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APPROXIMATION BY NÖRLUND MEANS OF DOUBLE WALSH–FOURIER SERIES FOR LIPSCHITZ FUNCTIONS

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Abstract. For double trigonometric Fourier series, Móricz and Rhoades studied the rate of uniform approximation by Nörlund means of the rectangular partial sums of double Fourier series of a function belonging to the class $\mathrm{Lip}\alpha$ $(0<\alpha\leqslant 1)$ [12] and the class of continuous functions [13], on the two-dimensional torus. As a special case, they obtained the rate of uniform approximation by double Cesàro means.

The main aim of this paper is to investigate the rate of the approximation by the Nörlund means $T^{w}_{m,n}(f)$ of double Walsh-Fourier series of a function in L^p, in particular, in $\mathrm{Lip}(\alpha,p)$, where $\alpha>0$ and $1\leqslant p\leqslant\infty$. In case $p=\infty$, by L^p we mean C_W, the collection of the uniform W-continuous functions.

Earlier results on one-dimensional Nörlund means of the Walsh-Fourier series was given by Móricz and Siddiqi [16].

1. Nörlund means for double Fourier series

Let $\{q_{j,k} : j,k=0,1,\ldots\}$ be a double sequence of nonnegative numbers, $q_{0,0}>0$. Set

$$Q_{m,n} := \sum_{j=0}^{m}\sum_{k=0}^{n} q_{j,k} \quad (m,n=0,1,\ldots).$$

The Nörlund means and kernels of rectangular partial sums of double Walsh-Fourier series are defined by

$$T_{m,n}(f,x^1,x^2) := \frac{1}{Q_{m,n}}\sum_{j=0}^{m}\sum_{k=0}^{n} q_{m-j,n-k}S_{j,k}(f,x^1,x^2),$$

$$\mathscr{L}_{m,n}(x^1,x^2) := \frac{1}{Q_{m,n}}\sum_{j=0}^{m}\sum_{k=0}^{n} q_{m-j,n-k}D_j(x^1)D_k(x^2).$$

The Nörlund method generated by the double sequence $\{q_{j,k}\}$ is regular [12] iff

$$\lim_{n,m\to\infty}\frac{1}{Q_{m,n}}\sum_{k=0}^{n} q_{m-j,k}=0 \quad (j=0,1,\ldots;m\geqslant j)$$

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and

$$\lim_{n,m\to\infty} \frac{1}{Q_{m,n}} \sum_{j=0}^{m} q_{j,n-k} = 0 \quad (k = 0,1,\ldots; n \geqslant k).$$

In particular case, set

$$q_{j,k} := A_j^{\beta-1} A_k^{\gamma-1} \quad (j,k = 0,1,\ldots), \tag{1}$$

where $A_l^\alpha = \binom{\alpha+l}{l} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+l)}{l!}$ for $l = 1,2,\ldots$ and $A_0^\alpha = 1$ $(\alpha \neq -1,-2,\ldots)$. Then $Q_{m,n} = A_m^\beta A_n^\gamma$ and $T_{m,n}(f)$ are the (C,β,γ)-means of double Fourier series.

For double trigonometric Fourier series, Móricz and Rhoades studied the rate of uniform approximation by Nörlund means of the rectangular partial sums of double Fourier series of a function belonging to the class Lip α $(0 < \alpha \leqslant 1)$ [12] and the class of continuous functions [13], on the two-dimensional torus. As a special case, they obtained the rate of uniform approximation by double Cesàro means.

The main aim of this paper is to investigate the rate of the approximation by the Nörlund mean $T_{m,n}^w(f)$ of double Walsh-Fourier series of a function in L^p, in particular, in $\mathrm{Lip}(\alpha,p)$, where $\alpha > 0$ and $1 \leqslant p \leqslant \infty$. In case $p = \infty$, by L^p we mean C_W, the collection of the uniform W-continuous functions. As a special case, we obtain the rate of uniform approximation by double Walsh-Cesàro means, as Móricz and Rhoades did in the trigonometric case.

Now, we make a few historical comments on Walsh-Nörlund means. The one-dimensional Nörlund mean $t_n(f)$ and kernel L_n for Walsh-Fourier series are defined by

$$t_n(f,x) := \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(f,x), \quad L_n(x) := \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} D_k(x),$$

where $P_n := \sum_{k=0}^n p_k$ and $\{p_j\}$ is a sequence of nonnegative numbers, with $p_0 > 0$. The approximation behavior of the Walsh-Nörlund mean t_n in L^p was discussed by Móricz and Siddiqi [16] and recently, in dyadic homogeneous Banach space and dyadic Hardy space by Fridli, Manchanda and Siddiqi [3].

As special cases Móricz and Siddiqi obtained the earlier results given by Yano [20], Jastrebova [11] and Skvortsov [18] on the rate of the approximation by Cesàro means. The approximation properties of the Cesàro means of negativ order was studied by Goginava in 2002 [9].

The case when $p_k = \frac{1}{k}$ is not discussed in the paper of Móricz and Siddiqi, in this case t_n are called the Nörlund logarithmic means. Recently, the Nörlund logarithmic means for the Walsh-Fourier series was investigated by Gát, Goginava and Tkebuchava [4, 10], for unbounded Vilenkin system by Blahota and Gát [2]. Moreover, the two-dimensional Walsh-Nörlund logarithmic means was studied in [5, 6, 7, 8].

Our paper is motivated by the work of Móricz and Rhoades [12, 13] on classical trigonometric system and the results of Gát, Goginava, Tkebuchava [4, 8] and Fridli, Manchanda, Siddiqi [3] on the Walsh system in present days.

We will use the notations

$$\Delta_{1,0} q_{j,k} := q_{j,k} - q_{j+1,k},$$

$$\Delta_{0,1}q_{j,k} := q_{j,k} - q_{j,k+1},$$
$$\Delta_{1,1}q_{j,k} := q_{j,k} - q_{j+1,k} - q_{j,k+1} + q_{j+1,k+1}$$

for $j,k = 0,1,\ldots$. The double sequence $\{q_{j,k}\}$ is nondecreasing, in sign $q_{j,k}\uparrow$, if $\Delta_{1,0}q_{j,k} \leqslant 0$ and $\Delta_{0,1}q_{j,k} \leqslant 0$, and is nonincreasing, in sign $q_{j,k}\downarrow$, if $\Delta_{1,0}q_{j,k} \geqslant 0$ and $\Delta_{0,1}q_{j,k} \geqslant 0$ for every $j,k = 0,1,\ldots$.

2. Definitions and notations

Let G be the Walsh group. The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure is denoted by μ. Dyadic intervals are defined by

$$I_0(x) := G, \;\; I_n(x) := \{y \in G : y = (x_0,\ldots,x_{n-1},y_n,y_{n+1}\ldots)\}$$

for $x \in G, n \in \mathbf{P}$. They form a base for the neighborhoods of G. Let $0 = (0 : i \in \mathbf{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbf{N}$. Set $e_i := (0,\ldots,0,1,0,\ldots)$, where the ith coordinate is 1 and the rest are 0 ($i \in \mathbf{N}$).

Let L^p denote the usual Lebesgue spaces on G^2 (with the corresponding norm $\|.\|_p$). For the sake of brevity in notation, we agree to write L^∞ instead of C_W and set $\|f\|_\infty := \sup\{|f(x)| : x \in G^2\}$.

Next, we define the modulus of continuity in $L^p, 1 \leqslant p \leqslant \infty$, of a function $f \in L^p$ by

$$\omega_p(\delta, f) := \sup_{|t|<\delta} \|f(.+t) - f(.)\|_p, \quad \delta > 0.$$

The Lipschitz classes in L^p for each $\alpha > 0$ are defined by

$$\text{Lip}\,(\alpha, p) := \{f \in L^p : \omega_p(\delta, f) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

For $x \in G$ we define $|x|$ by $|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1}$, for $x = (x^1,x^2) \in G^2$ by $|x|^2 := (x^1)^2 + (x^2)^2$. Thus, for $f \in L^p(G^2)$ $(1 \leqslant p \leqslant \infty)$ the modulus of continuity $\omega_p(\delta, f)$ is well defined for $\delta > 0$. We define the mixed modulus of continuity as follows

$$\omega_{1,2}^p(\delta_1,\delta_2,f) :=$$
$$\sup\{\|f(.+x^1,.+x^2) - f(.+x^1,.) - f(.,.+x^2) + f(.,.)\|_p : |x^1| \leqslant \delta_1, |x^2| \leqslant \delta_2\},$$

where $\delta_1,\delta_2 > 0$.

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \;\; (x \in G, k \in \mathbf{N}).$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i$, $n_i \in \{0,1\}$ $(i \in \mathbf{N})$, where only a finite number of n_i's different from zero. Let the order of

$n > 0$ be denoted by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 = 1$ and for $n \geqslant 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k}.$$

The Dirichlet kernels are defined by

$$D_n := \sum_{k=0}^{n-1} w_k,$$

where $n \in \mathbf{P}$, $D_0 := 0$. The 2^nth Dirichlet kernels have a closed form (see e.g. [17])

$$D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n, \\ 0, & \text{otherwise } (n \in \mathbf{N}). \end{cases} \tag{2}$$

The nth Fejér mean and the Fejér kernel of the Fourier series of a function f is defined by

$$\sigma_n(f;x) := \frac{1}{n} \sum_{k=0}^{n} S_k(f;x), \quad K_n(x) := \frac{1}{n} \sum_{k=0}^{n} D_k(x) \ \ (x \in G),$$

and $K_0 = 0$.

On G^2 we consider the two-dimensional system as $\{w_{n^1}(x^1) \times w_{n^2}(x^2) : n := (n^1, n^2) \in \mathbf{N}^2\}$. The two-dimenional Fourier coefficients, the rectangular partial sums of the Fourier series and Dirichlet kernels are defined in the usual way.

3. The rate of the approximation

Our main results read as follow.

THEOREM 1. *Let $f \in L^p$, $1 \leqslant p \leqslant \infty$, and $|m| = A \geqslant 1$, $|n| = B \geqslant 1$. Let $\{q_{j,k}\}$ be a double sequence of nonnegative numbers such that $\Delta_{1,1} q_{j,k}$ is of fixed sign.*

If $\{q_{j,k}\}$ is nondecreasing, in sign $\uparrow$, then

$$\|T_{m,n}(f) - f\|_p \leqslant \frac{c}{Q_{m,n}} \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} 2^{k+l} q_{m-2^k, n-2^l} (\omega_p(2^{-k}, f) + \omega_p(2^{-l}, f)) \\ + O(\omega_p(2^{-A}, f)) + O(\omega_p(2^{-B}, f)). \tag{3}$$

If $\{q_{j,k}\}$ is nonincreasing, in sign $\downarrow$, and

$$\frac{(mn)^{\gamma-1}}{Q_{m,n}^{\gamma}} \sum_{j=0}^{m} \sum_{k=0}^{n} q_{j,k}^{\gamma} = O(1) \quad \text{for some } 1 < \gamma \leqslant 2, \tag{4}$$

then

$$\|T_{m,n}(f) - f\|_p \leqslant \frac{c}{Q_{m,n}} \sum_{k=0}^{A-1} \sum_{l=0}^{B-1} 2^{k+l} q_{m-2^{k+1}+1, n-2^{l+1}+1} (\omega_p(2^{-k}, f) + \omega_p(2^{-l}, f))$$

$$
\begin{aligned}
&+\frac{c}{Q_{m,n}}\sum_{l=0}^{B-1}(m-2^A)2^l q_{0,n-2^{l+1}+1}(\omega_p(2^{-A},f)+\omega_p(2^{-l},f))\\
&+\frac{c}{Q_{m,n}}\sum_{k=0}^{A-1}(n-2^B)2^k q_{m-2^{k+1}+1,0}(\omega_p(2^{-B},f)+\omega_p(2^{-k},f))\\
&+\frac{c}{Q_{m,n}}\sum_{i=0}^{n-2^B}(m-2^A)q_{0,n-2^B-i}\omega_p(2^{-A},f)+O(\omega_p(2^{-A},f))\\
&+\frac{c}{Q_{m,n}}\sum_{j=0}^{m-2^A}(n-2^B)q_{m-2^A-j,0}\omega_p(2^{-B},f)+O(\omega_p(2^{-B},f)). \qquad (5)
\end{aligned}
$$

We will discuss the following case:
The nondecreasing $\{q_{j,k}\}$, in sign $q_{j,k}\uparrow$, satiesfies the condition

$$\frac{(m+1)(n+1)q_{m,n}}{Q_{m,n}}=O(1) \qquad (6)$$

(the conditions of regularity are satiesfied). In particular, (6) is true if $q_{j,k}$ has a power growth both in j and in k. Namely,

$$q_{j,k}:=j^\beta k^\gamma \quad \text{for some } \beta,\gamma\geqslant 0.$$

The one-dimensional analogue of the following theorem was proved by Móricz and Siddiqi in [16] for Walsh system. We mention that as special case $T_{m,n}$ is the (C,β,γ)-mean with respect to the double Walsh-Fourier series and the approximation rate for (C,β,γ)-summability immediately follows from Theorem 1 (for $\beta,\gamma\geqslant 1$). We note that the approximation rate of (C,β,γ)-summability for trigonometric system was given in [12, Corollary 2].

THEOREM 2. *Let $f\in Lip(\alpha,p)$ for some $\alpha>0$ and $1\leqslant p\leqslant\infty$.*
Let $\{q_{j,k}\}$ be a double sequence of nonnegative numbers such that in case $q_{j,k}\uparrow$ the condition (6) *is satisfied, then*

$$\|T_{m,n}(f)-f\|_p=\begin{cases}O(m^{-\alpha}+n^{-\alpha}), & \text{if } 0<\alpha<1,\\ O(m^{-1}\log m+n^{-1}\log n), & \text{if } \alpha=1,\\ O(m^{-1}+n^{-1}), & \text{if } \alpha>1.\end{cases}$$

At last, we note that if $\{q_{j,k}\}$ is nonincreasing, in sign $q_{j,k}\downarrow$, then condition (4) is satiesfied if,

i.) $q_{j,k}:=j^{-\beta}k^{-\gamma}$ for some $0<\beta,\gamma<1$, or

ii.) $q_{j,k}:=(\log j)^{-\beta}(\log k)^{-\gamma}$ for some $0<\beta,\gamma$.

4. Auxiliary results

A Sidon type inequality proved by Móricz and Schipp [14, 15] implies that the Nörlund kernels L_n and $\mathscr{L}_{m,n}$ are quasi-positive. Namely, the following Lemma is true.

LEMMA 1. (Schipp and Móricz) *If the condition (4) is satisfied, then there exists a constant C such that*

$$\|\mathscr{L}_{m,n}\|_1 \leqslant C, \quad (n \geqslant 1).$$

We note that, the analogue of Lemma 1 can be stated for Nörlund kernel L_n [14] with the condition

$$\frac{n^{\delta-1}}{P_n^{\delta}} \sum_{k=0}^{n} p_k^{\delta} = O(1) \quad \text{for some } 1 < \delta \leqslant 2. \tag{7}$$

For Walsh-Fejér kernel Yano [19] proved that

LEMMA 2. (Yano) *Let $n \geqslant 1$, then*

$$\|K_n\|_1 \leqslant 2.$$

We denote by $\mathscr{P}_m$ the collection of Walsh polynomials of order less than m, that is, functions of the form

$$P(x) := \sum_{k=0}^{m-1} a_k w_k(x),$$

where $m \geqslant 1$ and $\{a_k\}$ is any sequence of real (or complex) numbers.

On G^2 we consider the two-dimensional Walsh polynomials of order less than (m,n) as $\mathscr{P}_{m,n} := \{P_1(x^1) \times P_2(x^2) : P_1 \in \mathscr{P}_m, P_2 \in \mathscr{P}_n\}$.

The one-dimensional analogue of the following Lemmas was given by Móricz and Siddiqi [16].

LEMMA 3. *Let $P \in \mathscr{P}_{2^A,2^B}, f \in L^p$, where $A,B \in \mathbf{P}$ and $1 \leqslant p \leqslant \infty$. Then*

$$\left\| \int_{G^2} (f(.+x) - f(.)) r_A(x^1) r_B(x^2) P(x) d\mu(x) \right\|_p \leqslant c\|P\|_1 \omega_{1,2}^p(2^{-A}, 2^{-B}, f),$$

with the notation $x = (x^1, x^2) \in G^2$.

Proof. We make the proof for $1 \leqslant p < \infty$, for $p = \infty$ the proof goes in a similar way (where $L^\infty = C_W$). We introduce the notation

$$\Delta_{A,B} f(x,y) := |f(x+y) - f(x+y+e_B^2) - f(x+y+e_A^1) + f(x+y+e_A^1+e_B^2)|,$$

where $e_A^1 := (e_A, 0)$ and $e_B^2 := (0, e_B)$.

We write the following for any $\varepsilon,\rho \in G, x,y \in G^2$ and $A,B \in \mathbf{P}$

$$\begin{aligned}
&\left|\int_{I_A(\varepsilon)\times I_B(\rho)}(f(y+x)-f(y))r_A(x^1)r_B(x^2)d\mu(x)\right| = \left|\int_{I_A(\varepsilon)\times I_B(\rho)} f(y+x)r_A(x^1)r_B(x^2)d\mu(x)\right| \\
&= \left|\int_{I_{A+1}(\varepsilon)\times I_{B+1}(\rho)} f(y+x)r_A(x^1)r_B(x^2)d\mu(x) + \int_{I_{A+1}(\varepsilon+e_A)\times I_{B+1}(\rho)} f(y+x)r_A(x^1)r_B(x^2)d\mu(x)\right. \\
&\quad \left. + \int_{I_{A+1}(\varepsilon)\times I_{B+1}(\rho+e_B)} f(y+x)r_A(x^1)r_B(x^2)d\mu(x) + \int_{I_{A+1}(\varepsilon+e_A)\times I_{B+1}(\rho+e_B)} f(y+x)r_A(x^1)r_B(x^2)d\mu(x)\right| \\
&= \left|\int_{I_{A+1}(\varepsilon)\times I_{B+1}(\rho)} f(y+x)-f(y+x+e_A^1)-f(y+x+e_B^2)+f(y+x+e_A^1+e_B^2)d\mu(x)\right| \\
&= \left|\int_{I_{A+1}(\varepsilon)\times I_{B+1}(\rho)} \Delta_{A,B}f(x,y)d\mu(x)\right|. \qquad (8)
\end{aligned}$$

We note that the polynomial $P(x^1,x^2)$ is constant on the sets $I_A(\varepsilon)\times I_B(\rho)$ $(\varepsilon,\rho \in G)$. This and the generalized Minkowski inequality give

$$\begin{aligned}
F^{A,B} &:= \left\|\int_{G^2}(f(.+x)-f(.))r_A(x^1)r_B(x^2)P(x^1,x^2)d\mu(x)\right\|_p \\
&= \left\|\sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\dots,A-1\}}}^{1}\sum_{\substack{\rho_j=0 \\ j\in\{0,\dots,B-1\}}}^{1}\int_{I_A(\varepsilon)\times I_B(\rho)}(f(.+x)-f(.))r_A(x^1)r_B(x^2)P(x^1,x^2)d\mu(x)\right\|_p \\
&\leqslant \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\dots,A-1\}}}^{1}\sum_{\substack{\rho_j=0 \\ j\in\{0,\dots,B-1\}}}^{1}|P(\varepsilon,\rho)|\left\|\int_{I_A(\varepsilon)\times I_B(\rho)}(f(.+x)-f(.))r_A(x^1)r_B(x^2)d\mu(x)\right\|_p \\
&\leqslant \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\dots,A-1\}}}^{1}\sum_{\substack{\rho_j=0 \\ j\in\{0,\dots,B-1\}}}^{1}|P(\varepsilon,\rho)|\times \\
&\qquad \times\left(\int_{G^2}\left|\int_{I_A(\varepsilon)\times I_B(\rho)}(f(y+x)-f(y))r_A(x^1)r_B(x^2)d\mu(x)\right|^p d\mu(y)\right)^{1/p}
\end{aligned}$$

The equation (8) gives

$$\begin{aligned}
F^{A,B} &\leqslant \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\dots,A-1\}}}^{1}\sum_{\substack{\rho_j=0 \\ j\in\{0,\dots,B-1\}}}^{1}|P(\varepsilon,\rho)|\left(\int_{G^2}\left(\int_{I_{A+1}(\varepsilon)\times I_{B+1}(\rho)}\Delta_{A,B}f(x,y)d\mu(x)\right)^p d\mu(y)\right)^{1/p} \\
&\leqslant \sum_{\substack{\varepsilon_i=0 \\ i\in\{0,\dots,A-1\}}}^{1}\sum_{\substack{\rho_j=0 \\ j\in\{0,\dots,B-1\}}}^{1}|P(\varepsilon,\rho)|\int_{I_{A+1}(\varepsilon)\times I_{B+1}(\rho)}\left(\int_{G^2}(\Delta_{A,B}f(x,y))^p d\mu(y)\right)^{1/p} d\mu(x)
\end{aligned}$$

$$\leqslant c \sum_{\substack{\varepsilon_i=0\\ i\in\{0,\dots,A-1\}}}^{1} \sum_{\substack{\rho_j=0\\ j\in\{0,\dots,B-1\}}}^{1} \int_{I_{A+1}(\varepsilon)\times I_{B+1}(\rho)} |P(\varepsilon,\rho)|\, d\mu(x)\ \ \omega_{1,2}^{p}(2^{-A},2^{-B},f)$$
$$\leqslant c\|P\|_1 \omega_{1,2}^{p}(2^{-A},2^{-B},f).$$

This completes the proof of Lemma 3.

LEMMA 4. *Let* $P\in\mathscr{P}_{2^A}$, $f\in L^p$ $(1\leqslant p\leqslant\infty)$ *and* $A,B\in\mathbf{P}$. *Then*

$$\left\|\int_{G^2}(f(.+x)-f(.))D_{2^B}(x^2)r_A(x^1)P(x^1)\,d\mu(x)\right\|_p \leqslant c\|P\|_1\omega_p(2^{-A},f).$$

The proof goes analogously as we did above. Thus, it is left to the readers.

For two-dimensional variable $(x^1,x^2)\in G^2$ we use the notations

$$w_n^1(x^1,x^2):=w_n(x^1),\quad D_n^1(x^1,x^2):=D_n(x^1),\quad K_n^1(x^1,x^2):=K_n(x^1),$$
$$w_n^2(x^1,x^2):=w_n(x^2),\quad D_n^2(x^1,x^2):=D_n(x^2),\quad K_n^2(x^1,x^2):=K_n(x^2),$$

for any $n\in\mathbf{N}$.

Now, we give a decomposition Lemma of the kernel $\mathscr{L}_{m,n}$.

LEMMA 5. *Let* $|m|:=A$ *and* $|n|:=B$, *then*

$$\begin{aligned}
Q_{m,n}\mathscr{L}_{m,n} &= \sum_{k=0}^{A-1}\sum_{l=0}^{B-1} w^1_{2^{k+1}-1}w^2_{2^{l+1}-1}\sum_{s=1}^{2^k-1}\sum_{t=1}^{2^l-1}\Delta_{1,1}q_{m-2^{k+1}+s,n-2^{l+1}+t}\,stK_s^1K_t^2\\
&+\sum_{k=0}^{A-1}\sum_{l=0}^{B-1} w^1_{2^{k+1}-1}w^2_{2^{l+1}-1}\sum_{t=1}^{2^l-1}\Delta_{0,1}q_{m-2^k,n-2^{l+1}+t}\,t2^kK_{2^k}^1K_t^2\\
&+\sum_{k=0}^{A-1}\sum_{l=0}^{B-1} w^1_{2^{k+1}-1}w^2_{2^{l+1}-1}\sum_{s=1}^{2^k-1}\Delta_{1,0}q_{m-2^{k+1}+s,n-2^l}\,s2^lK_s^1K_{2^l}^2\\
&+\sum_{k=0}^{A-1}\sum_{l=0}^{B-1} w^1_{2^{k+1}-1}w^2_{2^{l+1}-1}2^k2^l q_{m-2^k,n-2^l}K_{2^k}^1K_{2^l}^2\\
&-\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1}\sum_{s=1}^{2^k-1}\Delta_{1,0}q_{m-2^{k+1}+s,n-2^l-j}\,w^1_{2^{k+1}-1}sK_s^1D^2_{2^{l+1}}\\
&-\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1} q_{m-2^k,n-2^l-j}\,w^1_{2^{k+1}-1}2^kK_{2^k}^1D^2_{2^{l+1}}\\
&-\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{t=1}^{2^l-1}\Delta_{0,1}q_{m-i-2^k,n-2^{l+1}+t}\,w^2_{2^{l+1}-1}tK_t^2D^1_{2^{k+1}}\\
&-\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1} q_{m-i-2^k,n-2^l}\,w^2_{2^{l+1}-1}2^lK_{2^l}^2D^1_{2^{k+1}}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} D^1_{2^{k+1}} D^2_{2^{l+1}} \\
&+\sum_{k=1}^{2^A-1}\sum_{l=2^B}^{n} q_{m-k,n-l} D^1_k D^2_l + \sum_{k=2^A}^{m}\sum_{l=1}^{2^B-1} q_{m-k,n-l} D^1_k D^2_l \\
&+\sum_{j=0}^{m-2^A}\sum_{i=0}^{n-2^B} q_{m-2^A-j,n-2^B-i} D^1_{2^A} D^2_{2^B} + \sum_{j=0}^{m-2^A} Q^{m-2^A-j}_{n-2^B} D^1_{2^A} r^2_B L^{m-2^A-j,2}_{n-2^B} \\
&+\sum_{i=0}^{n-2^B} Q^{n-2^B-i}_{m-2^A} D^2_{2^B} r^1_A L^{n-2^B-i,1}_{m-2^A} + Q_{m-2^A,n-2^B} r^1_A r^2_B \mathscr{L}_{m-2^A,n-2^B}.
\end{aligned}
$$

with the notation $Q^{m-2^A-j}_{n-2^B} L^{m-2^A-j}_{n-2^B} := \sum_{i=0}^{n-2^B} q_{m-2^A-j,n-2^B-i} D_i$ *and* $Q^{n-2^B-i}_{m-2^A} L^{n-2^B-i}_{m-2^A} := \sum_{j=0}^{m-2^A} q_{m-2^A-j,n-2^B-i} D_j$.

Proof. Set $|m| := A$ and $|n| := B$. During the proof of Lemma 5 we will use the following equations [16, 17]:

$$D_{2^A+j} = D_{2^A} + r_A D_j, \quad j = 0,1,...,2^A-1. \tag{9}$$

$$D_{2^j+i} - D_{2^{j+1}} = -w_{2^{j+1}-1} D_{2^j-i} \quad (0 \leqslant i < 2^j). \tag{10}$$

Now, we decompose the kernel $\mathscr{L}_{m,n}$.

$$
\begin{aligned}
Q_{m,n}\mathscr{L}_{m,n} &= \sum_{k=1}^{2^A-1}\sum_{l=1}^{2^B-1} q_{m-k,n-l} D^1_k D^2_l + \sum_{k=1}^{2^A-1}\sum_{l=2^B}^{n} q_{m-k,n-l} D^1_k D^2_l \\
&\quad + \sum_{k=2^A}^{m}\sum_{l=1}^{2^B-1} q_{m-k,n-l} D^1_k D^2_l + \sum_{k=2^A}^{m}\sum_{l=2^B}^{n} q_{m-k,n-l} D^1_k D^2_l \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

First, we decompose I_1. By the equation (10) we have that

$$
\begin{aligned}
I_1 &= \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} (D^1_{2^k+i} - D^1_{2^{k+1}})(D^2_{2^l+j} - D^2_{2^{l+1}}) \\
&+\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} (D^1_{2^k+i} - D^1_{2^{k+1}}) D^2_{2^{l+1}} \\
&+\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} D^1_{2^{k+1}} (D^2_{2^l+j} - D^2_{2^{l+1}}) \\
&+\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} D^1_{2^{k+1}} D^2_{2^{l+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} w^1_{2^{k+1}-1} w^2_{2^{l+1}-1} D^1_{2^k-i} D^2_{2^l-j}\\
&- \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} w^1_{2^{k+1}-1} D^1_{2^k-i} D^2_{2^{l+1}}\\
&- \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} D^1_{2^{k+1}} w^2_{2^{l+1}-1} D^2_{2^l-j}\\
&+ \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1} q_{m-2^k-i,n-2^l-j} D^1_{2^{k+1}} D^2_{2^{l+1}}\\
&=: I_1^1 - I_1^2 - I_1^3 + I_1^4.
\end{aligned}
$$

To decompose I_1^1, we set $s := 2^k - i$, $t := 2^l - j$ and use double Abel's transformation

$$
\begin{aligned}
I_1^1 &= \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{s=1}^{2^k}\sum_{t=1}^{2^l} q_{m-2^{k+1}+s,n-2^{l+1}+t} w^1_{2^{k+1}-1} w^2_{2^{l+1}-1} D^1_s D^2_t\\
&= \sum_{k=0}^{A-1}\sum_{l=0}^{B-1} w^1_{2^{k+1}-1} w^2_{2^{l+1}-1} \Bigg(\sum_{s=1}^{2^k-1}\sum_{t=1}^{2^l-1} \Delta_{1,1} q_{m-2^{k+1}+s,n-2^{l+1}+t} st K^1_s K^2_t\\
&\qquad + \sum_{t=1}^{2^l-1} \Delta_{0,1} q_{m-2^k,n-2^{l+1}+t} t 2^k K^1_{2^k} K^2_t\\
&\qquad + \sum_{s=1}^{2^k-1} \Delta_{1,0} q_{m-2^{k+1}+s,n-2^l} s 2^l K^1_s K^2_{2^l} + 2^k 2^l q_{m-2^k,n-2^l} K^1_{2^k} K^2_{2^l} \Bigg)\\
&=: I_1^{1,1} + I_1^{1,2} + I_1^{1,3} + I_1^{1,4}.
\end{aligned}
$$

Now, we turn our attention to I_1^2 (and I_1^3 goes analogously). Set $s := 2^k - i$,

$$
I_1^2 = \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{s=1}^{2^k}\sum_{j=0}^{2^l-1} q_{m-2^{k+1}+s,n-2^l-j} w^1_{2^{k+1}-1} D^1_s D^2_{2^{l+1}}.
$$

By Abel's transformation we get that

$$
\begin{aligned}
I_1^2 &= \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1}\sum_{s=1}^{2^k-1} \Delta_{1,0} q_{m-2^{k+1}+s,n-2^l-j} w^1_{2^{k+1}-1} s K^1_s D^2_{2^{l+1}}\\
&\quad + \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1} q_{m-2^k,n-2^l-j} w^1_{2^{k+1}-1} 2^k K^1_{2^k} D^2_{2^{l+1}} =: I_1^{2,1} + I_1^{2,2}.
\end{aligned}
$$

Now, we decompose I_3 (I_2 goes analogously). But, for the simplicity we do not write our result into the statement of Lemma 5. By the help of (9) and (10) we write

$$
I_3 = \sum_{i=0}^{m-2^A}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1} q_{m-2^A-i,n-2^l-j} D^1_{2^A+i} (D^2_{2^l+j} - D^2_{2^{l+1}})
$$

$$
\begin{aligned}
&+\sum_{i=0}^{m-2^A}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1} q_{m-2^A-i,n-2^l-j}D^1_{2^A+i}D^2_{2^{l+1}} \\
&=\sum_{i=0}^{m-2^A}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1} q_{m-2^A-i,n-2^l-j}\times \\
&\qquad \times\left(-D^1_{2^A}w^2_{2^{l+1}-1}D^2_{2^l-j}-r^1_AD^1_i w^2_{2^{l+1}-1}D^2_{2^l-j}+D^1_{2^A}D^2_{2^{l+1}}+r^1_AD^1_iD^2_{2^{l+1}}\right) \\
&=: -I^1_3-I^2_3+I^3_3+I^4_3.
\end{aligned}
$$

To discuss I^1_3 we set $s=2^l-j$ and use Abel's transformation.

$$
\begin{aligned}
I^1_3&=\sum_{i=0}^{m-2^A}\sum_{l=0}^{B-1}D^1_{2^A}w^2_{2^{l+1}-1}\left(\sum_{s=1}^{2^l-1}\Delta_{0,1}q_{m-2^A-i,n-2^{l+1}+s}sK^2_s+q_{m-2^A-i,n-2^l}2^lK^2_{2^l}\right) \\
&=: I^{1,1}_3+I^{1,2}_3. \qquad (11)
\end{aligned}
$$

To decompose I^2_3 we set $s=2^l-j$ and use double Abel's transformation.

$$
\begin{aligned}
I^2_3&=\sum_{l=0}^{B-1}r^1_Aw^2_{2^{l+1}-1}\Bigg(-\sum_{s=1}^{2^l-1}\sum_{i=1}^{m-2^A-1}\Delta_{1,1}q_{m-2^A-i-1,n-2^{l+1}+s}isK^1_iK^2_s \\
&\qquad +\sum_{s=1}^{2^l-1}\Delta_{0,1}q_{0,n-2^{l+1}+s}(m-2^A)sK^1_{m-2^A}K^2_s \\
&\qquad -\sum_{i=1}^{m-2^A-1}\Delta_{1,0}q_{m-2^A-i-1,n-2^l}i2^lK^1_iK^2_{2^l}+q_{0,n-2^l}(m-2^A)2^lK^1_{m-2^A}K^2_{2^l}\Bigg) \\
&=: -I^{2,1}_3+I^{2,2}_3-I^{2,3}_3+I^{2,4}_3. \qquad (12)
\end{aligned}
$$

By the help of Abel's transformation we write for I^4_3 that

$$
I^4_3=\sum_{l=0}^{B-1}r^1_AD^2_{2^{l+1}}\sum_{j=0}^{2^l-1}\left(-\sum_{i=1}^{m-2^A-1}\Delta_{1,0}q_{m-2^A-i-1,n-2^l-j}iK^1_i+q_{0,n-2^l-j}(m-2^A)K^1_{m-2^A}\right). \qquad (13)
$$

At last, we discuss I_4 by the help of (9).

$$
\begin{aligned}
I_4&=\sum_{j=0}^{m-2^A}\sum_{i=0}^{n-2^B}q_{m-2^A-j,n-2^B-i}D^1_{2^A+j}D^2_{2^B+i} \\
&=\sum_{j=0}^{m-2^A}\sum_{i=0}^{n-2^B}q_{m-2^A-j,n-2^B-i}D^1_{2^A}D^2_{2^B}+\sum_{j=0}^{m-2^A}\sum_{i=0}^{n-2^B}q_{m-2^A-j,n-2^B-i}D^1_{2^A}r^2_BD^2_i \\
&+\sum_{j=0}^{m-2^A}\sum_{i=0}^{n-2^B}q_{m-2^A-j,n-2^B-i}r^1_AD^1_jD^2_{2^B}+\sum_{j=0}^{m-2^A}\sum_{i=0}^{n-2^B}q_{m-2^A-j,n-2^B-i}r^1_Ar^2_BD^1_jD^2_i \\
&=: I^1_4+I^2_4+I^3_4+I^4_4.
\end{aligned}
$$

To discuss I_4^3, we fix $n-2^B-i$, and we introduce the notation $Q_{m-2^A}^{n-2^B-i}L_{m-2^A}^{n-2^B-i} := \sum_{j=0}^{m-2^A} q_{m-2^A-j,n-2^B-i}D_j$. This gives

$$I_4^3 = \sum_{i=0}^{n-2^B} Q_{m-2^A}^{n-2^B-i} D_{2^B}^2 r_A^1 L_{m-2^A}^{n-2^B-i,1} \tag{14}$$

(I_4^2 goes analogously). Now, we decompose I_4^3 (and I_4^4). But, for the simplicity we do not write our result into the statement of Lemma 5 (for nonincreasing double sequence and nondecreasing double sequence we need different decompositions of the expressions I_4^3, I_4^4). Abel's transformation yields

$$\begin{aligned} I_4^3 = &- \sum_{i=0}^{n-2^B} D_{2^B}^2 r_A^1 \sum_{j=1}^{m-2^A-1} \Delta_{1,0} q_{m-2^A-j-1,n-2^B-i} j K_j^2 \\ &+ \sum_{i=0}^{n-2^B} D_{2^B}^2 r_A^1 q_{0,n-2^B-i}(m-2^A)K_{m-2^A}^2 =: I_4^{3,1} + I_4^{3,2}. \end{aligned} \tag{15}$$

Double Abel's transformation gives

$$\begin{aligned} I_4^4 = & r_A^1 r_B^2 \sum_{j=1}^{m-2^A-1} \sum_{i=1}^{n-2^B-1} \Delta_{1,1} q_{m-2^A-j-1,n-2^B-i-1} i j K_j^1 K_i^2 \\ & - r_A^1 r_B^2 \sum_{i=1}^{n-2^B-1} \Delta_{0,1} q_{0,n-2^B-i-1} i K_i^2 (m-2^A) K_{m-2^A}^1 \\ & - r_A^1 r_B^2 \sum_{j=1}^{m-2^A-1} \Delta_{1,0} q_{m-2^A-j-1,0} j K_j^1 (n-2^B) K_{n-2^B}^2 \\ & + r_A^1 r_B^2 q_{0,0} (m-2^A)(n-2^B) K_{m-2^A}^1 K_{n-2^B}^2 =: I_4^{4,1} + I_4^{4,2} + I_4^{4,3} + I_4^{4,4}. \end{aligned} \tag{16}$$

This completes the proof of Lemma 5.

5. The proof of the main theorems

Proof of Theorem 1: During the proof of Theorem 1 we use the notations of Lemma 5.

Let $m,n \in \mathbf{P}$ be fixed and set $|m| := A, |n| := B$. We use the notations $x = (x^1, x^2) \in G^2$ and $y = (y^1, y^2) \in G^2$. By Minkowski inequality we may write that

$$\begin{aligned} Q_{m,n}\|T_{m,n}(f) - f\|_p &\leqslant \left\| \int_{G^2} (f(.+x) - f(.)) Q_{m,n} \mathscr{L}_{m,n}(x) d\mu(x) \right\|_p \\ &\leqslant \sum_{j=1}^{4} \left\| \int_{G^2} (f(.+x) - f(.)) I_j(x) d\mu(x) \right\|_p, \end{aligned}$$

and so on.

We will often use the following

$$\omega_{1,2}^{p}(\delta_1,\delta_2,f)\leqslant \omega_p(\delta_1,f)+\omega_p(\delta_2,f).$$

First, we discuss $\left\|\int_{G^2}(f(.+x)-f(.))I_4^i(x)d\mu(x)\right\|_p$ for $i=1,2,3,4$.
By (2) we find that

$$\begin{aligned}&\left\|\int_{G^2}(f(.+x)-f(.))D_{2^k}(x^1)D_{2^l}(x^2)d\mu(x)\right\|_p\leqslant\\&\leqslant\int_{I_k\times I_l}D_{2^k}(x^1)D_{2^l}(x^2)\left(\int_{G^2}|f(y+x)-f(y)|^p d\mu(y)\right)^{1/p}d\mu(x)\\&\leqslant c\omega_p(2^{-k},f)+c\omega_p(2^{-l},f).\end{aligned}\tag{17}$$

Thus, we immediately have

$$\begin{aligned}&\left\|\int_{G^2}(f(.+x)-f(.))I_4^1(x)d\mu(x)\right\|_p\leqslant\\&\leqslant c\sum_{j=0}^{m-2^A}\sum_{i=0}^{n-2^B}q_{m-2^A-j,n-2^B-i}(\omega_p(2^{-A},f)+\omega_p(2^{-B},f)).\end{aligned}$$

Next, we discuss $\left\|\int_{G^2}(f(.+x)+f(.))I_4^i(x)d\mu(x)\right\|_p$ for $i=2,3$. The equation 15, Lemma 4 and Lemma 2 yield

$$\begin{aligned}&\left\|\int_{G^2}(f(.+x)-f(.))I_4^3(x)d\mu(x)\right\|_p\leqslant\\&\leqslant c\sum_{i=0}^{n-2^B}\left(\sum_{j=0}^{m-2^A-1}|\Delta_{1,0}q_{m-2^A-j-1,n-2^B-i}|j+q_{0,n-2^B-i}(m-2^A)\right)\omega_p(2^{-A},f).\end{aligned}$$

For double sequence $q_{j,k}\uparrow$

$$\sum_{j=0}^{m-2^A-1}|\Delta_{1,0}q_{m-2^A-j-1,n-2^B-i}|j=\sum_{j=1}^{m-2^A-1}q_{m-2^A-j,n-2^B-i}-(m-2^A-1)q_{0,n-2^B-i}$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_4^3(x)d\mu(x)\right\|_p\leqslant c\sum_{i=0}^{n-2^B}\sum_{j=0}^{m-2^A}q_{m-2^A-j,n-2^B-i}\omega_p(2^{-A},f).$$

For double sequence $q_{j,k}\downarrow$

$$\sum_{j=0}^{m-2^A-1}|\Delta_{1,0}q_{m-2^A-j-1,n-2^B-i}|j\leqslant(m-2^A)q_{0,n-2^B-i}$$

and

$$\left\| \int_{G^2} (f(.+x) - f(.)) I_4^3(x) d\mu(x) \right\|_p \leqslant c \sum_{i=0}^{n-2^B} (m-2^A) q_{0,n-2^B-i} \omega_p(2^{-A}, f).$$

Now, we study $\left\| \int_{G^2} (f(.+x) - f(.)) I_4^4(x) d\mu(x) \right\|_p$.

For double sequence $q_{j,k} \downarrow$, Lemma 3, Lemma 1 and condition (4) immediately give that

$$\begin{aligned} \left\| \int_{G^2} (f(.+x) - f(.)) I_4^4(x) d\mu(x) \right\|_p &\leqslant c Q_{m-2^A, n-2^B} \| \mathscr{L}_{m-2^A, n-2^B} \|_1 \omega_{1,2}^p (2^{-A}, 2^{-B}, f) \\ &\leqslant c Q_{m,n} (\omega_p(2^{-A}, f) + \omega_p(2^{-B}, f)). \end{aligned}$$

For double sequence $q_{j,k} \uparrow$, we use the equation (16).

Lemma 3 and Lemma 2 yield that

$$\begin{aligned} &\left\| \int_{G^2} (f(.+x) - f(.)) I_4^{4,1}(x) d\mu(x) \right\|_p \leqslant \\ &\leqslant c \sum_{i=1}^{n-2^B-1} \sum_{j=1}^{m-2^A-1} |\Delta_{1,1} q_{m-2^A-j-1, n-2^B-i-1}| ij \omega_{1,2}^p (2^{-A}, 2^{-B}, f). \end{aligned}$$

Since $\Delta_{1,1} q_{j,k}$ is of fixed sign

$$\begin{aligned} &\sum_{i=1}^{n-2^B-1} \sum_{j=1}^{m-2^A-1} |\Delta_{1,1} q_{m-2^A-j-1, n-2^B-i-1}| ij \leqslant \\ &\leqslant (m-2^A)(n-2^B) \left| \sum_{i=1}^{n-2^B-1} \sum_{j=1}^{m-2^A-1} \Delta_{1,1} q_{m-2^A-j-1, n-2^B-i-1} \right| \\ &\leqslant (m-2^A)(n-2^B) |q_{m-2^A-1, n-2^B-1} - q_{0,n-2^B-1} - q_{m-2^A-1,0} + q_{0,0}| \\ &\leqslant c 2^A 2^B q_{m-2^A-1, n-2^B-1} \leqslant c Q_{m,n}. \end{aligned}$$

Lemma 3 and Lemma 2 imply again

$$\begin{aligned} &\left\| \int_{G^2} (f(.+x) - f(.)) I_4^{4,3}(x) d\mu(x) \right\|_p \leqslant \\ &\leqslant c \sum_{j=1}^{m-2^A-1} |\Delta_{1,0} q_{m-2^A-j-1,0}| j (n-2^B) \omega_{1,2}^p (2^{-A}, 2^{-B}, f). \end{aligned}$$

From

$$\sum_{j=1}^{m-2^A-1} |\Delta_{1,0} q_{m-2^A-j-1,0}| j \leqslant \sum_{j=1}^{m-2^A-1} q_{m-2^A-j,0}$$

we get that

$$\left\|\int_{G^2}(f(.+x)-f(.))I_4^{4,3}(x)d\mu(x)\right\|_p \leqslant cQ_{m,n}\omega_{1,2}^p(2^{-A},2^{-B},f)$$

(the discussion of $\left\|\int_{G^2}(f(.+x)-f(.))I_4^{4,2}(x)d\mu(x)\right\|_p$ goes analogously).

By

$$q_{0,0}(m-2^A)(n-2^B) \leqslant Q_{m,n}$$

and Lemma 3 we have that

$$\left\|\int_{G^2}(f(.+x)-f(.))I_4^{4,4}(x)d\mu(x)\right\|_p \leqslant cQ_{m,n}\omega_{1,2}^p(2^{-A},2^{-B},f).$$

Second, we discuss $\left\|\int_{G^2}(f(.+x)-f(.))I_1^i(x)d\mu(x)\right\|_p$ for $i=1,2,3,4$.

Now, we study $\left\|\int_{G^2}(f(.+x)-f(.))I_1^{1,1}(x)d\mu(x)\right\|_p$. By Lemma 3 and Lemma 2 we get

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{1,1}(x)d\mu(x)\right\|_p \leqslant$$
$$\leqslant \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{s=1}^{2^k-1}\sum_{t=1}^{2^l-1}|\Delta_{1,1}q_{m-2^{k+1}+s,n-2^{l+1}+t}|st\,\omega_{1,2}^p(2^{-k},2^{-l},f).$$

Using that $\Delta_{1,1}q_{j,k}$ is of fixed sign

$$\sum_{s=1}^{2^k-1}\sum_{t=1}^{2^l-1}|\Delta_{1,1}q_{m-2^{k+1}+s,n-2^{l+1}+t}|st \leqslant 2^{k+l}\left|\sum_{s=1}^{2^k-1}\sum_{t=1}^{2^l-1}\Delta_{1,1}q_{m-2^{k+1}+s,n-2^{l+1}+t}\right|$$
$$=2^{k+l}|q_{m-2^k,n-2^l}-q_{m-2^k,n-2^{l+1}+1}-q_{m-2^{k+1}+1,n-2^l}+q_{m-2^{k+1}+1,n-2^{l+1}+1}|$$

If $q_{j,k}\uparrow$, then

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{1,1}(x)d\mu(x)\right\|_p \leqslant c\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}2^{k+l}q_{m-2^k,n-2^l}\,\omega_{1,2}^p(2^{-k},2^{-l},f).$$

If $q_{j,k}\downarrow$, then

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{1,1}(x)d\mu(x)\right\|_p \leqslant c\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}2^{k+l}q_{m-2^{k+1}+1,n-2^{l+1}+1}\,\omega_{1,2}^p(2^{-k},2^{-l},f).$$

We discuss $\left\|\int_{G^2}(f(.+x)-f(.))I_1^{1,i}(x)d\mu(x)\right\|_p$ for $i=3$. (For $i=2$ the proof goes analogously.) Lemma 3 and Lemma 2 give

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{1,3}(x)d\mu(x)\right\|_p \leqslant$$

$$\leqslant c\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}2^l\sum_{s=1}^{2^k-1}|\Delta_{1,0}q_{m-2^{k+1}+s,n-2^l}|s\omega_{1,2}^p(2^{-k},2^{-l},f).$$

For $q_{j,k}\uparrow$ we have

$$\sum_{s=1}^{2^k-1}|\Delta_{1,0}q_{m-2^{k+1}+s,n-2^l}|s=(2^k-1)q_{m-2^k,n-2^l}-\sum_{i=1}^{2^k-1}q_{m-2^{k+1}+i,n-2^l}\leqslant 2^kq_{m-2^k,n-2^l}.$$

For $q_{j,k}\downarrow$ we get

$$\sum_{s=1}^{2^k-1}|\Delta_{1,0}q_{m-2^{k+1}+s,n-2^l}|s\leqslant\sum_{i=1}^{2^k-1}q_{m-2^{k+1}+i,n-2^l}\leqslant 2^kq_{m-2^{k+1}+1,n-2^{l+1}+1}.$$

Lemma 3 and Lemma 2 imply

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{1,4}(x)d\mu(x)\right\|_p\leqslant c\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}2^{k+l}q_{m-2^k,n-2^l}\omega_{1,2}^p(2^{-k},2^{-l},f).$$

Now, we discuss I_1^2 (and I_1^3 goes analogously). Lemma 4 and Lemma 2 immediately yield

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{2,2}(x)d\mu(x)\right\|_p\leqslant\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1}q_{m-2^k,n-2^l-j}2^k\omega_p(2^{-k},f).$$

If $q_{j,k}\uparrow$, then

$$\sum_{j=0}^{2^l-1}q_{m-2^k,n-2^l-j}\leqslant 2^lq_{m-2^k,n-2^l},$$

while for $q_{j,k}\downarrow$, we have

$$\sum_{j=0}^{2^l-1}q_{m-2^k,n-2^l-j}\leqslant 2^lq_{m-2^{k+1}+1,n-2^{l+1}+1}.$$

By Lemma 4 and Lemma 2 we have again

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{2,1}(x)d\mu(x)\right\|_p\leqslant$$
$$\leqslant\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1}\sum_{s=1}^{2^k-1}|\Delta_{1,0}q_{m-2^{k+1}+s,n-2^l-j}|s\omega_p(2^{-k},f).$$

If $q_{j,k}\uparrow$, then

$$\sum_{s=1}^{2^k-1}|\Delta_{1,0}q_{m-2^{k+1}+s,n-2^l-j}|s=(2^k-1)q_{m-2^k,n-2^l-j}-\sum_{s=1}^{2^k-1}q_{m-2^{k+1}+s,n-2^l-j}$$

$$\leqslant (2^k-1)q_{m-2^k,n-2^l-j}$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{2,1}(x)d\mu(x)\right\|_p \leqslant \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}2^{k+l}q_{m-2^k,n-2^l}\omega_p(2^{-k},f).$$

If $q_{j,k}\downarrow$, then

$$\sum_{s=1}^{2^k-1}|\Delta_{1,0}q_{m-2^{k+1}+s,n-2^l-j}|s \leqslant \sum_{s=1}^{2^k-1}q_{m-2^{k+1}+s,n-2^l-j} \leqslant 2^k q_{m-2^{k+1}+1,n-2^l-j}$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{2,1}(x)d\mu(x)\right\|_p \leqslant \sum_{k=0}^{A-1}\sum_{l=0}^{B-1}2^{k+l}q_{m-2^{k+1}+1,n-2^{l+1}+1}\omega_p(2^{-k},f).$$

By the inequality (17)

$$\left\|\int_{G^2}(f(.+x)-f(.))I_1^{4}(x)d\mu(x)\right\|_p \leqslant c\sum_{k=0}^{A-1}\sum_{l=0}^{B-1}Q_{m,n}^{k,l}(\omega_p(2^k,f)+\omega_p(2^l,f)),$$

where $Q_{m,n}^{k,l}:=\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^l-1}q_{m-2^k-i,n-2^l-j}$.

If $q_{m,n}\uparrow$, then

$$Q_{m,n}^{k,l}\leqslant 2^k2^l q_{m-2^k,n-2^l}.$$

If $q_{m,n}\downarrow$, then

$$Q_{m,n}^{k,l}\leqslant 2^k2^l q_{m-2^{k+1}+1,n-2^{l+1}+1}.$$

Third, we discuss $\|\int_{G^2}(f(.+x)-f(.))I_i(x)d\mu(x)\|_p$ for $i=3$. For $i=2$ the discussion goes analogously, so it is left to the readers.

The equation (11), Lemma 4 and Lemma 2 yield

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{1,1}(x)d\mu(x)\right\|_p \leqslant c\sum_{i=0}^{m-2^A}\sum_{l=0}^{B-1}\sum_{s=1}^{2^l-1}|\Delta_{0,1}q_{m-2^A-i,n-2^{l+1}+s}|s\omega_p(2^{-l},f).$$

For sequence $q_{j,k}\uparrow$ we get

$$\sum_{s=1}^{2^l-1}|\Delta_{0,1}q_{m-2^A-i,n-2^{l+1}+s}|s=(2^l-1)q_{m-2^A-i,n-2^l}-\sum_{s=1}^{2^l-1}q_{m-2^A-i,n-2^{l+1}+s}$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{1,1}(x)d\mu(x)\right\|_p \leqslant c\sum_{l=0}^{B-1}2^{A-1}2^l q_{m-2^{A-1},n-2^l}\omega_p(2^{-l},f).$$

Analogously,

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{1,2}(x)d\mu(x)\right\|_p \leqslant c\sum_{l=0}^{B-1}2^{A-1}2^l q_{m-2^{A-1},n-2^l}\omega_p(2^{-l},f).$$

For sequence $q_{j,k}\downarrow$ we get

$$\sum_{s=1}^{2^l-1}|\Delta_{0,1}q_{m-2^A-i,n-2^{l+1}+s}|s \leqslant \sum_{s=1}^{2^l-1}q_{m-2^A-i,n-2^{l+1}+s}$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{1,1}(x)d\mu(x)\right\|_p \leqslant c\sum_{l=0}^{B-1}(m-2^A)2^l q_{0,n-2^{l+1}+1}\omega_p(2^{-l},f).$$

Analogously,

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{1,2}(x)d\mu(x)\right\|_p \leqslant c\sum_{l=0}^{B-1}(m-2^A)2^l q_{0,n-2^{l+1}+1}\omega_p(2^{-l},f).$$

Now, we study $\left\|\int_{G^2}(f(.+x)-f(.))I_3^2(x)d\mu(x)\right\|_p$.

Lemma 3 implies

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,1}(x)d\mu(x)\right\|_p \leqslant$$
$$\leqslant c\sum_{l=0}^{B-1}\sum_{s=1}^{2^l-1}\sum_{i=1}^{m-2^A-1}|\Delta_{1,1}q_{m-2^A-i-1,n-2^{l+1}+s}|is\,\omega_{1,2}^p(2^{-A},2^{-l},f).$$

Since, $\Delta_{1,1}q_{j,k}$ is of fixed sign we have

$$\sum_{s=1}^{2^l-1}\sum_{i=1}^{m-2^A-1}|\Delta_{1,1}q_{m-2^A-i-1,n-2^{l+1}+s}|is \leqslant$$
$$\leqslant 2^l(m-2^A)|q_{m-2^A-1,n-2^{l+1}+1}-q_{m-2^A-1,n-2^l}-q_{0,n-2^{l+1}+1}+q_{0,n-2^l}|.$$

If $q_{j,k}\uparrow$, then

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,1}(x)d\mu(x)\right\|_p \leqslant$$
$$\leqslant c\sum_{l=0}^{B-1}2^l 2^{A-1}q_{m-2^{A-1},n-2^l}(\omega_p(2^{-A+1},f)+\omega_p(2^{-l},f)).$$

If $q_{j,k}\downarrow$, then

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,1}(x)d\mu(x)\right\|_p \leqslant c\sum_{l=0}^{B-1}2^l(m-2^A)q_{0,n-2^{l+1}+1}\omega_{1,2}^p(2^{-A},2^{-l},f).$$

Lemma 3 and Lemma 2 yield

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,3}(x)d\mu(x)\right\|_p \leqslant$$
$$\leqslant c\sum_{l=0}^{B-1}\sum_{i=1}^{m-2^A-1}|\Delta_{1,0}q_{m-2^A-i-1,n-2^l}|i2^l\omega_{1,2}^p(2^{-A},2^{-l},f).$$

For double sequence $q_{j,k}\uparrow$, we get

$$\sum_{i=1}^{m-2^A-1}|\Delta_{1,0}q_{m-2^A-i-1,n-2^l}|i=\sum_{i=1}^{m-2^A-1}q_{m-2^A-i,n-2^l}-(m-2^A-1)q_{0,n-2^l}$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,3}(x)d\mu(x)\right\|_p \leqslant c\sum_{l=0}^{B-1}q_{m-2^{A-1},n-2^l}2^{A-1}2^l\omega_{1,2}^p(2^{-A},2^{-l},f)$$
$$\leqslant c\sum_{l=0}^{B-1}q_{m-2^{A-1},n-2^l}2^{A-1}2^l(\omega_p(2^{-A+1},f)+\omega_p(2^{-l},f)).$$

For double sequence $q_{j,k}\downarrow$, we get

$$\sum_{i=1}^{m-2^A-1}|\Delta_{1,0}q_{m-2^A-i-1,n-2^l}|i\leqslant(m-2^A)q_{0,n-2^l}$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,3}(x)d\mu(x)\right\|_p \leqslant c\sum_{l=0}^{B-1}q_{0,n-2^{l+1}+1}(m-2^A)2^l\omega_{1,2}^p(2^{-A},2^{-l},f).$$

Lemma 3 and Lemma 2 immediately imply again

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,2}(x)d\mu(x)\right\|_p \leqslant$$
$$\leqslant c\sum_{l=0}^{B-1}\sum_{s=1}^{2^l-1}|\Delta_{0,1}q_{0,n-2^{l+1}+s}|s(m-2^A)\omega_{1,2}^p(2^{-A},2^{-l},f).$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,4}(x)d\mu(x)\right\|_p \leqslant c\sum_{l=0}^{B-1}q_{0,n-2^l}(m-2^A)2^l\omega_{1,2}^p(2^{-A},2^{-l},f).$$

For double sequences $q_{j,k}\uparrow$ and $q_{j,k}\downarrow$ we get the same estimations as for the expression $\left\|\int_{G^2}(f(.+x)-f(.))I_3^{2,3}(x)d\mu(x)\right\|_p$ we have.

By (17) and $D_{2^{l+1}} \leqslant 2D_{2^l}$ we have for $\left\|\int_{G^2}(f(.+x)-f(.))I_3^3(x)d\mu(x)\right\|_p$ that

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^3(x)d\mu(x)\right\|_p \leqslant cQ_{m,n}\omega_p(2^{-A},f) + c\sum_{i=0}^{m-2^A}\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1} q_{m-2^A-i,n-2^l-j}\omega_p(2^{-l},f).$$

If $q_{j,k}\uparrow$, then

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^3(x)d\mu(x)\right\|_p \leqslant cQ_{m,n}\omega_p(2^{-A},f) + c\sum_{l=0}^{B-1} 2^{A-1}2^l q_{m-2^{A-1},n-2^l}\omega_p(2^{-l},f).$$

If $q_{j,k}\downarrow$, then

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^3(x)d\mu(x)\right\|_p \leqslant cQ_{m,n}\omega_p(2^{-A},f) + c\sum_{l=0}^{B-1}(m-2^A)2^l q_{0,n-2^{l+1}+1}\omega_p(2^{-l},f).$$

At last, we discuss $\left\|\int_{G^2}(f(.+x)-f(.))I_3^4(x)d\mu(x)\right\|_p$. (13), Lemma 4 and Lemma 2 imply

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^4(x)d\mu(x)\right\|_p \leqslant$$
$$\leqslant c\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1}\sum_{i=1}^{m-2^A-1} |\Delta_{1,0}q_{m-2^A-i-1,n-2^l-j}| i\omega_p(2^{-A},f) + c\sum_{l=0}^{B-1}\sum_{j=0}^{2^l-1} q_{0,n-2^l-j}(m-2^A)\omega_p(2^{-A},f).$$

For double sequence $q_{j,k}\uparrow$, we have

$$\sum_{i=1}^{m-2^A-1} |\Delta_{1,0}q_{m-2^A-i-1,n-2^l-j}| i = \sum_{i=1}^{m-2^A-1} q_{m-2^A-i,n-2^l-j} - (m-2^A-1)q_{0,n-2^l-j}$$

and

$$\left\|\int_{G^2}(f(.+x)-f(.))I_3^4(x)d\mu(x)\right\|_p \leqslant cQ_{m,n}\omega_p(2^{-A},f).$$

For double sequence $q_{j,k}\downarrow$, we have

$$\sum_{i=1}^{m-2^A-1} |\Delta_{1,0}q_{m-2^A-i-1,n-2^l-j}| i \leqslant (m-2^A)q_{0,n-2^l-j}$$

and

$$\left\| \int_{G^2} (f(.+x) - f(.)) I_3^4(x) d\mu(x) \right\|_p \leqslant c \sum_{l=0}^{B-1} (m - 2^A) 2^l q_{0,n-2^{l+1}+1} \omega_p(2^{-A}, f).$$

This completes the proof of Theorem 1.

The proof of Theorem 2 goes analogously as Móricz and Siddiqi did for the one-dimensional Nörlund mean [16]. Thus, it is left to the readers.

We note that analogue results could be reached for Walsh-Kaczmarz system, more generally for piecewise linear rearrangements of Walsh-Paley system [17].

At last, we note that if the nonincreasing double sequence $\{q_{j,k}\}$ is choosed specially, as i.) or ii.) mentioned above, then we could reach better result than in Theorem 1. For example, set the double sequence $\{q_{j,k}\}$ as in i.), then the one-dimensional kernel $L_{m-2^A}^{n-2^B-i,1}$ in the equation (14) satiesfies the condition (7) with $1 < \delta < 1/\beta$ (and $\delta \leqslant 2$ can be choosed). This and Lemma 4 yield that $\left\| \int_{G^2} (f(.+x) - f(.)) I_4^3(x) d\mu(x) \right\|_p$ $\leqslant c Q_{m,n} \omega_p(2^{-A}, f)$. Thus, to reach a better result in this case i.) it is needed to discuss the expression $\| \int_{G^2} (f(.+x) - f(.)) I_3(x) d\mu(x) \|_p$. But, this is out of the main topic of this paper.

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