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OPPENHEIM'S INEQUALITY AND RKHS

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Abstract. Applying norm inequalities for RKHSs corresponding to the product of reproducing kernels and using the minimal norm of the Nevanlinna interpolation, we give the basic background and essences of the quite famous fundamental inequalities, Oppenheim's inequality, Hadamard's inequality and Schur's inequality on positive semidefinite matrices. In particular, as an application, we determine equality conditions for Oppenheim's inequality and Schur's inequality.

1. Introduction

Oppenheim's inequality [4]: for any positive semidefinite complex matrices A and B, we have

$$|A \circ B| \geqslant |A| b_{11} \cdots b_{nn}, \tag{1.1}$$

where $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n$, and $A \circ B = (a_{ij}b_{ij})_{i,j=1}^n$ is the *Hadamard product* of A and B. Our aim is to derive equality conditions for Oppenheim's inequality by using the theory of kernel functions.

A function $k\colon E \times E \to \mathbb{C}$ is called a *positive definite kernel* on the set E if, for any finite sequence $\{x_i\}_{i=1}^n \subset E$ and for any complex numbers ξ_i $(i = 1, \ldots, n)$, k satisfies the inequality

$$\sum_{i=1}^n k(x_i, x_j)\xi_i \overline{\xi}_j \geqslant 0.$$

One verifies easily that the reproducing kernel of a *reproducing kernel Hilbert space* (RKHS) on E is a positive definite kernel on E. The converse to this fact is important. Indeed, it is well-known that, for each positive definite kernel k on E, there exists a unique RKHS H_k on E whose reproducing kernel is k. By Schur's theorem the product of two positive definite kernels on E is also a positive definite kernel on E. Thus, if H_{k^1} and H_{k^2} are RKHSs on E, then $H_{k^1k^2}$ is a well-defined RKHS on E and the following norm inequality holds: for every $f \in H_{k^1}$ and $g \in H_{k^2}$, the product fg belongs to $H_{k^1k^2}$ and satisfies

$$\|fg\|_{k^1k^2} \leqslant \|f\|_{k^1} \|g\|_{k^2}. \tag{1.2}$$

Here $\|\cdot\|_k$ denotes the norm of H_k. The inequality (1.2) and its equality condition are the main tool of our paper. For the general theory of reproducing kernels, the reader is referred to [1, 5].

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2. Positive semidefinite matrix and its RKHS

Setting $a(i,j) = a_{ij}$, we may regard any positive semidefinite matrix $A = (a_{ij}) \in M_n$ as a positive definite kernel $a(i,j)$ on the set $\{1,\ldots,n\}$, where M_n is the set of $n \times n$ complex matrices. Moreover, we identify a column vector $(x_i)_{i=1}^n$ with a function $x(i) = x_i$ on $\{1,\ldots,n\}$. With these identifications it is interesting to know a concrete description of the RKHS H_A. We summarize well-known facts about H_A as follows (cf. [5, pp. 13–14]):

PROPOSITION 2.1. *Let $A = (a_{ij})$ be an $n \times n$ positive semidefinite complex matrix. By identifying the matrix A as a positive definite kernel on $\{1,\ldots,n\}$, the RKHS H_A on $\{1,\ldots,n\}$ is the vector space* $\operatorname{ran} A$ *equipped with the inner product*

$$\langle Ax, Ay\rangle = \sum_{i,j=1}^{n} x_i \overline{y}_j a_{ji}$$

for ${}^t x = (x_1 \ \cdots \ x_n) \in \mathbb{C}^n$ and ${}^t y = (y_1 \ \cdots \ y_n) \in \mathbb{C}^n$. The i-th column vector of A is the reproducing kernel of H_A at i, $(i = 1,\ldots,n)$.

Proof. We only note that from the identity

$$\langle Ax, Ay\rangle = \langle Ax, y\rangle_0 = \langle x, Ay\rangle_0,$$

the inner product is well-defined and positive definite, where $\langle\cdot,\cdot\rangle_0$ denotes the standard inner product of $\mathbb{C}^n$. The rest of the proof is omitted. □

From now on the reproducing kernel of H_A at i is denoted by k_i^A. Thus, $A = (k_1^A \ k_2^A \ \ldots \ k_n^A)$.

We next show an analogue of Bergman's formula for minimal integrals [2, p. 26].

PROPOSITION 2.2. *Let $\{x_j\}_{j=1}^n$ be a linearly independent subset of a complex Hilbert space H. Then, for any complex numbers $\{b_j\}_{j=1}^n$, there exists a unique element $f \in H$ which satisfies*

$$\langle f, x_j\rangle = b_j, \quad j = 1,\ldots,n, \tag{2.1}$$

and minimizes the norm $\|f\|$. Moreover, if f_n is the element which satisfies (2.1) *and minimizes the norm, then*

$$f_n = -\frac{1}{G_n}\begin{vmatrix} 0 & x_1 & \ldots & x_n \\ b_1 & \langle x_1, x_1\rangle & \ldots & \langle x_n, x_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \langle x_1, x_n\rangle & \ldots & \langle x_n, x_n\rangle \end{vmatrix}, \tag{2.2}$$

and

$$\|f_n\|^2 = -\frac{1}{G_n}\begin{vmatrix} 0 & \overline{b}_1 & \dots & \overline{b}_n \\ b_1 & \langle x_1,x_1\rangle & \dots & \langle x_n,x_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \langle x_1,x_n\rangle & \dots & \langle x_n,x_n\rangle \end{vmatrix},$$

where $G_n = \det(\langle x_j,x_i\rangle)_{i,j=1}^n$ *is the Gramian of* $\{x_j\}_{j=1}^n$.

Proof. Note that $G_n > 0$ since the elements $\{x_j\}_{j=1}^n$ are linearly independent. From (2.2) we obtain

$$\langle f_n,x_j\rangle = -\frac{1}{G_n}\begin{vmatrix} 0 & \langle x_1,x_j\rangle & \dots & \langle x_n,x_j\rangle \\ b_1 & \langle x_1,x_1\rangle & \dots & \langle x_n,x_1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \langle x_1,x_n\rangle & \dots & \langle x_n,x_n\rangle \end{vmatrix},$$

which immediately implies that f_n satisfies (2.1). For any element g of H which satisfies (2.1), we have $(g-f_n)\perp f_n$, since f_n is a linear combination of the elements $\{x_j\}_{j=1}^n$. Hence

$$\|g\|^2 = \|f_n\|^2 + \|g-f_n\|^2,$$

so that f_n is the unique element which satisfies (2.1) and minimizes the norm. □

REMARK 2.3. When $\{a_j\}$ is a sequence in a RKHS H_k, putting $x_j = k_{a_j}$ we see that the conditions (2.1) are rewritten as $f(a_j) = b_j$. Hence we may consider (2.1) as a Nevanlinna interpolation problem.

COROLLARY 2.4. *Let* $\{x_j\}_{j=1}^n \subset H$ *be linearly independent, and set* $b_1 = \ldots = b_{n-1} = 0$, *and* $b_n = 1$. *If* f_n *is the solution to* (2.1) *which minimizes the norm, then we have*

1. $\|f_n\|^2 = G_{n-1}/G_n$, *and*
2. $f_n = \Phi_n/G_n$,

where G_k $(k=1,\ldots,n)$ *is the Gramian of* $\{x_j\}_{j=1}^k$ ($G_0 = 1$), *and*

$$\Phi_n = \begin{vmatrix} \langle x_1,x_1\rangle & \dots & \langle x_n,x_1\rangle \\ \vdots & \ddots & \vdots \\ \langle x_1,x_{n-1}\rangle & \dots & \langle x_n,x_{n-1}\rangle \\ x_1 & \dots & x_n \end{vmatrix}.$$

3. Equality of Oppenheim's inequality

From the minimal solution to the Nevanlinna interpolation problem on a RKHS, we shall obtain necessary and sufficient conditions for equality of Oppenheim's inequality (cf. [7]). To this end we first recall the norm inequality concerning the tensor product RKHS $H_{k^1} \otimes H_{k^2}$ and the RKHS $H_{k^1 k^2}$. Let H_{k^j} $(j = 1,2)$ be RKHSs on E. Then the tensor product Hilbert space $H_{k^1} \otimes H_{k^2}$ is a RKHS on $E \times E$ whose reproducing kernel at $(x,y) \in E \times E$ is $k_x^1 \otimes k_y^2$, where k_x^j $(j = 1,2)$ denotes the reproducing kernel of H_{k^j} at $x \in E$. Now we have the following inequality (see [1, 5]): for any $f \in H_{k^1} \otimes H_{k^2}$,

$$\|f \circ \iota\|_{k^1 k^2} \leqslant \|f\|_{k^1 \otimes k^2}, \tag{3.1}$$

where the map $\iota \colon E \to E \times E$ denotes the natural inclusion of E to the diagonal of $E \times E$, that is, $\iota(x) = (x,x)$ for all $x \in E$.

DEFINITION 3.1. If equality holds in the above inequality (3.1), the element $f \in H_{k^1} \otimes H_{k^2}$ is called *extremal* ([6]).

LEMMA 3.2. *A function $f \in H_{k^1} \otimes H_{k^2}$ on $E \times E$ is extremal if and only if f belongs to the closed span of the set $\{k_x^1 \otimes k_x^2\}_{x \in E}$.*

Proof. Let H_0 be the subspace of functions f in $H_{k^1} \otimes H_{k^2}$ on $E \times E$ with $f \circ \iota = 0$ on E. By the reproducing property of the kernel function $k^1 \otimes k^2$, we obtain

$$H_0 = (\{k_x^1 \otimes k_x^2\}_{x \in E})^{\perp}.$$

But it is well-known that f is extremal if and only if $f \in H_0^{\perp}$ ([6]). Since $H_0^{\perp} = (\{k_x^1 \otimes k_x^2\}_{x \in E})^{\perp\perp}$, the function f must belong to the closed span of the set $\{k_x^1 \otimes k_x^2\}_{x \in E}$. $\square$

We use the following notation. For x, y in a complex vector space, we write $x \sim y$ if there exists a nonzero constant $\alpha \in \mathbb{C}$ with $x = \alpha y$. For a positive semidefinite matrix $X \in M_n$, if the set of solutions to the interpolation problem

$$f(1) = \ldots = f(m-1) = 0,\ f(m) = 1 \tag{3.2}$$

is nonempty for a RKHS H_X, let λ_m^X $(m = 1, \ldots, n)$ be the minimal norm of such solutions.

LEMMA 3.3. *Assume that $A \in M_n$ is positive definite and that $B \in M_n$ is positive semidefinite with $b_{mm} > 0$ $(m = 1, \ldots, n)$. Then, for $m = 1, \ldots, n$,* (3.2) *has a solution in $H_{A \circ B}$ and the following inequality holds:*

$$\lambda_m^{A \circ B} \leqslant \lambda_m^A / \sqrt{b_{mm}}. \tag{3.3}$$

Equality holds for (3.3) *if and only if the solution $f_m \in H_A$ which satisfies* (3.2) *and minimizes the norm is a linear combination of $\{k_i^A : k_i^B \sim k_m^B, 1 \leqslant i \leqslant m\}$.*

Proof. Putting $h = k_m^B/b_{mm}$, we have $h(m) = 1$ and $\|h\| = 1/\sqrt{b_{mm}}$. Hence the function $f_m h$ on $\{1,\ldots,n\}$ clearly satisfies the interpolation condition (3.2). Since $f_m h$ is obtained from the function $f_m \otimes h \in H_A \otimes H_B$ on $\{1,\ldots,n\} \times \{1,\ldots,n\}$ as $f_m h = (f_m \otimes h) \circ \iota$, the minimum property of $\lambda_m^{A\circ B}$ and the inequality (3.1) imply

$$\lambda_m^{A\circ B} \leqslant \|f_m h\| \leqslant \|f_m \otimes h\| = \|f_m\|\|h\| = \lambda_m^A/\sqrt{b_{mm}},$$

which gives the inequality (3.3).

Now assume that equality holds in (3.3). First we note that f_m is a linear combination of the set k_i^A $(1 \leqslant i \leqslant m)$, since f_m is the minimum solution to (3.2):

$$f_m = \sum_{i=1}^{m} c_i k_i^A. \tag{3.4}$$

Equality in (3.3) implies that $f_m \otimes h$ is extremal in $H_A \otimes H_B$, so we conclude from Lemma 3.2 that there exist complex numbers $\alpha_i \in \mathbb{C}$, $i = 1,\ldots,n$ such that

$$f_m \otimes h = \sum_{i=1}^{n} \alpha_i k_i^A \otimes k_i^B. \tag{3.5}$$

By (3.4) we obtain $f_m \otimes h = (b_{mm})^{-1} \sum_{i=1}^{m} c_i k_i^A \otimes k_m^B$. We now make use of an elementary fact that if $\{x_i\}$ and $\{y_j\}$ are linearly independent subsets of vector spaces V_1 and V_2 respectively, then the set of their tensor products $\{x_i \otimes y_j\}$ is also linearly independent in $V_1 \otimes V_2$. Indeed, by $|A| > 0$, the set $\{k_i^A\}_{i=1}^n$ forms a basis for H_A. As for H_B take a basis containing k_m^B. Then, expanding both sides of (3.5) by means of these bases and comparing coefficients, we conclude that $\alpha_i = 0$ for all $i > m$.

If $k_j^B \not\sim k_m^B$ $(j < m)$, then there exists a basis for H_B which contains k_j^B and k_m^B. Again, expanding the right-hand side of (3.5) by using these bases, and comparing coefficients, we see that $\alpha_j = 0$. Therefore, putting $I_m = \{i\colon k_i^B \sim k_m^B,\ 1 \leqslant i \leqslant m\}$, we have

$$f_m \otimes h = \sum_{i \in I_m} \alpha_i k_i^A \otimes k_i^B = \Big(\sum_{i \in I_m} c_i k_i^A\Big) \otimes h,$$

which implies that f_m is of the form as desired.

Conversely, assume that f_m is a linear combination of $\{k_i^A : k_i^B \sim k_m^B, 1 \leqslant i \leqslant m\}$. Then, it is clear that $f_m \otimes h$ is extremal by Lemma 3.2. The function $f_m h$ satisfies the interpolation condition (3.2), and is a linear combination of $k_i^A k_i^B = k_i^{A\circ B}$ $(i = 1,\ldots,m)$. Hence $f_m h$ is the solution with minimum norm in $H_{A\circ B}$. Thus equality holds in this case. □

For $A = (a_{ij}) \in M_n$, if σ is a permutation in the symmetric group S_n, we define the matrix A^σ by $A^\sigma = (a_{\sigma(i)\sigma(j)})$. When σ is the transposition $(i\ j)$ $(1 \leqslant i,j \leqslant n)$, the matrix A^σ is obtained from the matrix A by swapping i-th and j-th rows and, simultaneously, i-th and j-th columns. We call such operations of a matrix by *simultaneous exchanges of rows and columns*. In terms of this terminology, A^σ is obtained from the matrix A by a finite number of simultaneous exchanges. One verifies easily the following:

1. A is positive semidefinite if and only if A^σ is positive semidefinite.
2. $|A| = |A^\sigma|$.
3. The set of diagonal entries of A coincides with that of A^σ.

We remark that if

(a) the matrix A is diagonal, or

(b) the matrix B is of rank one,

then equality holds in Oppenheim's inequality. Indeed, if (a) holds, this is trivial. If (b) holds, since B is of rank one and positive semidefinite, B is of the form $B = (w_i\overline{w}_j)$ for some $(w_i) \in \mathbb{C}^n$. Thus $|A \circ B| = \det(a_{ij}w_i\overline{w}_j) = |A||w_1 \cdots w_n|^2 = |A|b_{11}\cdots b_{nn}$, as desired.

Our main theorem asserts that the condition for equality of Oppenheim's inequality is in general a blend of two conditions (a) and (b) stated in the above remark.

THEOREM 3.4. *If complex matrices $A, B \in M_n$ are positive semidefinite, then the following are equivalent:*

1. *Equality holds in Oppenheim's inequality* (1.1).
2. *$A \circ B$ is singular, or there exists $\sigma \in S_n$ such that A^σ is block diagonal, i.e.*

$$A^\sigma = \begin{pmatrix} A_{11} & & & \mathbf{0} \\ & A_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & A_{pp} \end{pmatrix},$$

with $A_{ii} \in M_{n_i}$ $(i = 1, \dots, p)$, $n_1 + \dots + n_p = n$, and that B^σ satisfies

$$k_1^{B^\sigma} \sim \dots \sim k_{n_1}^{B^\sigma},\ k_{n_1+1}^{B^\sigma} \sim \dots \sim k_{n_1+n_2}^{B^\sigma},\ \dots,\ k_{n_1+\dots+n_{p-1}+1}^{B^\sigma} \sim \dots \sim k_n^{B^\sigma}.$$

3. *$A \circ B$ is singular, or there exists $B' = (b'_{ij}) \in M_n$ such that*
   (a) B' is positive semidefinite and of rank one,
   (b) $A \circ B = A \circ B'$, and
   (c) $b'_{ii} = b_{ii}$ $(i = 1, \dots, n)$.
4. *$A \circ B$ is singular, or there exists a diagonal matrix $T = \operatorname{diag}(w_1, \dots, w_n)$ such that*
   (a) $A \circ B = TAT^*$, and
   (b) $|w_i|^2 = b_{ii}$ $(i = 1, \dots, n)$.

Proof. (1) $\Longrightarrow$ (2): We may assume that $A \circ B$ is nonsingular. Then, (1) implies that A is nonsingular and $b_{ii} = \|k_i^B\|^2 > 0$ for all $i = 1, \dots, n$. Since k_i^A and $k_i^{A\circ B}$ are column vectors of the matrices A and $A \circ B$ respectively, the set of vectors $\{k_i^A\}_{i=1}^n$ and

$\{k_i^{A\circ B}\}_{i=1}^n$ are both linearly independent, and $k_i^B \neq 0$ for all $i = 1,\ldots,n$. Therefore, applying Corollary 2.4 with $x_i = k_i^{A\circ B}$ and with $x_i = k_i^A$, we obtain

$$|A\circ B| = G_n = \frac{G_n}{G_{n-1}} \cdot \frac{G_{n-1}}{G_{n-2}} \cdots G_1 = (\lambda_n^{A\circ B}\lambda_{n-1}^{A\circ B}\cdots\lambda_1^{A\circ B})^{-2},$$

and

$$|A| = (\lambda_n^A\lambda_{n-1}^A\cdots\lambda_1^A)^{-2}.$$

By Lemma 3.3 these identities immediately imply Oppenheim's inequality (1.1). Since equality occurs in (1.1), we conclude that equality must occur in (3.3) for all $m = 1,\ldots,n$. Now we claim that, for $1 \leqslant i,j \leqslant n$, if $k_i^B \not\sim k_j^B$, then $k_i^A \perp k_j^A$. This is proved as follows. As we remarked above, by the invariance property of simultaneous exchanges of columns and rows, equality holds in Oppenheim's inequality for the pair of matrices $(A',B') = (A^\sigma, B^\sigma)$ for any $\sigma \in S_n$ whenever equality holds for the pair (A,B). Thus, taking a permutation $\sigma \in S_n$ with $\sigma(1) = i$ and $\sigma(2) = j$, we may assume without loss of generality that $i = 1$ and $j = 2$. Then, equality must hold for $m = 2$ of the inequality (3.3) of Lemma 3.3. Since $k_i^B \not\sim k_j^B$ if and only if $k_1^{B'} \not\sim k_2^{B'}$, we conclude that $k_2^{A'} \sim f_2$ where f_2 is the solution with minimum norm for the interpolation problem (3.2). Hence, we have $k_2^{A'}(1) = f_2(1) = 0$. Thus,

$$\langle k_j^A, k_i^A\rangle = a_{ij} = a_{\sigma(1)\sigma(2)} = \langle k_2^{A'}, k_1^{A'}\rangle = k_2^{A'}(1) = 0,$$

so that our claim is proved. Thus, choosing a suitable $\sigma \in S_n$ such that B^σ is of the form as in (2), we see that A^σ is a block diagonal matrix of the form stated in our Theorem.

(2) $\Longrightarrow$ (3): Assume that $A\circ B$ is nonsingular. By invariance of simultaneous exchanges of columns and rows we may assume without loss of generality that $\sigma = id$, so that A is block diagonal and B is of the form described in (2). Then all the corresponding diagonal blocks B_{ii}, $i = 1,\ldots,p$, of the matrix B are of rank one. Hence, there exists a complex vector $(w_i) \in \mathbb{C}^n$ such that

$$b_{ij} = w_i\overline{w}_j \quad (n_1 + \ldots + n_{l-1} + 1 \leqslant i,j \leqslant n_1 + \ldots + n_l,\ l = 1,\ldots,p).$$

Put $B' = (w_i\overline{w}_j) \in M_n$. Then B' is positive semidefinite of rank one and $b_{ii} = |w_i|^2$ $(i = 1,\ldots,n)$. Since A is block diagonal, it is clear that $A\circ B = A\circ B'$.

(3) $\Longrightarrow$ (4): If $A\circ B$ is singular, there is nothing to prove. Otherwise, B' is of the form $B' = (w_i\overline{w}_j)_{i,j=1}^n$. Put $T = \operatorname{diag}(w_1,\ldots,w_n)$. Then, clearly, $A\circ B' = TAT^*$.

(4) $\Longrightarrow$ (1): If $A\circ B$ is singular, then obviously equality holds. Otherwise, from (a) and (b) we have $|A\circ B| = |A||T|^2 = |A|b_{11}\ldots b_{nn}$. Thus equality holds in Oppenheim's inequality. $\square$

REMARK 3.5. By the reproducing property and Schwarz's inequality we obtain $\lambda_m^B \geqslant 1/\sqrt{b_{mm}}$. As in the above proof this inequality immediately yields Hadamard's inequality for positive semidefinite matrix:

$$|B| \leqslant b_{11}\cdots b_{nn}.$$

REMARK 3.6. In [7, Theorem 1.5] it is claimed that equality occurs for Oppenheim's inequality if and only if $A \circ B$ is singular, or there exists a diagonal matrix $T = \operatorname{diag}(w_1, \dots, w_n)$ such that

1. $A \circ B = TAT$, and
2. $|w_i|^2 = b_{ii} \ (i = 1, \dots, n)$.

However, for complex matrices this is false. A counter example is given by the following: for any complex number a with $0 < |a| < 1$, let

$$A = \begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix}.$$

Then, matrices A and B are positive semidefinite, $A \circ B$ is nonsingular, and equality holds for Oppenheim's inequality. But it is easy to see that there exists no diagonal matrix T satisfying (1) above.

REMARK 3.7. For positive semidefinite matrices $A = (a_{ij})$ and $B = (b_{ij}) \in M_n$, the following inequality holds (Schur's inequality):

$$|A \circ B| + |A||B| \geqslant |A| b_{11} \cdots b_{nn} + |B| a_{11} \cdots a_{nn}.$$

Oppenheim [4] gave an equality condition for Schur's inequality when both A and B are positive definite. Observe that, when either A or B is singular, Schur's inequality reduces to Oppenheim's inequality. Thus equality condition for Schur's inequality can be reduced to that of Oppenheim's inequality (cf. [7]).

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