Mathematical
Inequalities
& Applications
Volume 15, Number 2 (2012), 343–351

SCORES, INEQUALITIES AND REGULAR HYPERTOURNAMENTS

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(*Communicated by R. A. Brualdi*)

Abstract. A k-hypertournament is a complete k-hypergraph with each k-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the edge. In a k-hypertournament, the score s_i (losing score r_i) of a vertex v_i is the number of edges containing v_i in which v_i is not the last element (in which v_i is the last element). In this paper we obtain inequalities involving powers of scores and losing scores by using classical results from mathematical analysis (such as Hölder's inequality) and show that equality holds if and only if the hypertournament is regular. We then use these inequalities to give a short proof of a result on the existence of regular hypertournaments. We also obtain an upper bound on the number of directed paths of length 2 in tournaments and hypertournaments, prove that the bound is sharp and that it is realized by regular hypertournaments.

1. Introduction

A k-hypergraph is a pair $H = (V, E)$, where V is the set of vertices and E is the set of k-subsets of V, called edges [2].

A tournament is a complete oriented graph. In a tournament the score of a vertex is its out-degree and the sequence of scores listed in non-decreasing order is called the score sequence. Landau's theorem [9] gives a necessary and sufficient condition for a sequence of non-negative integers to be the score sequence of some tournament. Brualdi and Shen [4] have strengthened Landau's inequalities on scores of a tournament, giving better necessary and sufficient conditions for the existence of tournament score sequences.

Hypertournaments are generalizations of tournaments. Given two non-negative integers n and k with $n \geqslant k > 1$, a k-hypertournament on n vertices is a pair (V, A), where V is the set of vertices with $|V| = n$, and A is the set of k-tuples of vertices, called arcs, such that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. A hypertournament is said to be regular if all vertices have the same score (equivalently the same losing score) [8]. Koh and Ree [8] have proved the following necessary and sufficient condition for the existence of regular hypertournaments.

Mathematics subject classification (2010): 05C20, 05C65.

Keywords and phrases: tournament, hypertournament, score, losing score, Hölder's inequality, Minkowski's inequality, power mean, inner product, regular hypertournament, Mahler's inequality.

The first two authors would like to thank King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia for continuous support in research.

The third author would like to acknowledge Qatar University, Doha, Qatar for providing reseach facilities.

Paper MIA-15-28

THEOREM 1.1. *For $n \geqslant 3$ and $2 \leqslant k \leqslant n-1$, a regular k-hypertournament on n vertices exists if and only if n divides $\binom{n}{k}$.*

Several authors have generalized concepts and results from tournaments to hypertournaments. The recent work on reconstruction of complete interval tournaments due to Ivanyi [5] can be extended to hypertournaments. The concept of kings in tournaments has been introduced in hypertournaments by Brcanov and Petrovic [3], but is still in its infancy. Zhou et al. [18] extended the concept of scores in tournaments to that of scores and losing scores in hypertournaments, and derived a result analogous to Landau's theorem on tournaments. The score $s(v_i)$ or s_i of a vertex v_i is the number of arcs containing v_i in which v_i is not the last element, and the losing score $r(v_i)$ or r_i of a vertex v_i is the number of arcs containing v_i in which v_i is the last element. The score sequence (losing score sequence) is formed by listing the scores (losing scores) in non-decreasing order. The last decade has seen a growing interest in the score structure of k-hypertournaments [6, 10, 15], multipartite hypertournaments [11, 12, 13] and oriented k-hypergraphs [14, 17].

The following characterizations of losing score sequences and score sequences of k-hypertournaments can be found in [18].

PROPOSITION 1.2. *Given two non-negative integers n and k with $n \geqslant k > 1$, a non-decreasing sequence $R = [r_1, r_2, \ldots, r_n]$ of non-negative integers is a losing score sequence of some k-hypertournament if and only if for each $1 \leqslant j \leqslant n$,*

$$\sum_{i=1}^{j} r_i \geqslant \binom{j}{k},$$

with equality when $j = n$.

PROPOSITION 1.3. *Given non-negative integers n and k with $n \geqslant k > 1$, a non-decreasing sequence $S = [s_1, s_2, \ldots, s_n]$ of non-negative integers is a score sequence of some k-hypertournament if and only if for each $1 \leqslant j \leqslant n$,*

$$\sum_{i=1}^{j} s_i \geqslant j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k},$$

with equality when $j = n$.

Bang and Sharp [1] proved Landau's theorem using Hall's theorem on a system of distinct representatives of a collection of sets. Based on Bang and Sharp's ideas, Koh and Ree [8] have given a different proof of Proposition 1.2 and 1.3. Some more results on scores of k-hypertournaments can be found in [7, 16].

As is evident from Proposition 1.2 and 1.3, inequalities play a pivotal role in the theory of hypertournaments. In this paper we investigate how classical inequalities can provide more insight into the behaviour of score and losing score sequences and hence the structure of hypertournaments. In particular, we focus on the powers of scores and losing scores. In section 2, we prove that the sum of power means of score and losing score sequences is never less than $\binom{n-1}{k-1}$ while the sum of their geometric means never exceeds $\binom{n-1}{k-1}$. We then consider score and losing score sequences as vectors in Euclidean space and obtain an upper bound on their inner product that is best possible. On the way we make use of several classical inequalities such as Hölder's, Minkowski's and Mahler's inequality. We also discuss the case of equality in detail for the inequalities derived in this paper and prove that equality holds in all these inequalities if and only if the hypertournament is regular. Some applications of these results are presented in section 3. We give a proof of Theorem 1.1 that is much shorter than the original proof by Koh and Ree. We also derive sharp upper bounds on the number of directed paths of length 2 in tournaments and hypertournaments. The bounds are realized by regular hypertournaments.

2. Inequalities involving powers of scores and losing scores

In this section we apply some famous inequalities from mathematical analysis such as Hölder, Minkowski and Mahler's inequalities to k-hypertournaments to obtain results on the powers of scores and losing scores. It is noteworthy that equality holds in all the results obtained in this section if and only if the hypertournament is regular.

The following result gives a lower bound on $\sum_{i=1}^{j} r_i^g$, where $1 < g < \infty$ is a real number.

THEOREM 2.1. *Let n and k be two non-negative integers with $n \geqslant k > 1$. If $R = [r_i]_1^n$ is a losing score sequence of a k-hypertournament, then for $1 < g < \infty$*

$$\sum_{i=1}^{j} r_i^g \geqslant \frac{j}{k^g}\binom{j-1}{k-1}^g,$$

where $1 \leqslant j < n$. In particular,

$$\sum_{i=1}^{n} r_i^g \geqslant \frac{n}{k^g}\binom{n-1}{k-1}^g, \tag{1}$$

with equality in (1) *if and only if the hypertournament is regular.*

Proof. By Proposition 1.2, we have

$$\binom{j}{k} \leqslant \sum_{i=1}^{j} r_i,$$

for $1 \leqslant j < n$. So by using Hölder's inequality

$$\binom{j}{k} \leqslant \sum_{i=1}^{j} r_i = \sum_{i=1}^{j} r_i \cdot 1 \leqslant \left(\sum_{i=1}^{j} r_i^g\right)^{\frac{1}{g}} \left(\sum_{i=1}^{j} 1^h\right)^{\frac{1}{h}},$$

for $1 \leqslant j < n$, where $\frac{1}{g} + \frac{1}{h} = 1$.

Thus

$$\binom{j}{k} \leqslant \left(\sum_{i=1}^{j} r_i^g\right)^{\frac{1}{g}} \left(\sum_{i=1}^{j} 1^h\right)^{\frac{1}{h}} = \left(\sum_{i=1}^{j} r_i^g\right)^{\frac{1}{g}} j^{\frac{1}{h}}.$$

That is

$$j^{\frac{-1}{h}} \left(\frac{j!}{k!(j-k)!}\right) \leqslant \left(\sum_{i=1}^{j} r_i^g\right)^{\frac{1}{g}},$$

or

$$\frac{j^{1-\frac{1}{h}}}{k} \left(\frac{(j-1)!}{(k-1)!(j-k)!}\right) \leqslant \left(\sum_{i=1}^{j} r_i^g\right)^{\frac{1}{g}}.$$

Hence

$$\sum_{i=1}^{j} r_i^g \geqslant \frac{j}{k^g} \binom{j-1}{k-1}^g. \tag{2}$$

For $j = n$, Proposition 1.2 gives

$$\binom{n}{k} = \sum_{i=1}^{n} r_i,$$

So inequalities (2) now become

$$\sum_{i=1}^{n} r_i^g \geqslant \frac{n}{k^g} \binom{n-1}{k-1}^g,$$

with equality if and only if $r_1 = r_2 = \cdots = r_n$ (condition of equality for Hölder's inequality), that is if and only if the hypertournament is regular. $\square$

We now proceed to obtain results which combine scores and losing scores. The following theorem relates powers of scores and losing scores.

THEOREM 2.2. *Let n and k be two non-negative integers with $n \geqslant k > 1$. If $S = [s_i]_1^n$ in non-increasing order and $R = [r_i]_1^n$ in non-decreasing order are respectively the score and losing score sequences of a k-hypertournament, then for any real number $1 < g < \infty$*

$$\left(\frac{\sum_{i=1}^{j} s_i^g}{j}\right)^{\frac{1}{g}} + \left(\frac{\sum_{i=1}^{j} r_i^g}{j}\right)^{\frac{1}{g}} \geqslant \binom{n-1}{k-1},$$

where $1 \leqslant j < n$. In particular,

$$\left(\frac{\sum_{i=1}^{n} s_i^g}{n}\right)^{\frac{1}{g}} + \left(\frac{\sum_{i=1}^{n} r_i^g}{n}\right)^{\frac{1}{g}} \geqslant \binom{n-1}{k-1},$$

with equality if and only if the hypertournament is regular. Furthermore, for any positive integer $1 \leqslant j < n$

$$\prod_{i=1}^{j} s_i^{\frac{1}{j}} + \prod_{i=1}^{j} r_i^{\frac{1}{j}} \leqslant \binom{n-1}{k-1}.$$

In particular, for $j = n$
$$\prod_{i=1}^{n} s_i^{\frac{1}{n}} + \prod_{i=1}^{n} r_i^{\frac{1}{n}} \leqslant \binom{n-1}{k-1},$$
with equality if and only if the hypertournament is regular.

Proof. Since the score sequence is arranged in non-increasing order, therefore for any $1 \leqslant i \leqslant n$,
$$\binom{n-1}{k-1} = s_i + r_i. \tag{3}$$
So for $1 \leqslant j \leqslant n$ and any $g > 1$, we have
$$j\binom{n-1}{k-1}^{g} = \sum_{i=1}^{j} (s_i + r_i)^g.$$
Applying Minkowski's inequality
$$j^{\frac{1}{g}}\binom{n-1}{k-1} = \left(\sum_{i=1}^{j} (s_i + r_i)^g\right)^{\frac{1}{g}} \leqslant \left(\sum_{i=1}^{j} s_i^g\right)^{\frac{1}{g}} + \left(\sum_{i=1}^{j} r_i^g\right)^{\frac{1}{g}},$$
or
$$\binom{n-1}{k-1} \leqslant \left(\frac{\sum_{i=1}^{j} s_i^g}{j}\right)^{\frac{1}{g}} + \left(\frac{\sum_{i=1}^{j} r_i^g}{j}\right)^{\frac{1}{g}}.$$
For $j = n$, we have
$$\binom{n-1}{k-1} \leqslant \left(\frac{\sum_{i=1}^{n} s_i^g}{n}\right)^{\frac{1}{g}} + \left(\frac{\sum_{i=1}^{n} r_i^g}{n}\right)^{\frac{1}{g}}. \tag{4}$$
Equality holds in (4) if and only if for all $1 \leqslant i \leqslant n$, we have $s_i = cr_i$ for some fixed real number c (by the condition of equality in Minkowski's inequality). Substituting in (3) and summing upto n
$$n\binom{n-1}{k-1} = \sum_{i=1}^{n} (s_i + r_i) = (c+1)\sum_{i=1}^{n} r_i = (c+1)\binom{n}{k} \quad \text{(by Proposition 1.2)},$$
which holds if and only if $c = k-1$ or equivalently if $r_i = \frac{1}{k}\binom{n-1}{k-1}$ or equivalently if the hypertournament is regular.
Again from (3), for any $1 \leqslant j \leqslant n$
$$\binom{n-1}{k-1} = \prod_{i=1}^{j} (s_i + r_i)^{\frac{1}{j}} \geqslant \prod_{i=1}^{j} s_i^{\frac{1}{j}} + \prod_{i=1}^{j} r_i^{\frac{1}{j}} \quad \text{(by Mahler's inequality)}.$$
Again arguing as above, equality holds for $j = n$ if and only if the hypertournament is regular. □

The power mean with exponent g of non-negative real numbers $x_1,x_2,\dots,x_j$ is defined as

$$M_g(x_1,\dots,x_j) = \left(\frac{\sum_{i=1}^{j} x_i^g}{j}\right)^{\frac{1}{g}}.$$

Theorem 2.2 states that for any exponent $1 < g < \infty$ the sum of power means of first j terms of the score sequence and the losing score sequence of an n vertex k-hypertournament is never less than $\binom{n-1}{k-1}$. That is

$$M_g(s_1,\dots,s_j) + M_g(r_1,\dots,r_j) \geqslant \binom{n-1}{k-1}, \tag{5}$$

where $1 \leqslant j \leqslant n$.

Since $S = [s_i]_1^n$ is arranged in non-increasing order and $R = [r_i]_1^n$ is in non-decreasing order, conditions (5) hold for any j terms (not necessarily the first j terms) of S and R.

The geometric mean of non-negative real numbers $x_1,x_2,\dots,x_j$ is defined as

$$GM(x_1,\dots,x_j) = \prod_{i=1}^{j} x_i^{\frac{1}{j}}.$$

Theorem 2.2 also implies that the sum of geometric means of first j terms of the score sequence and the losing score sequence of an n vertex k-hypertournament cannot exceed $\binom{n-1}{k-1}$. That is

$$GM(s_1,\dots,s_j) + GM(r_1,\dots,r_j) \leqslant \binom{n-1}{k-1}. \tag{6}$$

Let $\Re^n$ denote the n-dimensional Euclidean space. The inner product of two vectors $A = (a_1,\dots,a_n)$ and $B = (b_1,\dots,b_n)$ in $\Re^n$ is defined as

$$\langle A,B\rangle = \sum_{i=1}^{n} a_i b_i.$$

The next theorem gives an upper bound for the inner product of score and losing score vectors in $\Re^n$. The bound given in Theorem 2.3 is best possible in the sense that it is realized by regular hypertournaments. It should also be noted that Theorem 2.3 does not depend on the order of s_i and r_i and holds for any arbitrary ordering of scores and losing scores.

THEOREM 2.3. *Let n and k be two non-negative integers with $n \geqslant k > 1$. If $S = [s_i]_1^n$ and $R = [r_i]_1^n$ are respectively the score sequence and losing score sequence of a k-hypertournament, then*

$$\langle S,R\rangle \leqslant \frac{k-1}{k}\binom{n}{k}\binom{n-1}{k-1}, \tag{7}$$

with equality if and only if the hypertournament is regular.

Proof. By the linearity of inner product, we have

$$\langle S+R,R\rangle = \langle S,R\rangle + \langle R,R\rangle,$$

$$\binom{n-1}{k-1}\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} s_i r_i + \sum_{i=1}^{n} r_i^2.$$

From Proposition 1.2

$$\binom{n-1}{k-1}\binom{n}{k} = \sum_{i=1}^{n} s_i r_i + \sum_{i=1}^{n} r_i^2.$$

The arithmetic mean of n non-negative real numbers never exceeds their root mean square. So

$$\sqrt{\frac{\sum_{i=1}^{n} r_i^2}{n}} \geqslant \frac{\sum_{i=1}^{n} r_i}{n},$$

with equality if and only if $r_1 = r_2 = \cdots = r_n$.

Or

$$\sum_{i=1}^{n} r_i^2 \geqslant \frac{\left(\sum_{i=1}^{n} r_i\right)^2}{n}.$$

Thus

$$\binom{n-1}{k-1}\binom{n}{k} \geqslant \sum_{i=1}^{n} s_i r_i + \frac{\left(\sum_{i=1}^{n} r_i\right)^2}{n} = \sum_{i=1}^{n} s_i r_i + \frac{\binom{n}{k}^2}{n}. \quad \text{(by Proposition 1.2)}$$

Rearranging

$$\sum_{i=1}^{n} s_i r_i \leqslant \binom{n-1}{k-1}\binom{n}{k} - \frac{\binom{n}{k}^2}{n} = \frac{k-1}{k}\binom{n}{k}\binom{n-1}{k-1}.$$

The equality holds if and only if $r_1 = r_2 = \cdots = r_n$, that is, if and only if the hypertournament is regular. □

3. Some applications to hypertournaments

The results obtained in the previous section provide much insight into the structure of hypertournaments. First we give a short proof of Theorem 1.1 based on Theorem 2.1.

Proof of Theorem 1.1. If there exists a regular k-hypertournament with losing score sequence $[r_i]_{i=1}^{n}$ then by inequalities (2),

$$\sum_{i=1}^{n} r^2 = \frac{n}{k^2}\binom{n-1}{k-1}^2, \quad \text{(case of equality with } g=2\text{)}$$

where $r_1 = r_2 = \cdots = r_n = r$. Thus $r^2 = \frac{1}{k^2}\binom{n-1}{k-1}^2$ and so k divides $\binom{n-1}{k-1}$. Now $\frac{\binom{n}{k}}{n} = \frac{\binom{n-1}{k-1}}{k}$ which implies that n divides $\binom{n}{k}$.

Conversely, assume that n divides $\binom{n}{k}$. Then k divides $\binom{n-1}{k-1}$. For $1 \leqslant i \leqslant n$, set $r_i = \frac{1}{k}\binom{n-1}{k-1}$. Then for $1 \leqslant j \leqslant n$

$$\sum_{i=1}^{j} r_i = \frac{j}{k}\binom{n-1}{k-1} = \frac{j(n-1)!}{(n-k)!k!} \geqslant \frac{j!}{(j-k)!k!},$$

with equality when $j = n$. Thus by Proposition 1.2, $[r_i = r]_{i=1}^{n}$ is the losing score sequence of a regular k-hypertournament. $\square$

Note that Theorem 1.1 can also be proved by using Theorem 2.3 together with the argument as seen above.

An (x,y)-path [18] in a k-hypertournament H is a sequence

$$(x=)v_1e_1v_2e_2v_3\cdots v_{t-1}e_{t-1}v_t(=y)$$

of distinct vertices $v_1, v_2, \ldots, v_t$, $t \geqslant 1$ and distinct arcs $e_1, e_2, \ldots, e_{t-1}$ such that for $1 \leqslant i \leqslant t-1$, vertex v_{i+1} occurs as the last element in e_i. A little consideration reveals that the inner product of score and losing score vectors is actually equal to the number of directed paths of length two in a k-hypertournament. So the upper bound derived in Theorem 2.3 actually bounds the number of paths of length two in a k-hypertournament.

THEOREM 3.1. *The number of directed paths of length two in a k-hypertournament never exceeds $\frac{k-1}{k}\binom{n}{k}\binom{n-1}{k-1}$. The bound is achieved if and only if the hypertournament is regular.*

Proof. For $1 \leqslant i \leqslant n$, the quantity $r_i s_i$ counts the number of distinct directed paths of length two of the form xv_iy. Thus $\sum_{i=1}^{n} r_i s_i$ is the total number of distinct directed paths of length two in the hypertournament. By (7) this quantity is bounded above by $\frac{k-1}{k}\binom{n}{k}\binom{n-1}{k-1}$ with equality if and only if the hypertournament is regular. $\square$

The case $k = 2$ leads to the corresponding bound for tournaments.

COROLLARY 3.2. *The number of directed paths of length two in a tournament never exceeds $\frac{n(n-1)^2}{4}$. The bound is achieved if and only if the tournament is regular.*

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(Received August 28, 2010)

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