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# A GEOMETRIC PROOF OF BLUNDON'S INEQUALITIES

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*Abstract.* A geometric approach of Blundon's inequality is presented. Theorem 2.1 gives the formula for $\cos\widehat{ION}$ in terms of the symmetric invariants $s$, $R$, $r$ of a triangle, implying Blundon's inequality (Theorem 2.2). A dual formula for $\cos\widehat{I_aON_a}$ is given in Theorem 3.1 and this implies the dual Blundon's inequality (Theorem 3.2). As applications, some inequalities involving the exradii of the triangle are presented in the last section.

## 1. Introduction

Given a triangle $ABC$, denote by $O$ the circumcenter, $I$ the incenter , $G$ the centroid, $N$ the Nagel point, $s$ the semiperimeter, $R$ the circumradius, and $r$ the inradius of $ABC$. In this note, we present a geometric proof to the so-called fundamental triangle inequality. This relation contains in fact two inequalities and it was first time proved by E. Rouché in 1851 (see [15]), answering a question of Ramus concerning the necessary and sufficient conditions for three positive real numbers $s$, $R$, $r$ to be the semiperimeter, circumradius, and inradius of a triangle (see Theorem 2.2). The standard simple proof was first time given by W. J. Blundon [3] and it is based on the following algebraic property of the roots of a cubic equation: The roots $x_1$, $x_2$, $x_3$ of the equation

$$x^3+a_1x^2+a_2x+a_3=0$$

are the side lengths of a (nondegenerate) triangle if, and only if, the following three conditions are verified: i) $18a_1a_2a_3+a_1^2a_2^2-27a_3^2-4a_2^3-4a_1^3a_3>0$ ; ii) $-a_1>0$, $a_2>0$, $-a_3>0$; iii) $a_1^3-4a_1a_2+8a_3>0$. For more details we refer to the monograph of D. Mitrinović, J. Pečarić, V. Volenec [12], and to the papers of C. Niculescu [13], [14], R. A. Satnoianu [16], and S. Wu [20]. We mention that G. Dospinescu, M. Lascu, C. Pohoaţă, M. Tetiva [6] have proposed a new algebraic proof to the weaker Blundon's inequality $s\leqslant 2R+(3\sqrt{3}-4)r$. This inequality is a direct consequence of the fundamental triangle inequality.

In this paper we present a geometric approach to Blundon's inequality. In Theorem 2.1 we give the formula for $\cos\widehat{ION}$ in terms of the symmetric invariants of triangle $s$,

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$R$, $r$. This formula contains in fact the fundamental triangle inequalities (Theorem 2.2) and it is connected to the famous Euler's determination problem. In Section 3 we prove a dual version of the Blundon's inequality,derived from Theorem 3.1, which gives a formula for $\cos\widehat{I_aON_a}$ in terms of $\alpha$, $R$, $r_a$, where $\alpha = \frac{a^2+b^2+c^2}{4}$. Here we discuss the possible duality between formulas (5) and (11). We show that it is much more natural to ask in the initial problem to find necessary and sufficient conditions such that the positive real numbers $\alpha$, $R$, $r$ are the corresponding elements of a triangle. In the last section, as consequences of the dual Blundon's inequality, some inequalities for $\alpha$, $R$ and the exradii of the triangle are derived.

## 2. The main result

In order to state our main results we need to recall some important distances in triangle $ABC$. The famous formula for the distance $OI$ is called Euler's relation and it is given by

$$OI^2 = R^2 - 2Rr \tag{1}$$

For the standard geometric proof of this relation we refer to the books of H. S. M. Coxeter and S. L. Greitzer [5, Theorem 2.12, p. 29] or T. Lalescu [11, Section 3.12, p. 25]. For a proof using complex numbers we mention the book of T. Andreescu and D. Andrica [1, Theorem 4, pp. 112–113]. The next important distance is $ON$ and it is given by

$$ON = R - 2r \tag{2}$$

The relation (2) gives in geometric way the difference between the quantities involved in the Euler's inequality $R \geqslant 2r$ and it will play an important role in the proof of our main results. A proof by using complex numbers is given in the book of T. Andreescu and D. Andrica [1, Theorem 6, pp. 113–114]. Another useful distance is $OG$ and the following relation holds

$$OG^2 = R^2 - \frac{a^2+b^2+c^2}{9}, \tag{3}$$

where $a, b$, and $c$ are the side lengths of triangle $ABC$. The standard proof uses Leibniz's relation combined with the median formula (see [1, pp. 142–143]).

The sum $a^2+b^2+c^2$ can be expressed in terms of the symmetric invariants $s, R, r$ of triangle $ABC$ as follows:

$$a^2+b^2+c^2 = 2(s^2 - r^2 - 4Rr) \tag{4}$$

This formula can be found in different references, for instance in [1, Corollary 2, p. 111] or in the excellent monograph of D. S. Mitrinović, J. E. Pečarić, V. Volenec [12].

The following result contains a simple geometric proof of the fundamental inequality of a triangle.

THEOREM 1. *Assume that the triangle ABC is not equilateral. The following relation holds :*

$$\cos\widehat{ION} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R-2r)\sqrt{R^2-2Rr}}. \tag{5}$$

*Proof.* It is known that the points $N$, $G$, and $I$ are collinear and the defined line is called Nagel's line of the triangle, and we have $NI = 3GI$ (see for instance [1, Theorem 3, p.108-109] or [10, pp 127-136]). If we use Stewart's Theorem in the triangle $ION$, then we get

$$ON^2 \cdot GI + OI^2 \cdot NG - OG^2 \cdot NI = GI \cdot GN \cdot NI,$$

and it follows that

$$ON^2 \cdot GI + OI^2 \cdot 2GI - OG^2 \cdot 3GI = 6GI^3.$$

This relation is equivalent to

$$ON^2 + 2 \cdot OI^2 - 3 \cdot OG^2 = 6GI^2. \tag{6}$$

Now, using formulas (1), (2), (3) in (6) we obtain

$$GI^2 = \frac{1}{6}\left(\frac{a^2+b^2+c^2}{3} - 8Rr + 4r^2\right) = \frac{1}{6}\left(\frac{2(s^2-r^2-4Rr)}{3} - 8Rr + 4r^2\right).$$

So we get

$$NI^2 = 9GI^2 = 5r^2 + s^2 - 16Rr.$$

We use the Law of Cosines in the triangle $ION$ and obtain

$$\cos\widehat{ION} = \frac{ON^2 + OI^2 - NI^2}{2ON \cdot OI}$$

$$= \frac{(R-2r)^2 + (R^2-2Rr) - (5r^2+s^2-16Rr)}{2(R-2r)\sqrt{R^2-2Rr}} = \frac{2R^2+10Rr-r^2-s^2}{2(R-2r)\sqrt{R^2-2Rr}} \tag{7}$$

and we are done.

If the triangle $ABC$ is equilateral, then the points $I, O, N$ coincide, i.e. triangle $ION$ degenerates to one point. In this case we can extend formula (5) by $\cos\widehat{ION} = 1$. □

THEOREM 2. *(Blundon's inequalities) The necessary and sufficient condition for the existence of a triangle with elements $s$, $R$ and $r$ is*

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr} \leqslant s^2 \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}. \tag{8}$$

*Proof.* If we have $R = 2r$, then the triangle must be equilateral and we are done. If we assume that $R - 2r \neq 0$, then inequalities (8) are direct consequences of the fact that $-1 \leqslant \cos\widehat{ION} \leqslant 1$.

□

Equilateral triangles give the trivial situation where we have equality in (8). Suppose that we are not working with equilateral triangles, i.e we have $R - 2r \neq 0$. Denote by $\mathscr{T}(R,r)$ the family of all triangles having the circumradius $R$ and the inradius $r$.

The inequalities (8) give in terms of $R$ and $r$ the exact interval for the semiperimeter $s$ of triangles in family $\mathscr{T}(R,r)$. We have $s_{\min}^2 = 2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr}$ and $s_{\max}^2 = 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr}$. If we fix the circumcenter $O$ and the incenter $I$ such that $OI = \sqrt{R^2 - 2Rr}$, then the triangle in the family $\mathscr{T}(R,r)$ with minimal semiperimeter corresponds to the equality case $\cos\widehat{ION} = 1$, i.e. points $I,O,N$ are collinear and $I$ and $N$ belong to the same ray with the origin $O$. Taking in to account the well-known property that points $O,G,H$ belong to Euler's line of triangle, this implies that $O,I,G$ must be collinear, hence in this case triangle $ABC$ is isosceles. In Figure 1 this triangle is denoted by $A_{\min}B_{\min}C_{\min}$. Also, the triangle in the family $\mathscr{T}(R,r)$ with maximal semiperimeter corresponds to the equality case $\cos\widehat{ION} = -1$, i.e. points $I,O,N$ are collinear and $O$ is situated between $I$ and $N$. Using again the Euler's line of the triangle, it follows that triangle $ABC$ is isosceles. In Figure 1 this triangle is denoted by $A_{\max}B_{\max}C_{\max}$. Note that we have $B_{\min}C_{\min} > B_{\max}C_{\max}$. The triangles in the family $\mathscr{T}(R,r)$ are "between" these two extremal triangles (see Figure 1). According to the Poncelet's closure Theorem, they are inscribed in the circle $\mathscr{C}(O;R)$ and their sides are tangent externally to the circle $\mathscr{C}(I;r)$.

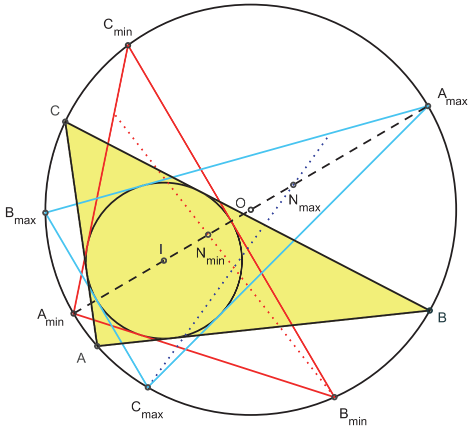


Figure 1

From Theorem 2.1 it follows that it is a natural and important problem to construct the triangle $ABC$ from its incenter $I$, circumcenter $O$, and its Nagel point $N$. Taking into account that points $I$, $G$, $N$ are collinear, determining the Nagel line of triangle, it follows that we get the centroid $G$ on the segment $IN$ such that $IG = \frac{1}{3}IN$. Then, using the Euler's line of the triangle, we get the orthocenter $H$ on the ray $(OG$ such that $OH = 3OG$. Now we have reduced the construction problem to the famous Euler's determination problem i.e. to construct a triangle from its incenter $I$, circumcenter $O$, and orthocenter $H$ (we refer to the original reference [8]). Some new approaches involving this problem are given by B. Scimemi [17], G. C. Smith [18], J. Stern [19], and P. Yiu [21].

## 3. A dual version of Blundon's inequalities

In this section we consider a triangle $ABC$ with circumcenter $O$, incenter $I$, excenters $I_a, I_b, I_c$, and $N_a, N_b, N_c$ the adjoint points to the Nagel point $N$. For the definition and some properties of the adjoint points $N_a$ ,$N_b$ ,$N_c$ we refer to the paper of D. Andrica and K. L. Nguyen [2]. Let $s$, $R$, $r$, $r_a$ ,$r_b$ ,$r_c$ be the semiperimeter, circumradius, inradius, and exradii of triangle $ABC$, respectively. We know that points $N_a$, $G$, $I_a$ are collinear and we have $N_aI_a = 3GI_a$. The similar properties hold for the triples of points $N_b$, $G$, $I_b$ and $N_c$, $G$, $I_c$. The following relations are similar to (1) and (2)

$$OI_a^2 = R^2 + 2Rr_a, OI_b^2 = R^2 + 2Rr_b, OI_c^2 = R^2 + 2Rr_c \tag{9}$$

and

$$ON_a = R + 2r_a, ON_b = R + 2r_b, ON_c = R + 2r_c \tag{10}$$

For a proof by using complex numbers we refer to the paper [2].

THEOREM 3. *The following relation holds :*

$$\cos\widehat{I_aON_a} = \frac{R^2 - 3Rr_a - r_a^2 - \alpha}{(R + 2r_a)\sqrt{R^2 + 2Rr_a}}, \tag{11}$$

*where* $\alpha = \frac{a^2+b^2+c^2}{4}$.

*Proof.* We know that points $N_a$, $G$, $I_a$ are collinear and we have $N_aI_a = 3GI_a$, $ON_a = R + 2r_a$, and $OI_a^2 = R^2 + 2Rr_a$. Using Stewart's Theorem in triangle $I_aON_a$ it follows

$$ON_a^2 \cdot GI_a + OI_a^2 \cdot N_aG - OG^2 \cdot N_aI_a = GI_a \cdot GN_a \cdot N_aI_a, \tag{12}$$

hence

$$ON_a^2 \cdot GI_a + OI_a^2 \cdot 2GI_a - OG^2 \cdot 3GI_a = 6GI_a^3. \tag{13}$$

Taking in to account the relation $N_aI_a = 3GI_a$ we get

$$I_aN_a^2 = 9GI_a^2 = \frac{3}{2}(\frac{a^2+b^2+c^2}{3} + 8Rr_a + 4r_a^2)$$

$$= \frac{a^2+b^2+c^2}{2} + 12Rr_a + 6r_a^2 = 2\alpha + 12Rr_a + 6r_a^2. \tag{14}$$

From Law of Cosines applied in triangle $I_aON_a$ it follows:

$$\cos\widehat{I_aON_a} = \frac{OI_a^2 + ON_a^2 - I_aN_a^2}{2OI_a \cdot ON_a} = \frac{R^2 + 2Rr_a + (R+2r_a)^2 - I_aN_a^2}{2OI_a \cdot ON_a}$$
$$= \frac{R^2 - 3Rr_a - r_a^2 - \alpha}{(R + 2r_a)\sqrt{R^2 + 2Rr_a}},$$

and the relation is proved. There are similar formulas for $r_a$ and $r_b$. □

THEOREM 4. (Dual form of Blundon's inequalities) *The following inequalities hold*

$$0 \leqslant \frac{a^2+b^2+c^2}{4} \leqslant R^2 - 3Rr_a - r_a^2 + (R+2r_a)\sqrt{R^2+2Rr_a}. \tag{15}$$

*Proof.* Let us begin by noticing that if we proceed as in the proof of Theorem 2.2, i.e. by using the inequalities $-1 \leqslant \cos \widehat{I_aON_a} \leqslant 1$, then we get

$$R^2 - 3Rr_a - r_a^2 - (R+2r_a)\sqrt{R^2+2Rr_a} \leqslant \alpha \leqslant R^2 - 3Rr_a - r_a^2 + (R+2r_a)\sqrt{R^2+2Rr_a}.$$

It is clear that we have

$$R^2 = R\sqrt{R^2} \leqslant (R+2r_a)\sqrt{R^2+2Rr_a} \leqslant 3Rr_a + r_a^2 + (R+2r_a)\sqrt{R^2+2Rr_a},$$

that is the expression in the left hand side of the previous inequalities is always negative. On the other hand, we have $\alpha \geqslant 0$.

The right inequality in (15) follows from to $\cos \widehat{I_aON_a} \leqslant 1$, where $\cos \widehat{I_aON_a}$ is given by formula (11) in Theorem 3.1. There are similar inequalities for $r_b$ and $r_c$. □

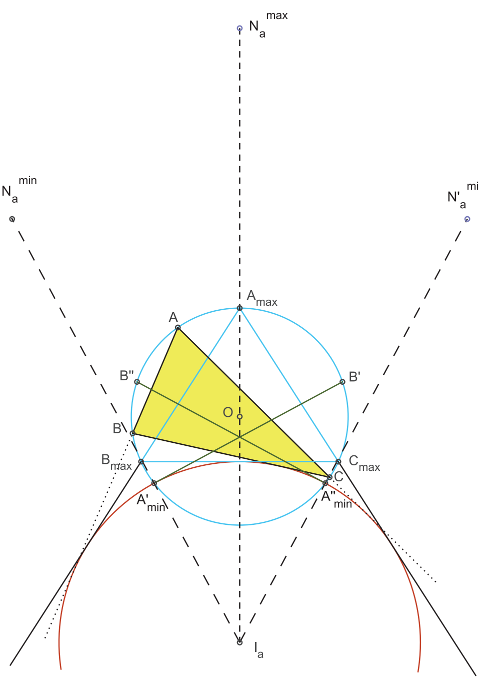


Figure 2

Denote by $\mathscr{T}(R, r_a)$ the family of all triangles having the circumradius $R$ and the exradius $r_a$. The inequalities (15) give in terms of $R$ and $r_a$ the exact interval for

$\alpha$, for triangles in family $\mathscr{T}(R,r_a)$. We have $\alpha_{\min}=0$ and $\alpha_{\max}=R^2-3Rr_a-r_a^2+(R+2r_a)\sqrt{R^2+2Rr_a}$. If we fix the circumcenter $O$ and the excenter $I_a$ such that we have $OI_a=\sqrt{R^2+2Rr_a}$, then the triangles in the family $\mathscr{T}(R,r_a)$ with minimal $\alpha$ are degenerated to a point and they correspond to the intersection points of the circles. In Figure 2 these points are denoted by $A'_{\min}$ and $A''_{\min}$. Also, the triangle in the family $\mathscr{T}(R,r_a)$ with maximal $\alpha$ corresponds to the equality case $\cos\widehat{I_aON_a}=-1$, i.e. points $I_a$, $O$, $N_a$ are collinear and $O$ is between $I_a$ and $N_a$. Using again the Euler's line of the triangle, it follows that triangle $ABC$ is isosceles. Figure 2 presents the standard geometric configuration illustrating the extremal triangles. Also, in Figure 2 other two degenerated triangles are illustrated, i.e. the "isosceles" triangle $A'_{\min}B'$ and $A''_{\min}B''$ defined by the intersections of tangents at points $A'_{\min}$ and $A''_{\min}$ to the circle $\mathscr{C}(I_a,r_a)$ with the circle $\mathscr{C}(I,r)$.

In this case we have also an exterior Poncelet's closure Theorem, that is the triangles "between" these two extremal triangles belong to the family $\mathscr{T}(R,r_a)$. We refer to the paper of L. Emelyanov and T. Emelyanov [7] for a complicate proof of this property.

From Theorem 3.1 it follows that it is a natural problem to construct the triangle $ABC$ from its excenter $I_a$, circumcenter $O$, and its adjoint Nagel point $N_a$. Taking in to account that points $I_a$, $G$, $N_a$ are collinear, it follows that we get the centroid $G$ on the segment $I_aN_a$ such that $I_aG=\frac{1}{3}I_aN_a$. Then, using the Euler's line of the triangle, we get the orthocenter $H$ on the ray $(OG$ such that $OH=3OG$. Now we have reduced again our construction problem to the famous Euler's determination problem: to construct a triangle from its incenter $I$, circumcenter $O$, and orthocenter $H$ (we refer to the original reference [7]).

REMARKS. 1) From Theorem 3.1 it follows that it is a natural question to express $\alpha$ in terms of $s$, $R$, $r_a$, in order to obtain a similar formula to (5). In order to answer to this question we shall prove that

$$ab+bc+ca=\frac{s^6+r_a(4R+3r_a)s^4+r_a^2(3r_a^2-16R^2)s^2+r_a^5(r_a-4R)}{(s^2+r_a^2)^2}. \tag{16}$$

Indeed, we have $\tan\frac{A}{2}=\frac{r_a}{s}$, hence we can write

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin^2\frac{A}{2}+\cos^2\frac{A}{2}}=\frac{2\tan\frac{A}{2}}{\tan^2\frac{A}{2}+1}=\frac{2\frac{r_a}{s}}{\frac{r_a^2}{s^2}+1}=\frac{2r_as}{r_a^2+s^2}.$$

It follows

$$a=2R\cdot\sin A=\frac{4Rr_as}{r_a^2+s^2}.$$

If $K$ denotes the area of triangle $ABC$, from $\frac{abc}{4R}=K=(s-a)r_a$, we get

$$bc=\frac{4(s-a)Rr_a}{a}=4Rr_a\cdot(\frac{s}{a}-1)=4Rr_a\cdot\frac{r_a^2+s^2-4Rr_a}{4Rr_a}=r_a^2+s^2-4Rr_a.$$

We combine these formulas to get

$$\begin{aligned}ab+bc+ca &= a(b+c)+bc = a(2s-a)+bc\\ &= \frac{4Rr_a s}{r_a^2+s^2}\cdot\left(2s-\frac{4Rr_a s}{r_a^2+s^2}\right)+r_a^2+s^2-4Rr_a\\ &= \frac{s^6+r_a(4R+3r_a)s^4+r_a^2(3r_a^2-16R^2)s^2+r_a^5(r_a-4R)}{(s^2+r_a^2)^2}.\end{aligned}$$

Now it is easy to find $\alpha$ in terms of $s$, $R$, $r_a$. We have

$$\begin{aligned}4\alpha &= a^2+b^2+c^2 = (a+b+c)^2-2(ab+bc+ca)\\ &= 4s^2-2\frac{s^6+r_a(4R+3r_a)s^4+r_a^2(3r_a^2-16R^2)s^2+r_a^5(r_a-4R)}{(s^2+r_a^2)^2}\\ &= \frac{2[s^6-r_a(4R-r_a)s^4+r_a^2(16R^2-r_a^2)s^2-r_a^5(r_a-4R)]}{(s^2+r_a^2)^2}.\end{aligned}\tag{17}$$

2) Formula (17) express $\alpha$ as a complicated function of $s$, $R$, $r_a$, so in order to have a duality between formulas (5) and (11) we have to express $\cos\widehat{ION}$ in terms of $\alpha, R, r$. In this respect, from formula $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$ we get $s^2 = 2\alpha+4Rr+r^2$. Replacing in formula (5) we obtain

$$\cos\widehat{ION} = \frac{2R^2+10Rr-r^2-s^2}{2(R-2r)\sqrt{R^2-2Rr}} = \frac{2R^2+10Rr-r^2-2\alpha-4Rr-r^2}{2(R-2r)\sqrt{R^2-2Rr}}$$

$$= \frac{2R^2+6Rr-2r^2-2\alpha}{2(R-2r)\sqrt{R^2-2Rr}}.$$

Then it follows

$$\cos\widehat{ION} = \frac{R^2+3Rr-r^2-\alpha}{(R-2r)\sqrt{R^2-2Rr}}.\tag{18}$$

The formal transformation $r \mapsto -r_a$ gives the duality between formulas (18) and (11). We have similar duality relations for $r_a, r_b$.

## 4. Applications to some inequalities in $s$, $R$, and the exradii

In this section we will obtain as consequences to Theorem 3.2 some inequalities involving $s, R$ and the exradii of the triangle.

COROLLARY 1. *In any triangle with semiperimeter $s$ the following inequalities hold*

$$4Rr+r^2 \leqslant s^2 \leqslant 2R^2+r^2+4Rr-6Rr_a-2r_a^2+2(R+2r_a)\sqrt{R^2+2Rr_a},\tag{19}$$

*and analogous relations for $r_b$ and $r_c$.*

*Proof.* Because $\frac{a^2+b^2+c^2}{2} = s^2 - r^2 - 4Rr$, the inequalities follow from the dual form of Blundon's inequality. □

Using the fact that $r < r_a$ from the right inequality in the previous result we obtain

$$\begin{aligned} s^2 &\leqslant 2R^2 + r^2 + 4Rr - 6Rr_a - 2r_a^2 + 2(R+2r_a)\sqrt{R^2+2Rr_a} \\ &< 2R^2 + r_a^2 + 4Rr_a - 6Rr_a - 2r_a^2 + 2(R+2r_a)\sqrt{R^2+2Rr_a} \\ &= 2R^2 - 2Rr_a - r_a^2 + 2(R+2r_a)\sqrt{R^2+2Rr_a} \end{aligned}$$

hence we get the following inequality between $s, R$ and $r_a$ :

$$s^2 < 2R^2 - 2Rr_a - r_a^2 + 2(R+2r_a)\sqrt{R^2+2Rr_a}. \tag{20}$$

COROLLARY 2. *In any triangle with semiperimeter $s$ the following inequality holds*

$$s^2 < \min\{4R^2 + 4Rr_a + 3r_a^2, 4R^2 + 4Rr_b + 3r_b^2, 4R^2 + 4Rr_c + 3r_c^2\}. \tag{21}$$

*Proof.* According to the inequality (20) we can write

$$\begin{aligned} s^2 &< 2R^2 - 2Rr_a - r_a^2 + 2(R+2r_a)\sqrt{R^2+2Rr_a} \\ &< 2R^2 - 2Rr_a - r_a^2 + 2(R+2r_a)(R+r_a) \\ &= 4R^2 + 4Rr_a + 3r_a^2, \end{aligned}$$

and the similar relations for $r_b$, $r_c$, and we are done. □

In what follows we will derive an inequality involving $R$, $r_a$ and $\alpha$.

COROLLARY 3. *The following inequalities hold*

$$a^2 + b^2 + c^2 < \min\{8R^2 + 4r_a^2, 8R^2 + 4r_b^2, 8R^2 + 4r_c^2\}. \tag{22}$$

*Proof.* From Theorem we have

$$\alpha = \frac{a^2+b^2+c^2}{4} \leqslant R^2 - 3Rr_a - r_a^2 + (R+2r_a)\sqrt{R^2+2Rr_a}$$

$$< R^2 - 3Rr_a - r_a^2 + (R+2r_a)(R+r_a) = 2R^2 + r_a^2,$$

and the inequality involving $r_a$ follows. We have the analogous relations for $r_b$ and $r_c$. □

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