two convex bodies. g should satisfy the following conditions:

(i). $g(K,L) \geqslant 0$ for all $K,\ L \in \mathscr{C}^n$;

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(ii). $g(K,L)=0$ if and only if $K=L$.

If Φ, $\mathscr{C}^n_\Phi$ and g are given, the stability problem associated with the geometric inequality (1.1) can now be formulated as follows:

Find positive constants c,α with property that whenever

$$\Phi(K)\leqslant\varepsilon, \tag{1.2}$$

(for some $\varepsilon\geqslant 0$), then there exists an $L\in\mathscr{C}^n_\Phi$ such that

$$g(K,L)\leqslant c\varepsilon^{\alpha}. \tag{1.3}$$

Or equivalently,

Find positive constants c,α with property that for each $K\in\mathscr{C}^n$, there exists an $L\in\mathscr{C}^n_\Phi$ (L may depend on K) such that

$$\Phi(K)\geqslant cg(K,L)^{\alpha}. \tag{1.4}$$

In this note we will focus our attention on the stability properties of a generalized Chernoff inequality in the plane $\mathbb{R}^2$, newly obtained by Ou-Pan [6], which states that if K is a convex domain with area $a(K)$, then one gets

$$a(K)\leqslant\frac{1}{k}\int_0^{\frac{\pi}{k}}w_k(\theta)w_k\Big(\theta+\frac{\pi}{k}\Big)d\theta, \tag{1.5}$$

where $w_k(\theta)(k\geqslant 2$ is an integer) is defined in §3 below ($w_k(\theta)$ is first introduced by Ou and Pan in [6]) and the equality in (1.5) holds if and only if K is a circular disc. We will prove in §3 that this inequality has stability properties with respect to both the Hausdorff distance and the L_2-metric on $\mathscr{C}^2$.

2. Preliminaries

We will first recall some basic facts about plane convex geometry which will be used later on. Firstly, let $K\in\mathscr{C}^2$ be a convex domain and assume that the origin O of $\mathbb{R}^2$ lies in the interior of K, and let $\vec{u}$ be a unit vector in $\mathbb{R}^2$ and $L(\vec{u})$ denote the supporting line of K that is perpendicular to $\vec{u}$ and on the same side of the origin. The oriented distance from O to $L(\vec{u})$, denoted by $H(\vec{u})$, is called the *Minkowski support function* of K. Since $\vec{u}$ is usually determined by the oriented angle, say θ, from the positive x-axis to $\vec{u}$, one also writes $H(\theta)$ instead of $H(\vec{u})$. It is clear that $H(\theta)$ is a continuous 2π-periodic function.

Let $p(K)$ denote the perimeter and $a(K)$ the area of K, one can find (see, for example, [3] or [5])

$$p(K)=\int_0^{2\pi}H(\theta)d\theta, \tag{2.1}$$

and if H is sufficiently smooth, then

$$a(K) = \frac{1}{2}\int_0^{2\pi} \left(H^2(\theta) - H'^2(\theta)\right) d\theta, \tag{2.2}$$

where $'$ denotes the derivative with respect to θ.

The *Steiner disc* of K, denoted by $S(K)$, is the circular disc with radius $p(K)/2\pi$ and center at the *Steiner point* which can be defined in terms of the Minkowski support function

$$\vec{s}(K) = \frac{1}{\pi}\int_0^{2\pi} \vec{u}(\theta)H(\theta)d\theta. \tag{2.3}$$

The Steiner disc $S(K)$ of K will play a role in our stability statement in §3 below.

The width of K in a direction $\vec{u}(\theta) = (\cos(\theta), \sin(\theta))$, denoted by $w(\theta)$, is defined to be the distance between two tangents to a perpendicular to $\vec{u}(\theta)$. It is clear that

$$w(\theta) = H(\theta) + H(\theta + \pi). \tag{2.4}$$

The closed convex curve K is said to be of constant width if its width in any direction is a positive constant w_0, then $w(\theta) = H(\theta) + H(\theta + \pi) = w_0$ for any $\theta \in [0, 2\pi]$.

Secondly, we wish to express $p(K), \vec{s}(K), a(K)$ in terms of the Fourier coefficients of $H(\theta)$. Since the support function of a given domain K is always continuous, bounded and 2π-periodic, it has a Fourier series of the form

$$H(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta).$$

Differentiation of this with respect to θ gives us

$$H'(\theta) = \sum_{n=1}^{\infty} n(-a_n \sin n\theta + b_n \cos n\theta).$$

From (2.1) and (2.3), it follows immediately that

$$p(K) = 2\pi a_0, \tag{2.5}$$

$$\vec{s}(K) = (a_1, b_1). \tag{2.6}$$

By the Parseval equality, one can get

$$\int_0^{2\pi} H^2(\theta)d\theta = \pi(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)), \qquad \int_0^{2\pi} H'^2(\theta)d\theta = \pi\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2),$$

which together with (2.2) give us

$$a(K) = \pi a_0^2 - \frac{\pi}{2}\sum_{n=2}^{\infty}(n^2 - 1)(a_n^2 + b_n^2). \tag{2.7}$$

Finally, let K and L be two convex domains with respective support functions H_K and H_L. The most frequently used function to measure the deviation between L and K is the Hausdorff distance,

$$h(K,L) = \max_{\vec{u}} \left| H_L(\vec{u}) - H_K(\vec{u}) \right|.$$

Another such measure which appears to be of particular value with respect to stability problems is the measure that corresponds to the L_2-metric in function space. It is defined by

$$h_2(K,L) = \left(\int_0^{2\pi} \left| H_L(\theta) - H_K(\theta) \right|^2 \mathrm{d}\theta \right)^{1/2}.$$

It is obvious that $h(K,L)=0$ (or $h_2(K,L)=0$) if and only if $K=L$.

3. The Main Results

For an integer $k \geqslant 2$, Ou-Pan[6] has introduced for a convex domain K a function $\omega_k(\theta)$ as follows

$$w_k(\theta) = H(\theta) + H\left(\theta + \frac{2\pi}{k}\right) + \cdots + H\left(\theta + \frac{2(k-1)\pi}{k}\right).$$

It is clear that $w_k(\theta)$ is a periodic function with period $\frac{2\pi}{k}$. If $k=2$, $\omega_2(\theta)$ is the usual width $w(\theta)$ (see (2.4) above) of a convex domain.

In [6], Ou and Pan have got the inequality (1.5) which involves the area $a(K)$ of K and the function $w_k(\theta)$ and generalizes the Chernoff inequality (see [1], the Chernoff inequality corresponds to the case of $k=2$). We consider now the stability properties of the so-called Chernoff-Ou-Pan inequality (1.5) with respect to the deviation measures h_2 and h.

THEOREM 3.1. *Let K be convex domain in the plane with smooth boundary, denote $a(K)$ the area of K, then*

$$\frac{1}{k}\int_0^{\frac{\pi}{k}} w_k(\theta) w_k\left(\theta + \frac{\pi}{k}\right) d\theta - a(K) \geqslant \begin{cases} h_2(K,S(K))^2, & \text{when} \quad k=2, \\ \frac{3}{2} h_2(K,S(K))^2, & \text{when} \quad k \geqslant 3. \end{cases} \tag{3.1}$$

Equality sign in (3.1) holds if and only if the support function of K is of the form

$$H(\theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + b_1 \sin\theta + b_2 \sin 2\theta,$$

where $a_0 = p(K)/2\pi$.

Proof. We may assume that $\vec{s}(K) = O$, then because of (2.5) and (2.6), the support functions H_K and $H_{S(K)}$ have the following Fourier series

$$H_K(\theta) = p(K)/2\pi + \sum_{n=2}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), \quad H_{S(K)} = p(K)/2\pi.$$

By using Parseval's equality one can obtain

$$h_2(K,S(K))^2=\int_0^{2\pi}|H_K(\theta)-H_{S(K)}|^2\mathrm{d}\theta=\pi\sum_{n=2}^{\infty}(a_n^2+b_n^2).$$

In Ou-Pan[6] it has proved that (see (3.10) of Ou-Pan[6])

$$\frac{1}{k}\int_0^{\frac{\pi}{k}}w_k(\theta)w_k\Big(\theta+\frac{\pi}{k}\Big)d\theta-a(K)=\frac{\pi}{2}\Big[\sum_{n=2}^{\infty}(a_n^2+b_n^2)(n^2-1)+\sum_{l=1}^{\infty}(-1)^l(a_{kl}^2+b_{kl}^2)\Big].\tag{3.2}$$

Therefore we get

$$\frac{1}{k}\int_0^{\frac{\pi}{k}}w_k(\theta)w_k\Big(\theta+\frac{\pi}{k}\Big)d\theta-a(K)\geqslant\begin{cases}\pi\sum\limits_{n=2}^{\infty}(a_n^2+b_n^2), & k=2,\\ \dfrac{3}{2}\pi\sum\limits_{n=2}^{\infty}(a_n^2+b_n^2), & k\geqslant 3.\end{cases}$$

$$\geqslant\begin{cases}h_2(K,S(K))^2, & k=2,\\ \dfrac{3}{2}h_2(K,S(K))^2, & k\geqslant 3.\end{cases}$$

And furthermore, it is easy to see that the equality holds if and only if

$$H_K(\theta)=a_0+a_1\cos\theta+a_2\cos 2\theta+b_1\sin\theta+b_2\sin 2\theta,$$

where a_2,b_2 are small in comparison with a_0. □

To obtain a stability statement in terms of the Hausdorff metric, one can observe that (3.2) gives an explicit expression for the quantity $\frac{1}{k}\int_0^{\frac{\pi}{k}}w_k(\theta)w_k(\theta+\frac{\pi}{k})d\theta-a(K)$.

Since it is easily seen that

$$|a_n\cos n\theta+b_n\sin n\theta|\leqslant\sqrt{a_n^2+b_n^2},$$

one can get

$$|H_K(\theta)-H_{S(K)}(\theta)|=|a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)-(a_0+a_1\cos\theta+b_1\sin\theta)|$$

$$\leqslant\sum_{n=2}^{\infty}|a_n\cos n\theta+b_n\sin n\theta|\leqslant\sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2}.$$

Using Hölder's inequality, one finds

$$h(K,S(K))\leqslant\sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2}\leqslant\Big(\sum_{n=2}^{\infty}\frac{1}{n^2-2}\Big)^{1/2}\Big(\sum_{n=2}^{\infty}(n^2-2)(a_n^2+b_n^2)\Big)^{1/2}.$$

One can calculate

$$\sum_{n=2}^{\infty}\frac{1}{n^2-2}<\frac{1}{2}+\frac{1}{7}+\sum_{n=4}^{\infty}\frac{1}{n^2-4}=\frac{1619}{1680}<1$$

which together with (3.2) implies that

$$\begin{aligned} h(K,S(K))^2 &\leqslant \sum_{n=2}^{\infty}\tfrac{1}{n^2-2}\sum_{n=2}^{\infty}(n^2-2)(a_n^2+b_n^2)\\ &< \sum_{n=2}^{\infty}(n^2-2)(a_n^2+b_n^2)\\ &\leqslant \sum_{n=2}^{\infty}(a_n^2+b_n^2)(n^2-1)+\sum_{l=1}^{\infty}(-1)^l(a_{kl}^2+b_{kl}^2)\\ &= \tfrac{2}{\pi}\left[\tfrac{1}{k}\int_0^{\frac{\pi}{k}} w_k(\theta)w_k(\theta+\tfrac{\pi}{k})d\theta-a(K)\right], \end{aligned}$$

where $k\in\mathbb{N}$ and thus one has arrived at the following result:

THEOREM 3.2. *Under the same assumptions of Theorem 3.1, one gets*

$$\frac{1}{k}\int_0^{\frac{\pi}{k}} w_k(\theta)w_k\left(\theta+\frac{\pi}{k}\right)d\theta-a(K)>\frac{\pi}{2}h(K,S(K))^2. \tag{3.3}$$

REMARKS. (i). Theorems 3.1 and 3.2 can be looked upon as strengthened forms of the generalized Chernoff inequality (1.5).

(ii). Observe that although (3.1) can not be improved for all $K\in\mathscr{C}^2$, it is possible to prove stronger inequalities for particular kinds of convex domains. For example, if K is of constant width, the Fourier expression of the support function of K has the property that $a_{2n}=b_{2n}=0$ for all $n\in\mathbb{N}$. Checking the proof of (3.1),

$$h_2(K,S(K))^2=\pi\sum_{n=1}^{\infty}(a_{2n+1}^2+b_{2n+1}^2),$$

$$\begin{aligned} &\frac{1}{k}\int_0^{\frac{\pi}{k}} w_k(\theta)w_k\left(\theta+\frac{\pi}{k}\right)d\theta-a(K)\\ =&\frac{\pi}{2}\left[4\sum_{n=1}^{\infty}n(n+1)(a_{2n+1}^2+b_{2n+1}^2)+\sum_{\substack{l=1\\ kl=2m+1,m\in\mathbb{N}}}^{\infty}(-1)^l(a_{kl}^2+b_{kl}^2)\right]\\ \geqslant&\begin{cases}\dfrac{7\pi}{2}\sum_{n=1}^{\infty}(a_{2n+1}^2+b_{2n+1}^2), & k=3,\\ 4\pi\sum_{n=1}^{\infty}(a_{2n+1}^2+b_{2n+1}^2), & k\neq 3.\end{cases}\\ =&\begin{cases}\dfrac{7}{2}h_2(K,S(K))^2, & k=3,\\ 4h_2(K,S(K))^2, & k\neq 3.\end{cases} \end{aligned}$$

where the equality sign holds if and only if

$$H_K(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_3\cos 3\theta + b_3\sin 3\theta.$$

Similarly, (3.3) can also be strengthened in this case. Since

$$\begin{aligned}
|H_K(\theta) - H_{S(K)}(\theta)| &\leqslant \sum_{n=1}^{\infty}\sqrt{a_{2n+1}^2 + b_{2n+1}^2} \\
&\leqslant \Big(\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2-2}\Big)^{1/2}\Big[\sum_{n=1}^{\infty}\big((2n+1)^2-2\big)^2(a_{2n+1}^2+b_{2n+1}^2)\Big]^{1/2} \\
&< \Big(\sum_{n=1}^{\infty}\frac{1}{4n^2+4n-3}\Big)^{1/2}\Big[\sum_{n=1}^{\infty}\big((2n+1)^2-2\big)^2(a_{2n+1}^2+b_{2n+1}^2)\Big]^{1/2} \\
&= \sqrt{\frac{1}{3}}\Big[\sum_{n=1}^{\infty}\big((2n+1)^2-2\big)^2(a_{2n+1}^2+b_{2n+1}^2)\Big]^{1/2},
\end{aligned}$$

one can obtain

$$\frac{1}{k}\int_0^{\frac{\pi}{k}} w_k(\theta)w_k\left(\theta+\frac{\pi}{k}\right)d\theta - a(K) > \frac{3\pi}{2}h(K,S(K))^2.$$

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