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SOME NEW GENERALIZATIONS OF ZYGMUND–TYPE INEQUALITIES FOR POLYNOMIALS

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Abstract. In this paper, we consider a problem of investigating the dependence of

$$\left\| P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} \right\|_p$$

on $\|P(z)\|_p$ for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$, $p > 0$ and present certain sharp compact generalizations of some well-known Zygmund-type inequalities for polynomials, from which a variety of interesting results follows as special cases.

1. Introduction and statements of results

Let $P_n(z)$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. For $P \in P_n$, define

$$\|P(z)\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p \right\}^{1/p}, \quad 1 \leqslant p < \infty$$

and

$$\|P(z)\|_\infty := \underset{|z|=1}{\operatorname{Max}} |P(z)|.$$

A famous result known as Bernstein's inequality (for reference, see [13, p. 531], [17, p. 508] or [19]) states that if $P \in P_n$, then

$$\left\| P'(z) \right\|_\infty \leqslant n \|P(z)\|_\infty, \tag{1}$$

whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have

$$\|P(Rz)\|_\infty \leqslant R^n \|P(z)\|_\infty, \tag{2}$$

(for reference, see [13, p. 442] or [14, vol. I, p. 137]). Inequalities (1) and (2) can be obtained by letting $p \to \infty$ in the inequalities

$$\left\| P'(z) \right\|_p \leqslant n \|P(z)\|_p, \quad p \geqslant 1 \tag{3}$$

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and

$$\|P(Rz)\|_p \leqslant R^n \|P(z)\|_p, \quad R>1, \quad p>0, \tag{4}$$

respectively. Inequality (3) was found by Zygmund [20] whereas inequality (4) is a simple consequence of a result of Hardy [10] (see also [15, Th. 5.5]). Since inequality (3) was deduced from M. Riesz's interpolation formula [18] by means of Minkowski's inequality, it was not clear, whether the restriction on p was indeed essential. This question was open for a long time. Finally Arestov [2] proved that (3) remains true for $0<p<1$ as well.

If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z|<1$, then inequalities (1) and (2) can be respectively replaced by

$$\left\|P'(z)\right\|_\infty \leqslant \frac{n}{2}\|P(z)\|_\infty \tag{5}$$

and

$$\|P(Rz)\|_\infty \leqslant \frac{R^n+1}{2}\|P(z)\|_\infty. \tag{6}$$

Inequality (5) was conjectured by Erdös and later verified by Lax [12]. Ankeny and Rivlin [1] used inequality (5) to prove inequality (6).

Both the inequalities (5) and (6) can be obtain by letting $p \to \infty$ in the inequalities

$$\left\|P'(z)\right\|_p \leqslant n\frac{\|P(z)\|_p}{\|1+z\|_p}, \quad p \geqslant 0 \tag{7}$$

and

$$\|P(Rz)\|_p \leqslant \frac{\|R^n z+1\|_p}{\|1+z\|_p}\|P(z)\|_p, \quad R>1, \quad p>0. \tag{8}$$

Inequality (7) is due to De-Bruijn [8] for $p \geqslant 1$. Rahman and Schmeisser [16] extended it for $0<p<1$ whereas the inequality (8) was proved by Boas and Rahman [7] for $p \geqslant 1$ and later it was extended for $0<p<1$ by Rahman and Schmeisser [16].

Jain [11] generalized both the inequalities (5) and (6) and proved that if $P(z) \neq 0$ in $|z|<1$, then for every real or complex number β with $|\beta| \leqslant 1$, $|z|=1$ and $R \geqslant 1$,

$$\left|zP'(z)+\frac{n\beta}{2}P(z)\right| \leqslant \frac{n}{2}\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\|P(z)\|_\infty \tag{9}$$

and

$$\left|P(Rz)+\beta\left(\frac{R+1}{2}\right)^n P(z)\right| \leqslant \frac{1}{2}\left\{\left|R^n+\beta\left(\frac{R+1}{2}\right)^n\right|+\left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right\}\|P(z)\|_\infty. \tag{10}$$

Recently authors in [5] (see also [6]) investigated the dependence of

$$\|P(Rz)-\alpha P(z)\|_\infty \text{ on } \|P(z)\|_\infty$$

for every real or complex number α with $|\alpha| \leqslant 1$ and $R>1$. As a compact generalization of inequalities (1) and (2), they have shown that if $P \in P_n$, then for every real or complex number α with $|\alpha| \leqslant 1$, $R>1$ and $|z| \geqslant 1$,

$$|P(Rz)-\alpha P(z)| \leqslant |R^n-\alpha|\,|z|^n\,\|P(z)\|_\infty. \tag{11}$$

As a compact generalization of inequalities (5) and (6), authors [5] have also proved that if $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number α with $|\alpha| \leqslant 1$, $|z| \geqslant 1$ and $R > 1$,

$$|P(Rz) - \alpha P(z)| \leqslant \frac{|R^n - \alpha|\,|z|^n + |1-\alpha|}{2} \|P(z)\|_\infty . \tag{12}$$

In the present paper, we consider a more general problem of investigating the dependence of

$$\left\| P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} P(rz) \right\|_p \text{ on } \|P(z)\|_p$$

for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$, $p > 0$ and develop a unified method for arriving at these results. We first present the following interesting result, which is a compact generalization of inequalities (1), (2), (3), (4), (9), (10) and an extension of (11) to L^p mean of $|P(z)|$.

THEOREM 1. *If $P \in P_n$, then for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$ and $p > 0$,*

$$\|P(Rz) + \phi(R,r,\alpha,\beta) P(rz)\|_p \leqslant |R^n + \phi(R,r,\alpha,\beta) r^n|\, \|P(z)\|_p , \tag{13}$$

where

$$\phi(R,r,\alpha,\beta) := \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} - \alpha. \tag{14}$$

The result is best possible and equality in (13) holds for $P(z) = az^n$, $a \neq 0$.

A variety of interesting results can be deduced from Theorem 1 as special cases. Here we mention a few of these.

Taking $\beta = 0$ in (13), we get the following compact generalization of inequalities (3), (4) and (11).

COROLLARY 1. *If $P \in P_n$, then for every real or complex number α with $|\alpha| \leqslant 1$, $R > r \geqslant 1$ and $p > 0$,*

$$\|P(Rz) - \alpha P(rz)\|_p \leqslant |R^n - \alpha r^n|\, \|P(z)\|_p . \tag{15}$$

The result is best possible and equality in (15) holds for $P(z) = az^n$, $a \neq 0$.

REMARK 1. For $\alpha = 0$, Corollary 1 reduces to inequality (4). If we divide the two sides of (15) by $R - r$ with $\alpha = 1$ and let $R \to r$, we get

$$\left\| P'(rz) \right\|_p \leqslant n r^{n-1} \|P(z)\|_p , \quad p > 0. \tag{16}$$

For $r = 1$, it reduces to inequality (3) due to Zygmund [20] for each $p > 0$.

Next, if we divide the two sides of (13) by $R - r$ with $\alpha = 1$ and let $R \to r$, we obtain:

COROLLARY 2. *If $P \in P_n$, then for every real or complex number β with $|\beta| \leqslant 1$, $r \geqslant 1$ and $p > 0$,*

$$\left\| zP'(rz) + \frac{n\beta}{r+1} P(rz) \right\|_p \leqslant n \left| r^{n-1} + \frac{\beta r^n}{r+1} \right| \|P(z)\|_p . \tag{17}$$

The result is best possible.

The following result immediately follows from Theorem 1 by letting $p \to \infty$ in (13).

COROLLARY 3. *If $P \in P_n$, then for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$ and $|z| = 1$,*

$$|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)| \leqslant |R^n + \phi(R,r,\alpha,\beta)r^n| \operatorname*{Max}_{|z|=1} |P(z)| . \tag{18}$$

The result is sharp and equality in (18) holds for $P(z) = az^n$, $a \neq 0$.

REMARK 2. For $\beta = 0$, $r = 1$, we get inequality (11). If we divide the two sides of (18) by $R - r$ with $\alpha = 1$ and let $R \to r$, we obtain:

COROLLARY 4. *If $P \in P_n$, then for every real or complex number β with $|\beta| \leqslant 1$, $r \geqslant 1$ and $|z| = 1$,*

$$\left| zP'(rz) + \frac{n\beta}{r+1} P(rz) \right| \leqslant n \left| r^{n-1} + \frac{\beta r^n}{r+1} \right| \operatorname*{Max}_{|z|=1} |P(z)| . \tag{19}$$

The result is best possible.

For $\beta = 0$ and $r = 1$, inequality (19) reduces to inequality (1) due to Bernstein.

We next present the following multifaceted generalization for polynomials $P \in P_n$ not vanishing in $|z| < 1$ which among other interesting results include inequalities (5), (6), (7), (8), (9), (10) and (12) as special cases.

THEOREM 2. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1, |\beta| \leqslant 1$, $R > r \geqslant 1$ and $p > 0$,*

$$\|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)\|_p \leqslant \frac{C_p}{\|1+z\|_p} \|P(z)\|_p , \tag{20}$$

where

$$C_p = \|(R^n + \phi(R,r,\alpha,\beta)r^n)z + (1 + \phi(R,r,\alpha,\beta))\|_p \tag{21}$$

and $\phi(R,r,\alpha,\beta)$ is defined by (14). The result is best possible and equality in (20) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

REMARK 3. For $\alpha = \beta = 0$, Theorem 2 reduces to inequality (8). If we divide the two sides of (20) by $R - r$ with $\alpha = 1$ and let $R \to r$, we easily get:

COROLLARY 5. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leqslant 1$, $r \geqslant 1$ and $p > 0$,*

$$\left\| zP'(rz) + \frac{n\beta}{1+r} P(rz) \right\|_p \leqslant n \left\| \left(r^{n-1} + \frac{\beta r^n}{1+r} \right) z + \frac{\beta}{1+r} \right\|_p \frac{\|P(z)\|_p}{\|1+z\|_p}. \tag{22}$$

Next corollary follows by taking $\alpha = 0$ in Theorem 2.

COROLLARY 6. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leqslant 1$, $r \geqslant 1$ and $p > 0$,*

$$\|P(Rz) + \psi(R,r,\beta)P(rz)\|_p \leqslant \frac{\|(R^n + \psi(R,r,\beta)r^n)z + (1 + \psi(R,r,\beta)r^n)\|_p}{\|1+z\|_p} \|P(z)\|_p, \tag{23}$$

where $\psi(R,r,\beta) = \phi(R,r,0,\beta)$.

For $r = 1$, inequalities (22) and (23) extend inequalities (9) and (10) to the L^p mean of $|P(z)|$.

The following corollary immediately follows from Theorem 2 by taking $\beta = 0$.

COROLLARY 7. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \leqslant 1$, $R > r \geqslant 1$ and $p > 0$,*

$$\|P(Rz) - \alpha P(rz)\|_p \leqslant \frac{\|(R^n - \alpha r^n)z + (1-\alpha)\|_p}{\|1+z\|_p} \|P(z)\|_p. \tag{24}$$

The result is sharp and equality in (24) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

REMARK 4. For $\alpha = r = 1$, if we divide the two sides of (24) by $R-1$ and let $R \to 1$, we immediately get De-Bruijn's theorem (inequality (7)) for each $p > 0$. For $\alpha = 0$, Corollary 7 reduces to inequality (8) for each $p > 0$.

Next, we mention the following compact generalization of a theorem of Erdös and Lax [12] and a result of Ankeny and Rivlin [1], which immediately follows from Corollary 7 by letting $p \to \infty$ in (19).

COROLLARY 8. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex numbers α with $|\alpha| \leqslant 1$ and $R > r \geqslant 1$,*

$$|P(Rz) - \alpha P(rz)| \leqslant \frac{|R^n - \alpha r^n| + |1-\alpha|}{2} \max_{|z|=1} |P(z)| \quad \text{for } |z| = 1. \tag{25}$$

The result is best possible and equality in (25) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

A polynomial $P \in P_n$ is said to be self-inversive if $P(z) = uQ(z)$ for all $z \in C$ where $|u| = 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. It is known [3, 9] that if $P \in P_n$ is self-inversive polynomial, then for every $p \geqslant 1$,

$$\left\| P'(z) \right\|_p \leqslant n \frac{\|P(z)\|_p}{\|1+z\|_p}. \tag{26}$$

Finally, we present the following result which include some well-known results for self-inversive polynomials as special cases.

THEOREM 3. *If $P \in P_n$ is self-inversive polynomial, then for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$ and $p > 0$,*

$$\|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)\|_p \leqslant \frac{C_p}{\|1+z\|_p}\|P(z)\|_p,$$

where

$$C_p = \|(R^n + \phi(R,r,\alpha,\beta)r^n)z + (1+\phi(R,r,\alpha,\beta))\|_p$$

and $\phi(R,r,\alpha,\beta)$ is defined by (14).

COROLLARY 9. *If $P \in P_n$ is self-inversive polynomial, then for every real or complex number α with $|\alpha| \leqslant 1$, $R > r \geqslant 1$ and $p > 0$,*

$$\|P(Rz) - \alpha P(rz)\|_p \leqslant \frac{\|(R^n - \alpha r^n)z + (1-\alpha)\|_p}{\|1+z\|_p}\|P(z)\|_p. \tag{27}$$

The result is best possible and equality in (27) holds for $P(z) = z^n + 1$.

REMARK 5. Many interesting results can be deduced from Theorem 3 in exactly the same way as we have deduced from Theorem 2.

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

LEMMA 1. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for every $R \geqslant r \geqslant 1$ and $|z| = 1$,*

$$|P(Rz)| \geqslant \left(\frac{R+k}{r+k}\right)^n |P(rz)|. \tag{28}$$

Proof of Lemma 1. Since all the zeros of P(z) lie in $|z| \leqslant k$, we write

$$P(z) = C\prod_{j=1}^{n}\left(z - r_j e^{i\theta_j}\right),$$

where $r_j \leqslant k$. Now for $0 \leqslant \theta < 2\pi$, $R > r \geqslant 1$, we have

$$\begin{aligned}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| &= \left\{\frac{R^2 + r_j^2 - 2Rr_jCos(\theta-\theta_j}{r^2 + r_j^2 - 2rr_jCos(\theta-\theta_j}\right\}^{1/2}\\ &\geqslant \left\{\frac{R+r_j}{r+r_j}\right\} \geqslant \left\{\frac{R+k}{r+k}\right\}, \quad j = 1,2,\cdots,n.\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| &= \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right|\\ &\geqslant \prod_{j=1}^{n}\left(\frac{R+k}{r+k}\right) = \left(\frac{R+k}{r+k}\right)^n\end{aligned}$$

for $0 \leqslant \theta < 2\pi$. This implies for $|z| = 1$ and $R > r \geqslant 1$,

$$|P(Rz)| \geqslant \left(\frac{R+k}{r+k}\right)^n |P(rz)|,$$

which completes the proof of Lemma 1. $\square$

LEMMA 2. *If $F \in P_n$ has all its zeros in $|z| \leqslant 1$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leqslant |F(z)| \text{ for } |z| = 1,$$

then for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$, and $|z| \geqslant 1$,

$$|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)| \leqslant |F(Rz) + \phi(R,r,\alpha,\beta)F(rz)|, \tag{29}$$

where $\phi(R,r,\alpha,\beta)$ is defined by (14).

Proof of Lemma 2. In case $R = r$, we have nothing to prove. Henceforth, we assume that $R > r$. Since the polynomial $F(z)$ of degree n has all its zeros in $|z| \leqslant 1$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \leqslant |F(z)| \text{ for } |z| = 1, \tag{30}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = e^{i\theta_0}$, then $P(z)$ has a zero of multiplicity at least s at $z = e^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then the inequality (29) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \ for\ |z| > 1\ .$$

Suppose $F(z)$ has m zeros on $|z| = 1$ where $0 \leqslant m \leqslant n$, so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = 1$ and $F_2(z)$ is a polynomial of degree exactly $n-m$ having all its zeros in $|z| < 1$. This implies with the help of inequality (30) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n-m$. Now, from inequality (30), we get

$$|P_1(z)| \leqslant |F_2(z)| \text{ for } |z| = 1$$

where $F_2(z) \neq 0$ for $|z| = 1$. Therefore for every real or complex number λ with $|\lambda| > 1$, a direct application of Rouche's theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n-m \geqslant 1$ lie in $|z| < 1$ Hence the polynomial

$$f(z) = F_1(z)\left(P_1(z) - \lambda F_2(z)\right) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leqslant 1$ with at least one zero in $|z| < 1$, so that we can write

$$f(z) = (z - te^{i\delta})H(z)$$

where $t < 1$ and $H(z)$ is a polynomial of degree $n-1$ having all its zeros in $|z| \leqslant 1$. Applying Lemma 1 to the polynomial f(z) with $k = 1$, we obtain for every $R > r \geqslant 1$ and $0 \leqslant \theta < 2\pi$,

$$\begin{aligned}
|f(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\
&\geqslant |Re^{i\theta} - te^{i\delta}| \left(\frac{R+1}{r+1}\right)^{n-1} |H(re^{i\theta})| \\
&= \left(\frac{R+1}{r+1}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|} |(re^{i\theta} - te^{i\delta})H(re^{i\theta})| \\
&\geqslant \left(\frac{R+1}{r+1}\right)^{n-1} \left(\frac{R+t}{r+t}\right) |f(re^{i\theta})|.
\end{aligned}$$

This implies for $R > r \geqslant 1$ and $0 \leqslant \theta < 2\pi$,

$$\left(\frac{r+t}{R+t}\right) |f(Re^{i\theta})| \geqslant \left(\frac{R+1}{r+1}\right)^{n-1} |f(re^{i\theta})|. \tag{31}$$

Since $R > r \geqslant 1 > t$ so that $f(Re^{i\theta}) \neq 0$ for $0 \leqslant \theta < 2\pi$ and $\frac{1+r}{1+R} > \frac{r+t}{R+t}$, from inequality (31), we obtain

$$|f(Re^{i\theta}| > \left(\frac{R+1}{r+1}\right)^{n} |f(re^{i\theta})| \quad R > r \geqslant 1 \text{ and } 0 \leqslant \theta < 2\pi. \tag{32}$$

Equivalently,

$$|f(Rz)| > \left(\frac{R+1}{r+1}\right)^{n} |f(rz)|$$

for $|z| = 1$ and $R > r \geqslant 1$. Hence for every real or complex number α with $|\alpha| \leqslant 1$ and $R > r \geqslant 1$, we have

$$\begin{aligned}
|f(Rz) - \alpha f(rz)| &\geqslant |f(Rz)| - |\alpha||f(rz)| \\
&> \left\{ \left(\frac{R+1}{r+1}\right)^{n} - |\alpha| \right\} |f(rz)|, \quad |z| = 1.
\end{aligned} \tag{33}$$

Also, inequality (32) can be written in the form

$$|f(re^{i\theta})| < \left(\frac{r+1}{R+1}\right)^{n} |f(Re^{i\theta})| \tag{34}$$

for every $R > r \geqslant 1$ and $0 \leqslant \theta < 2\pi$. Since $f(Re^{i\theta}) \neq 0$ and $\left(\frac{r+1}{R+1}\right)^{n} < 1$, from inequality (34), we obtain for $0 \leqslant \theta < 2\pi$ and $R > r \geqslant 1$,

$$|f(re^{i\theta}| < |f(Re^{i\theta}).$$

Equivalently,

$$|f(rz)| < |f(Rz)| \text{ for } |z| = 1.$$

Since all the zeros of $f(Rz)$ lie in $|z| \leqslant (1/R) < 1$, a direct application of Rouche's theorem shows that the polynomial $f(Rz) - \alpha f(rz)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leqslant 1$. Applying Rouche's theorem again, it follows from (33) that for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1, |\beta| \leqslant 1$ and $R > r \geqslant 1$, all the zeros of the polynomial

$$\begin{aligned}
T(z) &= f(Rz) - \alpha f(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} f(rz) \\
&= \left[P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} P(rz) \right] \\
&\quad -\lambda \left[F(Rz) - \alpha F(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} F(rz) \right] \\
&= [P(Rz) + \phi(R,r,\alpha,\beta)P(rz)] - \lambda \left[F(Rz) + \phi(R,r,\alpha,\beta)F(rz) \right]
\end{aligned}$$

lie in $|z| < 1$ with $|\lambda| \geqslant 1$. This implies

$$|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)| \leqslant |F(Rz) + \phi(R,r,\alpha,\beta)F(rz)| \tag{35}$$

for $|z| \geqslant 1$ and $R > r \geqslant 1$. If inequality (35) is not true, then there a point $z = w$ with $|w| \geqslant 1$ such that

$$|P(Rw) + \phi(R,r,\alpha,\beta)P(rw))| > |F(Rw) + \phi(R,r,\alpha,\beta)F(rw)|.$$

But all the zeros of $F(Rz)$ lie in $|z| < 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $F(Rz) + \phi(R,r,\alpha,\beta)F(rz)$ lie in $|z| < 1$. Hence

$$F(Rw) + \phi(R,r,\alpha,\beta)F(rw) \neq 0$$

with $|w| \geqslant 1$. We take

$$\lambda = \frac{P(Rw) + \phi(R,r,\alpha,\beta)P(rw)}{F(Rw) + \phi(R,r,\alpha,\beta)F(rw)},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $T(w) = 0$ where $|w| \geqslant 1$. This contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus

$$|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)| \leqslant |F(Rz) + \phi(R,r,\alpha,\beta)F(rz)|$$

for $|z| \geqslant 1$ and $R > r \geqslant 1$. This proves Lemma 2. □

Next we describe a result of Arestov.

For $\delta = (\delta_0, \delta_1, \cdots, \delta_n)$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in P_n$, we define

$$\Lambda_\delta P(z) = \sum_{j=0}^{n} \delta_j a_j z^j.$$

The operator Λ_δ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in C : |z| \leqslant 1\}$,

(ii) $P(z)$ has all its zeros in $\{z \in C : |z| \geqslant 1\}$.

The result of Arestov may now be stated as follows.

LEMMA 3. *[2, Th. 4] Let $\phi(x) = \psi(logx)$ where ψ is a convex nondecreasing function on $\boldsymbol{R}$. Then for all $P \in P_n$ and each admissible operator Λ_δ,*

$$\int_0^{2\pi} \phi(|\Lambda_\delta P(e^{i\theta})|)d\theta \leqslant \int_0^{2\pi} \phi(C(\delta,n)|P(e^{i\theta})|)d\theta,$$

where $C(\delta,n) = max(|\delta_0|,|\delta_n|)$.

In particular, Lemma 3 applies with $\phi : x \to x^p$ for every $p \in (0,\infty)$. Therefore, we have

$$\left\{\int_0^{2\pi} (|\Lambda_\delta P(e^{i\theta})|^p)d\theta\right\}^{1/p} \leqslant C(\delta,n)\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{36}$$

We use (36) to prove the following interesting result.

LEMMA 4. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$, $p > 0$ and γ real,*

$$\begin{aligned}\int_0^{2\pi} &|(P(Re^{i\theta}) + \phi(R,r,\alpha,\beta)P(re^{i\theta}))\\ &+e^{i\gamma}(R^nP(e^{i\theta}/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^nP(e^{i\theta}/r))|^p d\theta\\ \leqslant\ &|(R^n + \phi(R,r,\alpha,\beta)r^n) + e^{i\gamma}(1 + \phi(R,r,\overline{\alpha},\overline{\beta}))|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.\end{aligned}$$

Proof of Lemma 4. By hypothesis $P(z)$ does not vanish in $|z| < 1$, therefore, the polynomial $Q(z) = z^n\overline{P(1/\overline{z})}$ of degree n has all its zeros in $|z| \leqslant 1$ and $|P(z)| = |Q(z)|$ and $|z| = 1$. Applying Lemma 2 with $F(z)$ replaced by $Q(z)$, we get for arbitrary real or complex numbers α, β with $|\alpha \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$ and $|z| = 1$,

$$\begin{aligned}|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)| &\leqslant |Q(Rz) + \phi(R,r,\alpha,\beta)Q(rz)|\\ &= \left|R^nP(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^nP(z/r)\right|.\end{aligned}$$

Now (as in the proof of Lemma 2), the polynomial

$$\begin{aligned}H(z) &= Q(Rz) + \phi(R,r,\alpha,\beta)Q(rz)\\ &= R^nz^n\overline{P(1/R\overline{z})} + \phi(R,r,\alpha,\beta)r^nz^n\overline{P(1/r\overline{z})}\end{aligned}$$

has all its zeros in $|z| < 1$ for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $R > r$, therefore, it follows that the polynomial

$$z^n\overline{H(1/\overline{z})} = R^nP(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^nP(z/r)$$

has all its zeros in $|z| > 1$. Hence the function

$$f(z) = \frac{P(Rz) + \phi(R,r,\alpha,\beta)P(rz)}{R^n P(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n P(z/r)}$$

is analytic in $|z| \leqslant 1$ and $|f(z)| \leqslant 1$ for $|z| = 1$. Since $f(z)$ is not a constant, it follows by the maximum modulus principle that

$$|f(z)| < 1 \quad \text{for } |z| < 1,$$

or equivalently, for $|z| < 1$,

$$|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)| < \left|R^n P(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n P(z/r)\right|. \tag{37}$$

A direct application of Rouche's theorem shows that

$$\begin{aligned}\Lambda_\delta P(z) &= (P(Rz) + \phi(R,r,\alpha,\beta)P(rz)) + e^{i\gamma}(R^n P(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n P(z/r)) \\ &= (R^n + \phi(R,r,\alpha,\beta)r^n + e^{i\gamma}(1 + \phi(R,r,\overline{\alpha},\overline{\beta})))a_n z^n + \cdots \\ &\quad + (1 + \phi(R,r,\alpha,\beta) + e^{i\gamma}(R^n + \phi(R,r,\overline{\alpha},\overline{\beta})r^n))a_0\end{aligned}$$

does not vanish in $|z| < 1$ for every α, β with $|\alpha| \leqslant 1, |\beta| \leqslant 1$, $R > r \geqslant 1$ and γ real. Therefore, Λ_δ is an admissibe operator. Applying (36) of Lemma 3, the desired result follows immediately for each $p > 0$. This completes the proof of Lemma 4. □

From lemma 4, we deduce the following more general lemma which is a result of independent interest with variety of applications.

LEMMA 5. *If $P \in P_n$, then for arbitrary real or complex numbers α, β with $|\alpha| \leqslant 1, |\beta| \leqslant 1, R > r \geqslant 1$, $p > 0$ and γ real,*

$$\begin{aligned}&\int_0^{2\pi} |(P(Re^{i\theta}) + \phi(R,r,\alpha,\beta)P(re^{i\theta})) \\ &\qquad + e^{i\gamma}(R^n P(e^{i\theta}/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n P(e^{i\theta}/r))|^p d\theta \\ &\quad \leqslant |(R^n + \phi(R,r,\alpha,\beta)r^n) + e^{i\gamma}(1 + \phi(R,r,\overline{\alpha},\overline{\beta})r^n)|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta.\end{aligned}$$

The result is sharp and the extremal polynomial is $P(z) = \lambda z^n$, $\lambda \neq 0$.

Proof of Lemma 5. Since $P(z)$ is a polynomial of degree n, we can write

$$P(z) = P_1(z)P_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(z - z_j),\ k \geqslant 1,$$

where all the zeros of $P_1(z)$ lie in $|z| \geqslant 1$ and all the zeros of $P_2(z)$ lie in $|z| < 1$. First we suppose that $P_1(z)$ has no zero on $|z| = 1$ so that all the zeros of $P_1(z)$ lie in $|z| > 1$. Let $Q_2(z) = z^{n-k}\overline{P_2(1/\overline{z})}$, then all the zeros of $Q_2(z)$ lie in $|z| > 1$ and

$|Q_2(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$G(z) = P_1(z)Q_2(z) = \prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(1-z\overline{z_j}), k \geqslant 1,$$

then all the zeros of $G(z)$ lie in $|z| > 1$ and for $|z| = 1$,

$$|G(z)| = |P_1(z)|\,|Q_2(z)| = |P_1(z)|\,|P_2(z)| = |P(z)|. \tag{38}$$

By the Maximum Modulus Principle, it follows that

$$|P(z)| \leqslant |G(z)| \text{ for } |z| \leqslant 1. \tag{39}$$

We now claim that the polynomial $H(z) = P(z) + \lambda G(z)$ does not vanish in $|z| \leqslant 1$ for every λ with $|\lambda| > 1$. If this is not true, then there is some z_0 with $|z_0| \leqslant 1$ such that $H(z_0) = 0$. This gives

$$|P(z_0)| = |\lambda||G(z_0)|. \tag{40}$$

Since $G(z_0) \neq 0$ and $|\lambda| > 1$, (40) implies

$$|P(z_0)| > |G(z_0)|,$$

which clearly contradicts (39). Thus the polynomial $H(z)$ does not vanish in $|z| \leqslant 1$ for every λ with $|\lambda| > 1$ so that all the zeros of $H(z)$ lie in $|z| \geqslant \rho$ for some $\rho > 1$, or equivalently, all the zeros of $H(\rho z)$ lie in $|z| \geqslant 1$. Applying (37) to the polynomial $H(\rho z)$, we get

$$|H(R\rho z) + \phi(R,r,\alpha,\beta)H(r\rho z)| < |R^n H(\rho z/R) + \phi(R,r,\overline{\alpha},\overline{\beta}r^n H(\rho z/r)| \text{ for } |z| < 1.$$

Taking $z = e^{i\theta}/\rho, 0 \leqslant \theta < 2\pi$, then $|z| = (1/\rho) < 1$ as $\rho > 1$ and we get

$$|H(Re^{i\theta}) + \phi(R,r,\alpha,\beta)H(re^{i\theta})| < |R^n H(e^{i\theta}/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n H(e^{i\theta}/r)|,$$

for $0 \leqslant \theta < 2\pi, R > r \geqslant 1$ and $|\alpha| \leqslant 1, |\beta| \leqslant 1$. This implies

$$|H(Rz) + \phi(R,r,\alpha,\beta)H(rz)| < |R^n H(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n H(z/r)| \text{ for } |z| = 1.$$

An application of Rouche's theorem shows that the polynomial

$$T(z) = (H(Rz) + \phi(R,r,\alpha,\beta)H(rz)) + e^{i\gamma}(R^n H(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n H(z/r))$$

does not vanish in $|z| \leqslant 1$ for every α, β with $|\alpha \leqslant 1$, $|\beta| \leqslant 1$, $R > r \geqslant 1$ and γ real. Replacing $H(z)$ by $P(z) + \lambda G(z)$, it follows that the polynomial

$$T(z) = \left\{P(Rz) + \phi(R,r,\alpha,\beta)P(rz) + e^{i\gamma}(R^n P(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n P(z/r))\right\}$$
$$+\lambda\left\{(G(Rz) + \phi(R,r,\alpha,\beta)G(rz)) + e^{i\gamma}(R^n G(z/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n G(z/r))\right\} \tag{41}$$

does not vanish in $|z| \leqslant 1$ for every α, β, λ with $|\alpha| \leqslant 1, |\beta| \leqslant 1$ and $|\lambda| > 1$. This implies

$$|(P(Rz)+\phi(R,r,\alpha,\beta)P(rz))+e^{i\gamma}(R^nP(z/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nP(z/r))|$$

$$\leqslant |(G(Rz)+\phi(R,r,\alpha,\beta)G(rz))+e^{i\gamma}(R^nG(z/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nG(z/r))| \qquad (42)$$

for $|z| \leqslant 1$, $|\alpha| \leqslant 1, |\beta| \leqslant 1$, $R > r \geqslant 1$ and γ real. If inequality (42) is not true, then there is a point $z = z_0$ with $|z_0| \leqslant 1$ such that

$$|(P(Rz_0)+\phi(R,r,\alpha,\beta)P(rz_0))+e^{i\gamma}(R^nP(z_0/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nP(z_0/r))|$$

$$> |(G(Rz_0)+\phi(R,r,\alpha,\beta)G(rz_0))+e^{i\gamma}(R^nG(z_0/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nG(z_0/r))|.$$

Since all the zeros of polynomials $G(z)$ lie in $|z| > 1$, it follows (as before) that all the zeros of polynomial

$$(G(Rz)+\phi(R,r,\alpha,\beta)G(rz))+e^{i\gamma}(R^nG(z/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nG(z/r))$$

also lie in $|z| > 1$ for every α, β with $|\alpha| \leqslant 1, |\beta| \leqslant 1$, $R > r \geqslant 1$ and γ real. Hence

$$G(Rz_0)+\phi(R,r,\alpha,\beta)G(rz_0))+e^{i\gamma}(R^nG(z_0/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nG(z_0/r)) \neq 0,$$

$|z_0| \leqslant 1$. We take

$$\lambda = -\frac{(P(Rz_0)+\phi(R,r,\alpha,\beta)P(rz_0))+e^{i\gamma}(R^nP(z_0/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nP(z_0/r))}{(G(Rz_0)+\phi(R,r,\alpha,\beta)G(rz_0))+e^{i\gamma}(R^nG(z_0/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nG(z_0/r))}$$

so that λ is a well-defined real or complex number with $|\lambda| > 1$ and with this choice of λ, from (41) we get $T(z_0) = 0$ with $|z_0| \leqslant 1$. This clearly is a contradiction to the fact that $T(z)$ does not vanish in $|z| \leqslant 1$. Thus for every α, β with $|\alpha| \leqslant 1, |\beta| \leqslant 1$, $R > r$ and γ real,

$$|(P(Rz)P(rz))+e^{i\gamma}(R^nP(z/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nP(z/r))|$$

$$\leqslant |(G(Rz)+\phi(R,r,\alpha,\beta)G(rz))+e^{i\gamma}(R^nG(z/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nG(z/r))|$$

for $|z| \leqslant 1$, which in particular gives for each $p > 0$ and $0 \leqslant \theta < 2\pi$,

$$\begin{aligned}
&\int_0^{2\pi} |(P(Re^{i\theta})+\phi(R,r,\alpha,\beta)P(re^{i\theta})) \\
&\qquad +e^{i\gamma}(R^nP(e^{i\theta}/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nP(e^{i\theta}/r))|^p d\theta \\
&\leqslant \int_0^{2\pi} |(G(Re^{i\theta})+\phi(R,r,\alpha,\beta)G(re^{i\theta})) \\
&\qquad +e^{i\gamma}(R^nG(e^{i\theta}/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nG(e^{i\theta}/r))|^p d\theta.
\end{aligned}$$

Using lemma 4 and (38), it follows that for every α, β with $|\alpha| \leqslant 1, |\beta| \leqslant 1$, $R > r$, $p > 0$ and γ real,

$$\begin{aligned}\int_0^{2\pi} &|(P(Re^{i\theta}) + \phi(R,r,\alpha,\beta)P(re^{i\theta})) \\ &\quad + e^{i\gamma}(R^n P(e^{i\theta}/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n P(e^{i\theta}/r))|^p d\theta \\ &\leqslant |(R^n + \phi(R,r,\alpha,\beta)r^n) + e^{i\gamma}(1+\phi(R,r,\overline{\alpha},\overline{\beta}))|^p \int_0^{2\pi} |G(e^{i\theta})|^p d\theta \\ &= |(R^n + \phi(R,r,\alpha,\beta)r^n) + e^{i\gamma}(1+\phi(R,r,\overline{\alpha},\overline{\beta}))|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \end{aligned} \tag{43}$$

Now, if $P_1(z)$ has a zero on $|z| = 1$, then applying (43) to the polynomial $P^*(z) = P_1(tz)P_2(z)$ where $t < 1$, we get for every α, β with $|\alpha \leqslant 1, |\beta| \leqslant 1$, $R > r$, $p > 0$ and γ real,

$$\begin{aligned}\int_0^{2\pi} &|(P^*(Re^{i\theta}) + \phi(R,r,\alpha,\beta)P^*(re^{i\theta})) \\ &\quad + e^{i\gamma}(R^n P^*(e^{i\theta}/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n P^*(e^{i\theta}/r))|^p d\theta \\ &\leqslant |(R^n + \phi(R,r,\alpha,\beta)r^n) + e^{i\gamma}(1+\phi(R,r,\overline{\alpha},\overline{\beta}))|^p \int_0^{2\pi} |P^*(e^{i\theta})|^p d\theta. \end{aligned}$$

Letting $t \to 1$ in (44) and using continuity, the desired result follows immediately and this proves Lemma 5. □

3. Proofs of the theorems

Proof of Theorem 1. Since $P(z)$ is a polynomial of degree n, we can write

$$P(z) = P_1(z)P_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(z - z_j),$$

where all the zeros of $P_1(z)$ lie in $|z| \leqslant 1$ and all the zeros of $P_2(z)$ lie in $|z| > 1$. First we suppose that $P_1(z)$ has no zero on $|z| = 1$ so that all the zeros of lie in $|z| < 1$. Let $Q_2(z) = z^{n-k}\overline{P_2(1/\overline{z})}$, then all the zeros of $Q_2(z)$ lie in $|z| < 1$ and $|Q_2(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$F(z) = P_1(z)Q_2(z) = \prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(1 - z\overline{z_j}),$$

then all the zeros of $F(z)$ lie in $|z| < 1$ and for $|z| = 1$,

$$|F(z)| = |P_1(z)|\,|Q_2(z)| = |P_1(z)|\,|P_2(z)| = |P(z)|. \tag{44}$$

By the Maximum Modulus Principle, it follows that

$$|P(z)| \leqslant |F(z)| \text{ for } |z| \geqslant 1.$$

Since $F(z) \neq 0$ for $|z| \geqslant 1$, therefore, for every λ with $|\lambda| > 1$, a direct application of Rouche's theorem shows that the polynomial $H(z) = P(z) + \lambda F(z)$ has all its zeros in $|z| < 1$. Applying lemma 1 to the polynomial $H(z)$ and noting that all the zeros of $H(Rz)$ lie in $|z| < \frac{1}{R} < 1$, we deduce (as before) that for every α, β, λ with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$ and $|\lambda| > 1$, all the zeros of polynomial

$$\begin{aligned} G(z) &= H(Rz) + \phi(R,r,\alpha,\beta)H(rz) \\ &= (P(Rz) + \phi(R,r,\alpha,\beta)P(rz)) + \lambda(F(Rz) + \phi(R,r,\alpha,\beta)F(rz)) \end{aligned}$$

lie in $|z| < 1$. This implies (as in the case of Lemma 2)

$$|P(Rz) + \phi(R,r,\alpha,\beta)P(rz)| \leqslant |F(Rz) + \phi(R,r,\alpha,\beta)F(rz)|$$

for $|z| \geqslant 1$ and $R > r \geqslant 1$, which in particular gives for $R > r \geqslant 1$ and $p > 0$,

$$\begin{aligned} &\int_0^{2\pi} |P(Re^{i\theta}) + \phi(R,r,\alpha,\beta)P(re^{i\theta})|^p d\theta \\ &\leqslant \int_0^{2\pi} |F(Re^{i\theta}) + \phi(R,r,\alpha,\beta)F(re^{i\theta})|^p d\theta. \end{aligned} \tag{45}$$

Again, since all the zeros of $F(z)$ lie in $|z| < 1$, as before, the polynomial

$$F(Rz) + \phi(R,r,\alpha,\beta)F(rz)$$

has all its zeros in $|z| < 1$ for every real or complex number β with $|\beta| \leqslant 1$. Therefore, the operator Λ_γ defined by

$$\begin{aligned} \Lambda_\gamma F(z) &= F(Rz) + \phi(R,r,\alpha,\beta)F(rz) \\ &= (R^n + \phi(R,r,\alpha,\beta)r^n)b_n z^n + \cdots + (1 + \phi(R,r,\alpha,\beta)r^n)b_0 \end{aligned}$$

is admissible. Hence by (36) of Lemma (3), for each $p > 0$, we have

$$\int_0^{2\pi} |F(Re^{i\theta}) + \phi(R,r,\alpha,\beta)F(re^{i\theta})|^p d\theta \leqslant |R^n + \phi(R,r,\alpha,\beta)r^n|^p \int_0^{2\pi} |F(e^{i\theta})|^p d\theta. \tag{46}$$

Combining inequalities (45) and (46) and noting that $|F(e^{i\theta})| = |P(e^{i\theta})|$, we obtain

$$\begin{aligned} &\left\{\int_0^{2\pi} |P(Re^{i\theta}) + \phi(R,r,\alpha,\beta)P(re^{i\theta})|^p d\theta\right\}^{1/p} \\ &\leqslant |R^n + \phi(R,r,\alpha,\beta)r^n| \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{1/p}. \end{aligned} \tag{47}$$

In case $P_1(z)$ has a zero on $|z| = 1$, the inequality (47) follows by using similar argument as in the case of Lemma 5. This completes the proof of Theorem 1. □

Proof of Theorem 2. By hypothesis $P \in P_n$ and P(z) does not vanish in $|z| < 1$, therefore, by Lemma 2 for every real or complex number α, β with $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $0 \leqslant \theta < 2\pi$ and $R > r \geqslant 1$,

$$|P(Re^{i\theta}) + \phi(R,r,\alpha,\beta)P(re^{i\theta})| \leqslant |R^n P(e^{i\theta}/R) + \phi(R,r,\overline{\alpha},\overline{\beta})r^n P(e^{i\theta}/r)|. \tag{48}$$

Also, by Lemma 5,

$$\int_0^{2\pi} |F(\theta)+e^{i\gamma}G(\theta)|^p d\theta$$
$$\leqslant |(R^n+\phi(R,r,\alpha,\beta)r^n)e^{i\gamma}+(1+\phi(R,r,\overline{\alpha},\overline{\beta})|^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta, \tag{49}$$

where

$$F(\theta)=P(Re^{i\theta})+\phi(R,r,\alpha,\beta)P(re^{i\theta})$$

and

$$G(\theta)=R^nP(e^{i\theta}/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nP(e^{i\theta}/r).$$

Integrating both sides of (49) with respect to γ from 0 to 2π, we get for each $p>0$, $R>r\geqslant 1$ and γ real,

$$\int_0^{2\pi}\int_0^{2\pi} |F(\theta)+e^{i\gamma}G(\theta)|^p d\gamma d\theta$$
$$\leqslant \left\{\int_0^{2\pi} |(R^n+\phi(R,r,\alpha,\beta)r^n)e^{i\gamma}+(1+\phi(R,r,\overline{\alpha},\overline{\beta})|^p d\gamma\right\}\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}. \tag{50}$$

Now for every real γ, $t\geqslant 1$ and $p>0$, we have

$$\int_0^{2\pi} |t+e^{i\gamma}|^p d\gamma \geqslant \int_0^{2\pi} |1+e^{i\gamma}|^p d\gamma.$$

If $F(\theta)\neq 0$, we take $t=|G(\theta)|/|F(\theta)|$, then by (48) $t\geqslant 1$ and we get

$$\begin{aligned}\int_0^{2\pi} |F(\theta)+e^{i\gamma}G(\theta)|^p d\gamma &= |F(\theta)|^p \int_0^{2\pi} \left|1+e^{i\gamma}\frac{G(\theta)}{F(\theta)}\right|^p d\gamma\\ &= |F(\theta)|^p \int_0^{2\pi} \left|\frac{G(\theta)}{F(\theta)}+e^{i\gamma}\right|^p d\gamma\\ &= |F(\theta)|^p \int_0^{2\pi} \left|\left|\frac{G(\theta)}{F(\theta)}\right|+e^{i\gamma}\right|^p d\gamma\\ &\geqslant |F(\theta)|^p \int_0^{2\pi} \left|1+e^{i\gamma}\right|^p d\gamma.\end{aligned}$$

For $F(\theta)=0$, this inequality is trivially true. Using this in (50), we conclude that for arbitrary real or complex numbers α, β with $|\alpha|\leqslant 1, |\beta|\leqslant 1$, $R>r\geqslant 1$ and γ real,

$$\begin{aligned}&\left\{\int_0^{2\pi} \left|1+e^{i\gamma}\right|^p d\gamma\right\}\left\{\int_0^{2\pi} |P(Re^{i\theta})+\phi(R,r,\alpha,\beta)P(re^{i\theta})|^p d\theta\right\}\\ &\qquad\leqslant \left\{\int_0^{2\pi} |(R^n+\phi(R,r,\alpha,\beta)r^n)+e^{i\gamma}(1+\phi(R,r,\overline{\alpha},\overline{\beta}))|^p d\gamma\right\}\times\\ &\qquad\qquad\times\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}.\end{aligned} \tag{51}$$

Since

$$\begin{aligned}
&\left\{\int_0^{2\pi} |(R^n+\phi(R,r,\alpha,\beta)r^n)+e^{i\gamma}(1+\phi(R,r,\overline{\alpha},\overline{\beta}))|^p d\gamma\right\}\\
&=\left\{\int_0^{2\pi} ||R^n+\phi(R,r,\alpha,\beta)r^n|+e^{i\gamma}|1+\phi(R,r,\overline{\alpha},\overline{\beta})||^p d\gamma\right\}\\
&=\left\{\int_0^{2\pi} ||R^n+\phi(R,r,\alpha,\beta)r^n|+e^{i\gamma}|1+\phi(R,r,\alpha,\beta)||^p d\gamma\right\}\\
&=\left\{\int_0^{2\pi} ||R^n+\phi(R,r,\alpha,\beta)r^n|e^{i\gamma}+|1+\phi(R,r,\alpha,\beta)||^p d\gamma\right\}\\
&=\left\{\int_0^{2\pi} |(R^n+\phi(R,r,\alpha,\beta)r^n)e^{i\gamma}+(1+\phi(R,r,\alpha,\beta))|^p d\gamma\right\}, \qquad (52)
\end{aligned}$$

the desired result follows immediately by combining (51) and (52). This completes the proof of Theorem 2. □

Proof of Theorem 3. Since $P(z)$ is self-inversive polynomial, we have $P(z) = uQ(z)$ for all $z \in C$ where $|u| = 1$ and $Q(z) = z^n\overline{P}(1/\overline{z})$. Therefore, for arbitrary real or complex numbers α, β and $R > r \geqslant 1$,

$$|P(Rz)+\phi(R,r,\alpha,\beta)P(rz)| = |Q(Rz)+\phi(R,r,\alpha,\beta)Q(rz)| \quad for\ all\ z \in C$$

so that

$$|G(\theta)/F(\theta)| = \left|\frac{P(Re^{i\theta})+\phi(R,r,\alpha,\beta)P(re^{i\theta})}{R^nP(e^{i\theta}/R)+\phi(R,r,\overline{\alpha},\overline{\beta})r^nP(e^{i\theta}/r)}\right| = 1.$$

Using this in (50) and proceeding similarly as in the proof of Theorem 2, we get the desired result. This completes the proof of Theorem 3. □

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