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THE MULTIPLICATION OPERATOR FROM MIXED–NORM TO n–TH WEIGHTED–TYPE SPACES ON THE UNIT DISK

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Abstract. Let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$ and $u \in H(\mathbb{D})$. The boundedness and compactness of the multiplication operator from mixed-norm space to nth weighted-type spaces on the unit disk are investigated in this paper.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the open unit disc $\mathbb{D}$ of the finite complex plane $\mathbb{C}$, $\partial\mathbb{D}$ the boundary of $\mathbb{D}$, $\mathbb{N}_0$ the set of all nonnegative integers and $\mathbb{N}$ the set of all positive integers. Let $\mu(z)$ be a positive continuous function on $\mathbb{D}$ (weight) such that $\mu(z) = \mu(|z|)$ and $n \in \mathbb{N}_0$. The nth weighted-type spaces on the unit disk $\mathbb{D}$, denoted by $\mathscr{W}_\mu^{(n)}(\mathbb{D})$ which were introduced in [25], consists of all $f \in H(\mathbb{D})$ such that

$$b_{\mathscr{W}_\mu^{(n)}(\mathbb{D})}(f) = \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < \infty.$$

For $n = 0$ the space becomes the weighted-type space $H_\mu^\infty(\mathbb{D})$ (see, e.g. [1, 2, 17, 20]), for $n = 1$ the Bloch-type space $\mathscr{B}_\mu(\mathbb{D})$ and for $n = 2$ the Zygmund-type space $\mathscr{Z}_\mu(\mathbb{D})$. For $\mu(z) = 1-|z|^2$ we obtain correspondingly the classical weighted-type space, the Bloch space $\mathscr{B}(\mathbb{D}) = \mathscr{B}$ and the Zygmund space $\mathscr{Z}(\mathbb{D}) = \mathscr{Z}$. Some information on Zygmund-type spaces on the unit disk and some operators on them e.g. in [9, 10, 14, 15, 37] and on the unit ball, can be found, e.g. in [11, 12, 24]. From now on we will assume that $n \in \mathbb{N}$. Set

$$\|f\|_{\mathscr{W}_\mu^{(n)}(\mathbb{D})} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + b_{\mathscr{W}_\mu^{(n)}(\mathbb{D})}(f).$$

With this norm the nth weighted-type space becomes a Banach space.

The little nth weighted-type space, denoted by $\mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$ is a closed subspace of $\mathscr{W}_\mu^{(n)}(\mathbb{D})$ consisting of those f for which

$$\lim_{|z|\to 1} \mu(z)|f^{(n)}(z)| = 0.$$

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A positive continuous function ϕ on $[0,1)$ is called a normal if there exist positive numbers a, b, $0<a<b$ and $t_0 \in [0,1)$, such that

$$\frac{\phi(t)}{(1-t^2)^a} \quad \text{decreases for} \quad t_0 \leqslant t < 1 \quad \text{and} \quad \lim_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^a} = 0,$$

$$\frac{\phi(t)}{(1-t^2)^b} \quad \text{increases for} \quad t_0 \leqslant t < 1 \quad \text{and} \quad \lim_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^b} = \infty$$

(see [18]).

For $0<p,q\leqslant\infty$ and ϕ normal, let $H(p,q,\phi)$ denote the space of all analytic functions f on the unit disk $\mathbb{D}$ such that

$$\|f\|_{p,q,\phi} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}rdr\right)^{1/p} < \infty,$$

when $p\in(0,\infty)$, and

$$\|f\|_{\infty,q,\phi} = \sup_{0<r<1} \phi(r)M_q(f,r) < \infty,$$

where the integral means $M_q(f,r)$ are defined by

$$M_q(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^q d\theta\right)^{1/q},$$

if $q\in(0,\infty)$ and

$$M_\infty(f,r) = \sup_{\theta\in[0,2\pi]} |f(re^{i\theta})|.$$

For $1\leqslant p<\infty$, $H(p,q,\phi)$ equipped with the norm $\|\cdot\|_{p,q,\phi}$ is a Banach space. When $0<p<1$, $\|\cdot\|_{p,q,\phi}$ is a quasinorm on $H(p,q,\phi)$, $H(p,q,\phi)$ is a Fréchet space but not a Banach space. If $0<p=q<\infty$, then $H(p,p,\phi)$ is the Bergman-type space

$$H(p,p,\phi) = \left\{ f\in H(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p \frac{\phi^p(|z|)}{1-|z|} dA(z) < \infty \right\},$$

where $dA(z)$ denotes the normalized Lebesgue area measure on the unit disk $\mathbb{D}$ such that $A(\mathbb{D})=1$. Note that if $\phi(r)=(1-r)^{(\alpha+1)/p}$, then $H(p,p,\phi)$ is the weighted Bergman space $A_\alpha^p(\mathbb{D})$ defined for $0<p<\infty$ and $\alpha>-1$, as the space of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha dA(z) < \infty$$

(see, for example, [7]). The multiplication operator M_u is defined by $M_u f = uf$. It is interesting to provide a function theoretic characterization of when u induces a bounded or compact composition operator on various spaces (see, for example, [3, 5, 6, 33, 34, 35]). S. Stević in [23] studied the boundedness and compactness of weighted composition operators from mixed norm spaces into weighted Bloch spaces. S. Stević in

[28] studied the boundedness and compactness of the composition operator from the Hardy space to nth weighted-type spaces on the unit disk. S. Stević in [29] studied the boundedness and compactness of the product of the differentiation and composition operator from the space of bounded analytic functions, the Bloch space and the little Bloch space to nth weighted-type spaces on the unit disk. S. Stević in [30] studied the boundedness and compactness of weighted differentiation composition operators from the space of bounded analytic functions, the Bloch space and the little Bloch space to nth weighted-type spaces on the unit disk. For some other products, containing composition operators, see, for example [4, 9, 10, 13, 21, 22, 24, 26, 27, 31, 32, 38]. We consider the boundedness and compactness of the operators M_u from $H(p,q,\phi)$ to nth weighted-type spaces on the unit disk.

From now on, we will always assume that $p,q \in (0,\infty]$, ϕ is normal and $n \in \mathbb{N}$. In addition, we will denote $a \asymp b$ whenever there exist two positive universal constants c and C, such that $cb \leqslant a \leqslant Cb$. Further, for the sake of simplicity, C will always denote an independent constant, which can be different from one display to another.

2. Auxiliary results

In this section we formulate some auxiliary results which will be used in the proofs of the main results.

LEMMA 1. ([22]) *Assume that $f \in H(p,q,\phi)$. Then for each $n \in \mathbb{N}_0$, there is a positive constant C independent of f such that*

$$\left|f^{(n)}(z)\right| \leqslant C\frac{\|f\|_{p,q,\phi}}{\phi(|z|)(1-|z|^2)^{1/q+n}}, z \in \mathbb{D}.$$

LEMMA 2. ([7]) *For any real β, let*

$$J_\beta(z) = \int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^{1+\beta}}, z \in \mathbb{D}.$$

Then we have

$$J_\beta(z) \asymp \begin{cases} 1 & , \textit{if } \beta<0, \\ \log\frac{1}{1-|z|^2} & , \textit{if } \beta=0, \\ \frac{1}{(1-|z|^2)^\beta} & , \textit{if } \beta>0, \end{cases} \qquad \textit{as } |z| \to 1^-.$$

LEMMA 3. ([18]) *For $\beta > -1$ and $\gamma > 1+\beta$ we have*

$$\int_0^1 \frac{(1-r)^\beta}{(1-r\rho)^\gamma} dr \leqslant C(1-\rho)^{1+\beta-\gamma}, \ 0<\rho<1.$$

LEMMA 4. ([25]) *Assume* $a > 0$ *and*

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & (a+1) & \cdots & (a+n-1) \\ a(a+1) & (a+1)(a+2) & \cdots & (a+n-1)(a+n) \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j=0}^{n-2}(a+j) & \prod_{j=0}^{n-2}(a+j+1) & \cdots & \prod_{j=0}^{n-2}(a+j+n-1) \end{vmatrix}$$

Then $D_n(a) = \prod_{j=0}^{n-1} j!$.

By standard arguments (see, for example, [4] or Lemma 3 in [19]) the following lemma follows.

LEMMA 5. *Assume that* $u \in H(\mathbb{D})$. *Then* $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ *is compact if and only if* $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ *is bounded and for any bounded sequence* $\{f_k\}$ *in* $H(p,q,\phi)$ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $k \to \infty$, *we have* $\|M_u f_k\|_{\mathscr{W}_\mu^{(n)}(\mathbb{D})} \to 0$ *as* $k \to \infty$.

The following lemma was proved in [8] similar to the corresponding lemma in [16]. Hence we omit it.

LEMMA 6. *Suppose* $n \in \mathbb{N}_0$ *and* μ *is normal. A closed set* K *in* $\mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$ *is compact if and only if* K *is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z) \left| f^{(n)}(z) \right| = 0.$$

3. The boundedness and compactness of M_u from $H(p,q,\phi)$ to nth weighted-type spaces on $\mathbb{D}$

In this section, we characterize the boundedness and compactness of $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ (or $\mathscr{W}_{\mu,0}^{(\mathrm{n})}(\mathbb{D})$).

THEOREM 1. *Assume that* $u \in H(\mathbb{D})$ *and* μ *is normal. Then* $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)\,|u(z)|}{\phi(|z|)(1-|z|^2)^{1/q+n}} < \infty. \tag{1}$$

Proof. Assume that conditions (1) holds. Then $\forall z \in \mathbb{D}$

$$|u(z)| \leqslant C \frac{\phi(|z|)(1-|z|^2)^{1/q+n}}{\mu(|z|)}. \tag{2}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|e^{i\theta}-z|^2}d\theta=\frac{1}{1-|z|^2},\forall z\in\mathbb{D},$$

let $\delta_z=\frac{1+|z|}{2}$, then we have $|\frac{z}{\delta_z}|=\frac{2|z|}{1+|z|}<1$, so

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|\delta_z e^{i\theta}-z|^2}d\theta=\frac{1}{\delta_z^2-|z|^2}.$$

By the Cauchy integral formula and (2) we obtain

$$\begin{aligned}|u'(z)|&=\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{u(\delta_z e^{i\theta})}{(\delta_z e^{i\theta}-z)^2}\delta_z e^{i\theta}d\theta\right|\\&\leqslant C\frac{\phi(\delta_z)(1-\delta_z^2)^{1/q+n}}{\mu(\delta_z)}\frac{1}{2\pi}\int_0^{2\pi}\frac{\delta_z}{|\delta_z e^{i\theta}-z|^2}d\theta\\&=C\frac{\phi(\delta_z)(1-\delta_z^2)^{1/q+n}}{\mu(\delta_z)}\frac{\delta_z}{\delta_z^2-|z|^2}\\&\leqslant C\frac{\phi(\delta_z)(1-\delta_z^2)^{1/q+n}}{\mu(\delta_z)}\frac{1}{1-\delta_z}.\end{aligned}$$

Note that

$$\frac{1}{2}(1-|z|)\leqslant 1-\delta_z^2=(1+\delta_z)(1-\delta_z)\leqslant(1-|z|)$$

and ϕ,μ are normal, we have

$$|u'(z)|\leqslant C\frac{\phi(|z|)(1-|z|^2)^{1/q+n-1}}{\mu(|z|)},$$

hence

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|u'(z)|}{\phi(|z|)(1-|z|^2)^{1/q+n-1}}<\infty. \tag{3}$$

Similarly, for $j\in\{2,3,\cdots,n\}$ we have that

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|u^{(j)}(z)\right|}{\phi(|z|)(1-|z|^2)^{1/q+n-j}}<\infty. \tag{4}$$

Hence, by Lemma 1, the Leibnitz formula, (1), (3) and (4) we have that

$$\begin{aligned}\left|\mu(z)(M_u f)^{(n)}(z)\right|&=\mu(z)\left|(u(z)f(z))^{(n)}\right|\\&=\mu(z)\left|\sum_{j=0}^{n}C_n^j u^{(j)}(z)f^{(n-j)}(z)\right|\\&\leqslant C\sum_{j=0}^{n}\frac{\left|C_n^j\mu(z)u^{(j)}(z)\right|}{\phi(|z|)(1-|z|^2)^{1/q+n-j}}\|f\|_{p,q,\phi}\\&\leqslant C\|f\|_{p,q,\phi},\end{aligned} \tag{5}$$

for every $z \in \mathbb{D}$ and $f \in H(p,q,\phi)$. We also have that

$$|(M_uf)(0)| = |u(0)f(0)| \leqslant C\frac{|u(0)|}{\phi(0)}\|f\|_{p,q,\phi}, \tag{6}$$

and for each $s \in \{1,2,\cdots,n-1\}$

$$\begin{aligned}\left|(M_uf)^{(s)}(0)\right| &= \left|\sum_{j=0}^{s} C_s^j u^{(j)}(0) f^{(s-j)}(0)\right| \\ &\leqslant \frac{C}{\phi(0)}\sum_{j=0}^{s}\left|C_s^j u^{(j)}(0)\right| \|f\|_{p,q,\phi}. \end{aligned}\tag{7}$$

Using (5), (6) and (7) it follows that the operator $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ is bounded.

On the other hand, suppose that $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ is bounded, that is there exists a constant C such that

$$\|M_uf\|_{\mathscr{W}_\mu^{(n)}(\mathbb{D})} \leqslant C\|f\|_{p,q,\phi}$$

for all $f \in H(p,q,\phi)$. For a fixed $w \in \mathbb{D}$, set

$$f_w(z) = \frac{(1-|w|^2)^{b+1}}{\phi(|w|)}\sum_{j=0}^{n}\frac{c_j(1-|w|^2)^j}{(1-\overline{w}z)^{\alpha+j}}, \tag{8}$$

where the constant b is from the definition of the normality of the function ϕ, $\alpha = 1/q+b+1$ and $c_j\,(j=0,1,\cdots,n)$ are left undetermined.

By Lemma 2, we have

$$M_q(f_w,r) \leqslant C\frac{(1-|w|^2)^{b+1}}{\phi(|w|)(1-r|w|)^{b+1}}.$$

As ϕ is normal and by applying Lemma 3, we obtain (see [21, 22])

$$\sup_{w\in\mathbb{D}}\|f_w\|_{p,q,\phi} \leqslant C. \tag{9}$$

A straightforward calculation shows that

$$f_w^{(n)}(z) = \frac{(1-|w|^2)^{b+1}(\overline{w})^n}{\phi(|w|)}\sum_{j=0}^{n}\frac{c_j(\alpha+j)\cdots(\alpha+j+n-1)(1-|w|^2)^j}{(1-\overline{w}z)^{\alpha+j+n}}.$$

Thus

$$f_w^{(n)}(w) = \frac{(\overline{w})^n}{\phi(|w|)(1-|w|^2)^{1/q+n}}\sum_{j=0}^{n}c_j(\alpha+j)\cdots(\alpha+j+n-1).$$

By Lemma 4, using the same method in [31] we can choose $c_j \ (j=0,1,\cdots,n)$, the corresponding function is denoted by f_w, such that

$$f_w^{(n)}(w) = \frac{(\overline{w})^n}{\phi(|w|)(1-|w|^2)^{1/q+n}}, \tag{10}$$

and for $t \in \{0,1,\cdots,n-1\}$

$$f_w^{(t)}(w) = 0. \tag{11}$$

Therefore

$$\begin{aligned} C \geqslant \|M_u f_w\|_{\mathscr{W}_\mu^{(n)}(\mathbb{D})} &\geqslant \mu(w)\left|u(w) f_w^{(n)}(w)\right| \\ &= \frac{\mu(w)|w|^n\,|u(w)|}{\phi(|w|)(1-|w|^2)^{1/q+n}}. \end{aligned}$$

From this we obtain

$$\sup_{|w|>1/2} \frac{\mu(w)\,|u(w)|}{\phi(|w|)(1-|w|^2)^{1/q+n}} \leqslant C. \tag{12}$$

Since ϕ is normal, $\mu(z)$ is a positive continuous function on $\mathbb{D}$, $u \in H(\mathbb{D})$ and the fact $M_u(1) = u \in \mathscr{W}_\mu^{(n)}(\mathbb{D})$, we get

$$\sup_{|w|\leqslant 1/2} \frac{\mu(w)\,|u(w)|}{\phi(|w|)(1-|w|^2)^{1/q+n}} \leqslant C \sup_{|w|\leqslant 1/2} \mu(w)\,|u(w)| \leqslant C,$$

which along with (12) implies that (1), finishing the proof of the theorem. □

THEOREM 2. *Assume that $u \in H(\mathbb{D})$ and μ is normal. Then $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(z)\,|u(z)|}{\phi(|z|)(1-|z|^2)^{1/q+n}} = 0. \tag{13}$$

Proof. Assume that conditions (13) holds. By Theorem 1, $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ is bounded. For any bounded sequence $\{f_k\}$ in $H(p,q,\phi)$ with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. To establish the assertion, it suffices, in view of Lemma 5, to show that

$$\|M_u f_k\|_{\mathscr{W}_\mu^{(n)}(\mathbb{D})} \to 0 \ \text{ as } \ k \to \infty.$$

We assume that $\|f_k\|_{p,q,\phi} \leqslant 1$. From (13), given $\varepsilon > 0$, there exists a $\delta \in (0,1)$, when $\delta < |z| < 1$, we have

$$\frac{\mu(z)\,|u(z)|}{\phi(|z|)(1-|z|^2)^{1/q+n}} < \varepsilon, \tag{14}$$

so by the Cauchy integral formula we obtain

$$
\begin{aligned}
|u'(z)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{u(\delta_z e^{i\theta})}{(\delta_z e^{i\theta}-z)^2}\delta_z e^{i\theta} d\theta\right| \\
&< \varepsilon \frac{\phi(\delta_z)(1-\delta_z^2)^{1/q+n}}{\mu(\delta_z)} \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta_z}{|\delta_z e^{i\theta}-z|^2} d\theta \\
&= \varepsilon \frac{\phi(\delta_z)(1-\delta_z^2)^{1/q+n}}{\mu(\delta_z)} \frac{\delta_z}{\delta_z^2-|z|^2} \\
&\leqslant \varepsilon C \frac{\phi(|z|)(1-|z|^2)^{1/q+n-1}}{\mu(z)},
\end{aligned}
$$

when $\delta < |z| < 1$. Hence

$$
\frac{\mu(z)|u'(z)|}{\phi(|z|)(1-|z|^2)^{1/q+n-1}} < C\varepsilon,
$$

when $\tau < |z| < 1$. Similarly, for $j \in \{2,\cdots,n\}$ we have that

$$
\frac{\mu(z)\left|u^{(j)}(z)\right|}{\phi(|z|)(1-|z|^2)^{1/q+n-j}} < C\varepsilon, \tag{15}
$$

when $\delta < |z| < 1$. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that $f_k^{(j)}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ for each $j \in \{0,1,\cdots,n\}$, there exists a $K_0 \in \mathbb{N}$ such that $k > K_0$ implies that

$$
\sum_{j=0}^{n-1}\sum_{m=0}^{j-1} C_j^m \left|u^{(m)}(0) f_k^{(j-m)}(0)\right| + \sup_{|z|\leqslant\delta} \mu(z)\left|\sum_{j=0}^{n} C_n^j u^{(j)}(z) f_k^{(n-j)}(z)\right| < C\varepsilon. \tag{16}
$$

From (14-16) and Lemma 1 we have

$$
\begin{aligned}
\|M_u f_k\|_{\mathscr{W}_\mu^{(n)}(\mathbb{D})} &= \sum_{j=0}^{n-1}\left|(uf_k)^{(j)}(0)\right| + \sup_{z\in\mathbb{D}} \mu(z)\left|(uf_k)^{(n)}(z)\right| \\
&\leqslant \sum_{j=0}^{n-1}\sum_{m=0}^{j-1} C_j^m \left|u^{(m)}(0) f_k^{(j-m)}(0)\right| + \sup_{|z|\leqslant\delta} \mu(z)\left|\sum_{j=0}^{n} C_n^j u^{(j)}(z) f_k^{(n-j)}(z)\right| \\
&\quad + \sup_{|z|>\delta} \mu(z)\left|\sum_{j=0}^{n} C_n^j u^{(j)}(z) f_k^{(n-j)}(z)\right| \\
&< C\varepsilon + C\sup_{|z|>\delta}\sum_{j=0}^{n}\left|\frac{C_n^j \mu(z) u^{(j)}(z)}{\phi(|z|)(1-|z|^2)^{1/q+n-j}}\right| < \left((1 + C\sum_{j=0}^{n} C_n^j\right) C\varepsilon,
\end{aligned}
$$

when $k > K_0$. It follows that the operator $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ is compact.

Conversely, assume that $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ is compact. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|z_k| \to 1$ as $k \to \infty$. Taking the test functions f_{z_k}, where f_{z_k} is defined by (8), we shall write

$$f_k(z) = f_{z_k}(z).$$

From (9), (10) and (11) we have

$$\sup_{k\in\mathbb{N}} \|f_k\|_{p,q,\phi} \leqslant C$$

and for $t \in \{0,1,\cdots,n-1\}$

$$f_k^{(n)}(z_k) = \frac{(\overline{z_k})^n}{\phi(|z_k|)(1-|z_k|^2)^{1/q+n}},\ f_k^{(t)}(z_k) = 0.$$

For $|z| = r < 1$, using the fact that ϕ is normal, we have

$$|f_k(z)| \leqslant \frac{C}{(1-r)^{1/q+b+1}}(1-|z_k|) \to 0 \ (\text{as } k \to \infty),$$

that is, f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, using the compactness of $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$, we obtain

$$\frac{\mu(z_k)\,|u(z_k)|\,|z_k|^n}{\phi(|z_k|)(1-|z_k|^2)^{1/q+n}} \leqslant C\|M_u f_k\|_{\mathscr{W}_\mu^{(n)}(\mathbb{D})} \to 0 \quad \text{as} \quad k \to \infty,$$

and consequently (13). □

THEOREM 3. *Assume that $u \in H(\mathbb{D})$ and μ is normal. Then $M_u : H(p,q,\phi) \to \mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded if and only if $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ is bounded and*

$$\lim_{|z|\to 1} \mu(z)\left|u^{(n)}(z)\right| = 0. \tag{17}$$

Proof. Assume that $M_u : H(p,q,\phi) \to \mathscr{W}_\mu^{(n)}(\mathbb{D})$ is bounded and condition (17) holds. Since

$$u(z) - u(0) = \int_0^1 xu'(zx)dx,$$

it follows that

$$\begin{aligned}\mu(z)|u(z)| &\leqslant \mu(z)|u(0)| + \mu(z)\left|\int_0^1 xu'(zx)dx\right| \\ &\leqslant \mu(z)|u(0)| + \mu(z)\left|\int_0^{1/2} xu'(zx)dx\right| + \mu(z)\left|\int_{1/2}^1 xu'(zx)dx\right| \\ &\leqslant \mu(z)|u(0)| + \mu(z)\max_{|z|\leqslant 1/2}\left|u'(z)\right| + \mu(z)\left|\int_{1/2}^1 xu'(zx)dx\right|. \end{aligned}\tag{18}$$

Since μ is normal, by the monotonicity of $\frac{\mu(t)}{(1-t^2)^a}$, for $t_0 \leqslant t_1 < t < 1$, we have

$$\mu(t) = (1-t^2)^a \frac{\mu(t)}{(1-t^2)^a} \leqslant (1-t^2)^a \frac{\mu(t_1)}{(1-t_1^2)^a} < \mu(t_1),$$

that is, μ is decreasing on $[t_0,1)$, and for any $\varepsilon>0$, there is a $\tau>0$ such that

$$0<\mu(|z|)<\varepsilon(1-|z|^2)^a,\ \ (\tau<|z|<1),$$

which implies $\lim_{|z|\to 1}\mu(|z|)=0$. Since for $1/2<x<1$ and $2t_0<|z|<1$ we have $\mu(z)\leqslant\mu(xz)$. From (18) it follows that

$$\mu(z)|u(z)|\leqslant\mu(z)|u(0)|+\mu(z)\max_{|z|\leqslant 1/2}\left|u'(z)\right|+\int_{1/2}^{1}\mu(zx)\left|u'(zx)\right|dx. \tag{19}$$

For $j\in\{2,3,\cdots,n\}$ by applying formula (19) to the function $u^{(j-1)}$ we get

$$\mu(z)\left|u^{(j-1)}(z)\right|\leqslant\mu(z)\left|u^{(j-1)}(0)\right|+\mu(z)\max_{|z|\leqslant 1/2}\left|u^{(j)}(z)\right|+\int_{1/2}^{1}\mu(zx)\left|u^{(j)}(zx)\right|dx, \tag{20}$$

when $2t_0<|z|<1$. It follows from (17), (19) and (20) that $\lim\limits_{|z|\to 1}\mu(z)|u^{(j-1)}(z)|=0$ for $j\in\{2,3,\cdots,n\}$. Since for each polynomial p, we have

$$\begin{aligned}\left|\mu(z)\left(M_u p\right)^{(n)}(z)\right| &= \mu(z)\left|(u(z)p(z))^{(n)}\right| \\ &= \mu(z)\left|\sum_{j=0}^{n}C_n^j u^{(j)}(z)p^{(n-j)}(z)\right|\leqslant C\sum_{j=0}^{n}\left|C_n^j\mu(z)u^{(j)}(z)\right|\|p^{(n-j)}\|_\infty,\end{aligned}$$

hence $M_u p\in\mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$. Since the set of all polynomials is dense in $H(p,q,\phi)$ we have that for every $f\in H(p,q,\phi)$ there is a sequence of polynomials $\{p_k\}$ such that

$$\lim_{k\to\infty}\|p_k-f\|_{p,q,\phi}=0.$$

From this and since the operator $M_u:H(p,q,\phi)\to\mathscr{W}_{\mu}^{(n)}(\mathbb{D})$ is bounded, we have that

$$\|M_u p_k-M_u f\|_{\mathscr{W}_{\mu}^{(n)}(\mathbb{D})}\leqslant\|M_u\|\|p_k-f\|_{p,q,\phi}\to 0 \text{ as } k\to\infty.$$

Since $\mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$ is a closed subspace of $\mathscr{W}_{\mu}^{(n)}(\mathbb{D})$, therefore, we have $M_u(H(p,q,\phi))\subset\mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$, from which the boundedness of $M_u:H(p,q,\phi)\to\mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$ follows.

On the other hand, assume that $M_u:H(p,q,\phi)\to\mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$ is bounded, then $M_u:H(p,q,\phi)\to\mathscr{W}_{\mu}^{(n)}(\mathbb{D})$ is bounded. By taking the function given by $f(z)=1$, we obtain

$$\mu(z)\left|u^{(n)}(z)\right|=\left|\mu(z)(M_u f)^{(n)}(z)\right|\to 0\ (\text{as}\,|z|\to 1),$$

as desired. □

THEOREM 4. *Assume that $u\in H(\mathbb{D})$ and μ is normal. Then $M_u:H(p,q,\phi)\to\mathscr{W}_{\mu,0}^{(n)}(\mathbb{D})$ is compact if and only if $M_u:H(p,q,\phi)\to\mathscr{W}_{\mu}^{(n)}(\mathbb{D})$ is compact.*

Proof. First assume that $M_u:H(p,q,\phi)\to\mathscr{W}_{\mu}^{(n)}(\mathbb{D})$ is compact, then by Theorem 2, condition (13) holds. Taking the supremum in inequality (5) over all $f\in H(p,q,\phi)$ such that $\|f\|_{p,q,\phi}\leqslant 1$ and letting $|z|\to 1$, yields

$$\lim_{|z|\to 1}\sup_{\|f\|_{p,q,\phi}\leqslant 1}\mu(z)\left|(M_uf)^{(n)}(z)\right|=0.$$

Hence, by Lemma 6 we see that the operator $M_u : H(p,q,\phi) \to \mathscr{W}^{(n)}_{\mu,0}(\mathbb{D})$ is compact.

Conversely, the compactness of $M_u : H(p,q,\phi) \to \mathscr{W}^{(n)}_{\mu,0}(\mathbb{D})$ implies the compactness of $M_u : H(p,q,\phi) \to \mathscr{W}^{(n)}_{\mu}(\mathbb{D})$ is obvious. The proof is completed. □

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