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WEAK HARNACK INEQUALITY FOR THE NON–NEGATIVE WEAK SUPERSOLUTION OF QUASILINEAR ELLIPTIC EQUATION

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Abstract. We introduce and study the classes $\tilde{\mathscr{P}}_p(\mathbb{R}^n)$ as well as $\mathscr{P}_p(\mathbb{R}^n)$, which are generalization of the Kato class. We also obtain a Fefferman inequality for the class $\tilde{\mathscr{P}}_p(\mathbb{R}^n)$ and derive the weak Harnack inequality.

1. Introduction

The Harnack inequality, which state that, the supremum of the solution of an elliptic differential equation is bounded by its infimum, has been known since the famous works [13, 15, 16] of Moser.

Other authors have generalized these results to weak solutions of quasi-linear equations in divergence form in a very general setting. In particular, Trudinger discovered that weak supersolutions posses a weak Harnack inequality. That is, the infimum of a nonnegative supersolution over a ball can be bounded below by its integral average.

In this note, we show that if u is a weak supersolution of a quasilinear elliptic equation of the form

$$-\,divA(x,u,\nabla u)+B(x,u,\nabla u)=0, \tag{1}$$

then, for any $\sigma,\tau\in(0,1)$ and $\gamma\in(0,n(p-1)/(n-p))$ with $1<p<n$, there exists a constant C such that

$$\left(\frac{1}{|B(\sigma r)|}\int_{B(\sigma r)}u^{\gamma}dx\right)^{\frac{1}{\gamma}}\leqslant C\left(\inf_{B(\tau r)}u+k(r)\right), \tag{2}$$

assuming that suitable powers of the coefficients of (1) belongs to the function space $\tilde{\mathscr{P}}_p(\mathbb{R}^n)$, (see Theorem 2), where by $\tilde{\mathscr{P}}_p(\mathbb{R}^n)$ we denote a generalization of the Kato class (see Remark 1). Here $|B(\sigma r)|$ is the Lebesgue measure of $B(\sigma r)$, and $k(r)$ is a given function depending on the radius of the ball $B(r)$ (see section 4). For a discussion of recent result on Harnack inequality and its connection with equation 1, see the work by Difazio, G., Fanciullo, M.S., Zamboni P., ([5, 7, 8] and the reference Therein). Also see [4, 6, 11, 12].

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In this paper we obtain results similar to those obtained by Pietro Zamboni in [20]. However, it is not clear to us whether there is an inclusion (in either direction) between the space we consider and the one considered in [20].

The Kato class K_n was introduced and studied by Aizenman and Simon (see [18] and [1]). For $n \geqslant 3$, it consists of locally integrable functions f on $\mathbb{R}^n$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}} dy = 0.$$

We point out that the essential ingredients in the proof of the Theorem 2 is the use of the Fefferman inequality (in Moser's iteration scheme), that is

$$\int_{\mathbb{R}^n} |f(x)|\,|u(x)|^p\, dx \leqslant C_n \Delta_f(2r) \int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx$$

for any $u \in C_0^\infty(\mathbb{R}^n)$, where C is a positive constant depending on some norm of f, (see Theorem 1).

2. Definition and notation

We begin this section giving some definitions

DEFINITION 1. Let $f \in L^1_{loc}(\mathbb{R}^n)$. For $p \in (1,n)$ and $t, r > 0$, we set

$$\Delta_f(r) = \sup_{x\in\mathbb{R}^n} \left\{ \int_{B(x,r)} \int_0^t \frac{s\,ds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}} \times \left(\int_{B(x,r)} \int_0^s \frac{\lambda\, |f(z)|\, dz\, d\lambda}{(\lambda^2+|y-z|^2)^{\frac{n+1}{2}}} \right)^{\frac{1}{p-1}} dy \right\}^{p-1}.$$

We say that f belongs to the functions space $\tilde{\mathscr{P}}_p(\mathbb{R}^n)$ if

$$\Delta_f(r) < \infty, \quad \forall r > 0.$$

DEFINITION 2. We say that $f \in L^1_{loc}(\mathbb{R}^n)$ belong to the function space $\mathscr{P}_p(\mathbb{R}^n)$ if

$$\lim_{r\to 0} \Delta_f(r) = 0,$$

where $\Delta_f(r)$ is as in Definition 1.

Some comments are now in order.

REMARK 1. The following relations hold

$$i)\quad \mathscr{P}_p(\mathbb{R}^n) \subset \tilde{\mathscr{P}}_p(\mathbb{R}^n);$$
$$ii)\quad \mathscr{P}_2(\mathbb{R}^n) = K_n(\mathbb{R}^n).$$

As is commonplace throughout the literature, we write $A \sim B$ and say that A is asymptotically equivalent to B if and only if there is a positive constant k which does not depended on A and B such that $\frac{1}{k}A \leqslant B \leqslant kA$.

i) is obvious. Concerning *ii)*, for $r < t$, observe that, Fubini's theorem implies

$$\int_{B(x,r)} \int_0^t \frac{sds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}} \int_{B(x,r)} \int_0^s \frac{\lambda\,|f(z)|\,dzd\lambda}{\left(\lambda^2+|y-z|^2\right)^{\frac{n+1}{2}}} dy$$

$$= \int_{B(x,r)} |f(z)| \int_{B(x,r)} \int_0^t \frac{sds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}} \int_0^s \frac{\lambda d\lambda}{\left(\lambda^2+|y-z|^2\right)^{\frac{n+1}{2}}} dydz.$$

Since

$$\int_{B(x,r)} \int_0^t \frac{sds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}} \int_0^s \frac{\lambda d\lambda}{\left(\lambda^2+|y-z|^2\right)^{\frac{n+1}{2}}} dy$$

$$\sim \int_{B(x,r)} \frac{dy}{|x-y|^{n-1}\,|y-z|^{n-1}}$$

$$\sim \frac{1}{|z-x|^{n-2}},$$

we get the conclusion, that *ii)* in fact holds.

DEFINITION 3. The distribution function D_f of a measurable function f is given by

$$D_f(\lambda) = m(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})$$

where m denotes the Lebesgue measure on $\mathbb{R}^n$. The distribution function D_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from definition 3 that D_f is a decreasing function of λ (not necessarily strictly).

DEFINITION 4. Let f be a measurable function in $\mathbb{R}^n$. The decreasing rearrangement of f is the function f defined on $[0,\infty)$ by

$$f^*(t) = \inf\{\lambda : D_f(\lambda) \leqslant t\}(t \geqslant 0).$$

We use here the convention that $\inf \emptyset = \infty$.

DEFINITION 5. (Lorentz space) Let f be a measurable function, we say that f belongs to $L_{(\frac{n}{2},1)}$ if

$$\|f\|_{(\frac{n}{2},1)} = \int_0^\infty t^{n/2-1} f^*(t)dt < \infty.$$

And belongs to $L\left(\frac{n}{n-2},\infty\right)$ if

$$\|f\|_{\left(\frac{n}{n-2},\infty\right)} = \sup_{t>1} t^{1-2/n} f^*(t) < \infty.$$

LEMMA 1. $L_{\left(\frac{n}{2},1\right)} \subset \tilde{\mathscr{P}}(\mathbb{R}^n)$.

Proof. Let $f \in L_{\left(\frac{n}{2},1\right)}$, then

$$\int_0^\infty t^{n/2-1} f^*(t)dt < \infty.$$

Since $|f|\chi_{B(x,r)} \leqslant |f|$ we have

$$\left(|f|\chi_{B(x,r)}\right)^*(t) \leqslant f^*(t),$$

then

$$\int_0^\infty t^{n/2-1}\left(|f|\chi_{B(x,r)}\right)^*(t)dt \leqslant \int_0^\infty t^{n/2-1} f^*(t)dt < \infty.$$

Thus $|f|\chi_{B(x,r)} \in L_{\left(\frac{n}{2},1\right)}$.

On the other hand let $g(x) = |x|^{-(n+1)}$, then

$$\begin{aligned} m(\{x : |g(x)| > \lambda\}) &= m\left(\left\{x : |x|^{-(n+1)} > \lambda\right\}\right) \\ &= m\left(\left\{x : |x| < \left(\frac{1}{\lambda}\right)^{\frac{1}{n+1}}\right\}\right) \\ &= C_n\left(\frac{1}{\lambda}\right)^{\frac{n}{n+1}}, \end{aligned}$$

where $C_n = m(B(0,1))$.

Next we set $t = C_n\left(\frac{1}{\lambda}\right)^{\frac{n}{n+1}}$, then $\lambda = C_n t^{\frac{1}{n}+1}$. Thus $g^*(t) = C_n t^{\frac{1}{n}+1}$. From this we obtain

$$\begin{aligned} \|g\|_{\left(\frac{n}{n-2},\infty\right)} &= \left\|\frac{1}{|\cdot|^{n+1}}\right\|_{\left(\frac{n}{n-2},\infty\right)} \\ &= \sup_{t>1} C_n t^{1-\frac{2}{n}} t^{\frac{1}{n}+1} \\ &= \sup_{t>1} C_n t^{2-\frac{1}{n}} \\ &\leqslant C_n < \infty. \end{aligned}$$

Which means that $g \in L_{\left(\frac{n}{n-2},\infty\right)}$. Finally, by Fubini's Theorem and Hölder's inequality we have

$$\Delta_f(r) = \sup_{x\in\mathbb{R}^n}\left\{\int_{B(x,1)}\int_0^t \frac{sds}{(s^2+|x-y|^2)^{\frac{n+1}{2}}}\left(\int_{B(x,1)}\int_0^s \frac{\lambda|f(z)|dzd\lambda}{(\lambda^2+|y-z|^2)^{\frac{n+1}{2}}}\right)^{\frac{1}{p-1}} dy\right\}^{p-1}$$
$$\leqslant \sup_{x\in\mathbb{R}^n}\left\{\int_{B(x,1)}\int_0^t \frac{sds}{|x-y|^{n+1}}\left(\int_{B(y,2r)}\int_0^s \frac{\lambda|f(z)|}{|y-z|^{n+1}}dzd\lambda\right)^{\frac{1}{p-1}} dy\right\}^{p-1}$$
$$\leqslant \sup_{x\in\mathbb{R}^n}\left(\int_{B(x,1)}\int_0^t \frac{sdsdy}{|x-y|^{n+1}}\right)^{p-1}\frac{s^2}{2}\int_{\mathbb{R}^n}\frac{|f(z)|\chi_{B(0,2r)}(y-y)dz}{|y-z|^{n+1}}$$
$$\leqslant C(n,s,t,p)\|f\chi_{B(0,2r)}\|_{(\frac{n}{2},1)}\left\|\frac{1}{|\cdot|^{n+1}}\right\|_{(\frac{n}{n-2},\infty)} < \infty,$$

where

$$C(n,s,t,p) = \sup_{x\in\mathbb{R}^n}\left\{\left(\int_{B(x,1)}\int_0^t \frac{sdsdy}{|x-y|^{n+1}}\right)^{p-1}\frac{s^2}{2}\right\}.$$

which means that $f \in \tilde{\mathscr{P}}(\mathbb{R}^n)$ and the proof is complete. □

3. On Fefferman's Inequality

Fefferman proved in [10] that if $f \in L^{r,n-2r}(\mathbb{R}^n)$, where $L^{r,n-2r}(\mathbb{R}^n)$ denotes the Morrey space, with $1 < r \leqslant n/2$, then there exists a constant C such that

$$\int_{\mathbb{R}^n} |f(x)|\,|u(x)|^2\,dx \leqslant C\int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx \tag{3}$$

for any $u \in C_0^\infty(\mathbb{R}^n)$.

Later, Chiarenza and Frasca [3] extended Fefferman's result with a different proof, assuming $f \in L^{r,n-pr}(\mathbb{R}^n)$, $1 < r \leqslant n/p$, $1 < p < n$.

A different approach to the inequality (3) was started with Schechter's proof in [16] of the inequality under the assumption that f is in the Kato class. In [20] inequality (3) was proved with $1 < p < n$ and f in a more general class of functions.

REMARK 2. Observe that

$$\begin{aligned} C_n\int_{B(x,r)}\int_0^t &\frac{s|\nabla u(y)|ds}{(s^2+|x-y|^2)^{\frac{n+1}{2}}} \\ &= C_n\int_{B(x,r)}\left(\frac{|\nabla u(y)|}{|x-y|^{n-1}}\int_0^{\frac{t}{|x-y|}}\frac{udu}{(u^2+1)^{\frac{n+1}{2}}}\right)dy \\ &\geqslant C_n\int_{B(x,r)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}\int_0^1\frac{udu}{(u^2+1)^{\frac{n+1}{2}}}dy \\ &= C_nA_n\int_{B(x,r)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}dy \\ &\geqslant A_n|u(x)| \end{aligned}$$

where C_n is a convenient constant and $A_n = \int_0^1 \frac{udu}{(u^2+1)^{\frac{n+1}{2}}}$. Thus

$$|u(x)| \leqslant \frac{C_n}{A_n} \int_{B(x,r)} \int_0^t \frac{s|\nabla u(y)|}{(s^2+|x-y|^2)^{\frac{n+1}{2}}}$$

In the following theorem using the ideas from [20] we provide a generalization of Schechter's result, assuming $f \in \tilde{\mathscr{P}}_p(\mathbb{R}^n)$.

THEOREM 1. *Assume $f \in \tilde{\mathscr{P}}_p(\mathbb{R}^n)$. Then for any r,t with $0<r<t$ there exists a positive constant $C(n,p)$ such that*

$$\int_{\mathbb{R}^n} |f(x)|\,|u(x)|^p\,dx \leqslant C\Delta_f(2r)\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx$$

for any $u \in C_0^\infty(\mathbb{R}^n)$ supported in $B(x_0,r)$.

Proof. For any $u \in C_0^\infty(\mathbb{R}^n)$ supported in $B(x_0,r)$, using the inequality from Remark 2 and Fubini's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |f(x)|\,|u(x)|^p\,dx \\
&\quad= \int_{B(x_0,r)} |f(x)|\,|u(x)|^p\,dx \qquad (4)\\
&\quad\leqslant C_n \int_{B(x_0,r)} |f(x)|\,|u(x)|^{p-1}\left(\int_{B(x_0,r)}\int_0^t \frac{s\,|\nabla u(y)|}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}dsdy\right)dx\\
&\quad= C_n \int_{B(x_0,r)} |\nabla u(y)|\left(\int_{B(x_0,r)} |f(x)|\,|u(x)|^{p-1}\int_0^t \frac{sds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}dx\right)dy
\end{aligned}$$

by Hölder inequality

$$\begin{aligned}
&\leqslant C_n\left(\int_{B(x_0,r)} |\nabla u(y)|^p\,dy\right)^{1/p}\\
&\qquad\times\left[\int_{B(x_0,r)}\left(\int_{B(x_0,r)} |f(x)|\,|u(x)|^{p-1}\int_0^t \frac{sds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}dx\right)^{\frac{p}{p-1}} dy\right]^{\frac{p-1}{p}}
\end{aligned}$$

we also have

$$\int_{B(x_0,r)}\left(\int_{B(x_0,r)}|f(x)|\,|u(x)|^{p-1}\int_0^t \frac{sds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}dx\right)^{\frac{p}{p-1}} dy$$

$$= \int_{B(x_0,r)}\left(\int_{B(x_0,r)}\left[\int_0^t \frac{s\,|f(x)|\,ds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}\right]^{\frac{1}{p}}\right)^{\frac{p}{p-1}} \tag{5}$$

$$\times\left(\left[\int_0^t \frac{s\,|f(x)|\,ds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}\right]^{\frac{1}{q}} |u(x)|^{p-1}dx\right)^{\frac{p}{p-1}} dy$$

using Hölder inequality one more time we have

$$\leqslant \int_{B(x_0,r)}\left[\int_0^t \frac{s\,|f(x)|\,|u(x)|^p\,ds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}\left(\int_{B(x_0,r)}\int_0^t \frac{s\,|f(x)|\,ds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}\right)^{\frac{1}{p-1}}\right]dy$$

$$= \int_{B(x_0,r)}|f(x)|\,|u(x)|^p\left[\int_{B(x_0,r)}\int_0^t \frac{sds}{\left(s^2+|x-y|^2\right)^{\frac{n+1}{2}}}\right]$$

$$\times\left[\left(\int_{B(x_0,r)}\int_0^t \frac{\lambda\,|f(z)|\,d\lambda dz}{\left(\lambda^2+|y-z|^2\right)^{\frac{n+1}{2}}}\right)^{\frac{1}{p-1}} dy\right] dx$$

$$\leqslant \left[\Delta_f(2r)\right]^{\frac{1}{p-1}}\int_{B(x_0,r)}|f(x)|\,|u(x)|^p\,dx.$$

By (4) and (5) we obtain the desired conclusion. □

The next Corollary is an easy consequence of the previous Theorem. It can be obtained via a standard partition of unity. The proof of this Corollary follows along the same lines as the proof of Corollary 2.2 in [17]. We have include its proof for sake of completeness and the convenice of the reader.

COROLLARY 1. *Let* $f\in\tilde{\mathscr{P}}_p(\mathbb{R}^n)$ *and* Ω *a bounded subset of* $\mathbb{R}^n$, *supp* $f\subseteq\Omega$. *Then for any* $\sigma>0$ *there exists a positive constant* k *depending on* σ *such that*

$$\int_\Omega |f(x)||u(x)|^p dx \leqslant \sigma\int_\Omega |\nabla u(x)|^p dx + k(\sigma)\int_\Omega |u(x)|^p dx,$$

for all $u\in C_0^\infty(\Omega)$.

Proof. Let $\sigma > 0$. Let r be a positive number that will be choosen later. Let $\{\alpha_k^p\}, k = 1, 2, \cdots, N(r)$, be a finite partition of unity of $\bar{\Omega}$, such that $suppf \subseteq B(x_k, r)$ with $x_k \in \bar{\Omega}$. We apply Theorem 1 to the functions $\alpha_k u$ and we get

$$\begin{aligned}
\int_\Omega |f(x)||u(x)|^p dx &= \int_\Omega |f(x)||u(x)|^p \sum_{k=1}^{N(r)} \alpha_k^p(x) dx \\
&= \sum_{k=1}^{N(r)} \int_\Omega |f(x)||\alpha_k(x) u(x)|^p dx \\
&\leqslant \sum_{k=1}^{N(r)} C\Delta_f(2r) \left(\int_\Omega |\nabla u(x)|^p \alpha_n^p(x) dx + \int_\Omega |\nabla \alpha_n(x)|^p |u(x)|^p dx \right) \\
&\leqslant C\Delta_f(2r) \left(\int_\Omega |\nabla u(x)|^p dx + \frac{N(r)}{r^p} \int_\Omega |u(x)|^p dx \right).
\end{aligned}$$

Finally, to obtain the result it is sufficient to choose r such that $C\Delta_f(2r) = \sigma$. After that we note that $N(r) \approx r^{-n}$ and the corollary follows. □

4. Preliminary results

Let Ω be a bounded open set in $\mathbb{R}^n$ and $1 < p < n$. We consider the equations form

$$-\,divA(x, u, \nabla u) + B(x, u, \nabla u) = 0, \tag{6}$$

where $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are given. We suppose that $A(\cdot, \zeta, \xi), B(\cdot, \zeta, \xi)$ are Lebesgue measurable, $A(x, \cdot, \cdot)$, $B(x, \cdot, \cdot)$ are Borel measurable, and the following structure conditions are satisfied:

$$\begin{aligned}
|A(x, \zeta, \xi)| &\leqslant a_1 |\xi|^{p-1} + a_2 |\zeta|^{p-1} + a_3 \\
|B(x, \zeta, \xi)| &\leqslant b_0 |\xi|^{p-1} + b_1 |\zeta|^{p-1} + b_2 \\
A(x, \zeta, \xi) \cdot \xi &\geqslant c_1 |\xi|^p - c_2 |\zeta|^p - c_3.
\end{aligned} \tag{7}$$

Here c_1 is a positive constant, a_1 and b_0 are nonnegative constants and the remaining coefficients are nonnegative functions assumed to lie in suitable function spaces. Specifically, we make the following assumption:

$$a_2^{p/p-1}, a_3^{p/p-1}, b_1, b_2, c_3 \in \tilde{\mathscr{P}}_p(\mathbb{R}^n). \tag{8}$$

Unless otherwise specified, reference to weak super solution of (6) will carry with it the assumption that either (7) or (8) are in force.

We will assume, without loss of generality, that $c_1 = 1$. We will frequently consider solutions defined in a ball $B(r)$. It is convenient to simplify the structure of (7) by introducing

$$k = k(r) = \left(\Delta_{a_3^{\frac{p}{p-1}}}(2r) + \Delta_{c_3}(2r) \right)^{\frac{1}{p}} + \left(\Delta_{b_3}(2r) \right)^{\frac{1}{p-1}}$$

and setting $v = |u| + k(r)$. Then from (7) we easily get

$$\begin{aligned} |A(x,\zeta,\xi)| &\leqslant a_1|\xi|^{p-1} + b|v|^{p-1} \\ |B(x,\zeta,\xi)| &\leqslant b_0|\xi|^{p-1} + d|v|^{p-1} \\ \xi A(x,\zeta,\xi) &\geqslant |\xi|^p - d|v|^p \end{aligned} \tag{9}$$

where $b = a_2 + k^{1-p}a_3, d = c_2 + k^{1-p}b_2 + k_{c_3}^{1-p}$.

Next, for $0 < \rho < 2r$

$$\begin{aligned} \Delta_b^{\frac{p}{p-1}}(\rho) &\leqslant C_p\left[\Delta_{a_2}^{\frac{p}{p-1}}(\rho) + k^{-p}\Delta_{a_3}^{\frac{p}{p-1}}(\rho)\right] \\ &\leqslant C_p\left[\Delta_{a_2}^{\frac{p}{p-1}}(\rho) + 1\right] \end{aligned}$$

and

$$\begin{aligned} \Delta_d(\rho) &\leqslant C_p\left[\Delta_{c_2}(\rho) + k^{1-p}\Delta_{b_2}(\rho) + k^p\Delta_{c_3}(\rho)\right] \\ &\leqslant C_p\left[\Delta_{c_2}(\rho) + 2\right]. \end{aligned}$$

Thus $b^{p/p-1}$ and d belong to the class $\tilde{\mathscr{P}}_p(\mathbb{R}^n)$, this means that, under our assumptions (8), the reduced structure assumptions (9) are of the same kind of the general structure assumption (7).

DEFINITION 6. Let Ω be a bounded open set in $\mathbb{R}^n$. We say that a function $u \in W^{1,p}_{loc}(\Omega)$ is a local weak super solution of (6) in Ω if

$$\int_\Omega A(x,u,\nabla u)\nabla\phi + B(x,u,\nabla u)\phi dx \geqslant 0,$$

for every $\phi \in W_C^{1,p}(\Omega)$.

In order to obtain our main result we recall the following Lemma, proved in [19] (Theorem 1.66, p. 40).

LEMMA 2. *Let $u \in W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is convex. Suppose there is a constant M such that*

$$\left(\int_{\Omega\cap B(r)} |\nabla u|^p dx\right)^{1/p} \leqslant Mr^{\frac{n-p}{p}},$$

for all balls $B(r)$. Then there exist positive constants σ_0 and C depending only on n such that

$$\int_\Omega exp\left(\frac{\sigma}{M}|u - \widetilde{u}_\Omega|\right)dx \leqslant C(diam\Omega)^n,$$

whenever $\sigma < \sigma_0|\Omega|(diam\Omega)^{-n}$.

5. Main result

In this section we show how Corollary 1 is used to obtain regularity results for a class of subelliptic quasilinear PDE,

THEOREM 2. *Let u be a weak supersolution of (6) defined in Ω. Assume $0 \leqslant u \leqslant M \leqslant \infty$ in some ball $B(r) \subset \Omega$ where $M = \infty$ is allowed if $b_0 = 0$. Then, for any $\sigma, \tau \in (0,1)$ and $\gamma \in (0, n(p-1)/(n-p))$, there exists a constant C such that*

$$\left(\frac{1}{|B(\sigma r)|}\int_{B(\sigma r)} u^{\gamma} dx\right)^{\frac{1}{\gamma}} \leqslant C\left(\inf_{B(\tau r)} u + k(r)\right). \tag{10}$$

Here,

$$C = C\left(p, n, \gamma, \sigma, \tau, \varepsilon, b_0, M, a, \Delta_{a_2}^{\frac{p}{p-1}}(r), \Delta_{b_1}(r), \Delta_{b_2}(r)\right).$$

Furthermore, in the case when $p = n$, (10) holds for any $\gamma > 0$.

Proof. It suffices to consider the case $r = 1$ since rescaling will yield the general result. We will assume throughout that $u \geqslant \varepsilon > 0$ and then let $\varepsilon \to 0$ to establish the general result.

Let $G : (0,\infty) \to \mathbb{R}$ be a smooth nonincreasing function and let $\eta \in C_c^{\infty}(B(1))$. Now set $\overline{u} = u + k$, and define a test function φ as

$$\varphi = G(u)\eta^p.$$

Then

$$\nabla\varphi = G'(u)\nabla u \eta^p + pG(u)\eta^{p-1}\nabla\eta$$

and we obtain

$$\int_{B(1)} A(x,u,\nabla u)\nabla u G'(u)\eta^p dx +$$
$$+ p\int_{B(1)} A(x,u,\nabla u)\nabla u G(u)\eta^{p-1} dx + \int_{B(1)} B(x,u,\nabla u)G(u)\eta^p dx \geqslant 0. \tag{11}$$

Taking (9) into account, we get

$$-\int_{B(1)} A(x,u,\nabla u)\nabla u G'(u)\eta^p dx \geqslant \int_{B(1)} \left(|\nabla u|^p - \overline{b}\overline{u}^p\right)\left|G'(u)\right|\eta^p dx, \tag{12}$$

$$p\int_{B(1)} A(x,u,\nabla u)\nabla\eta G'(u)\eta^{p-1} dx$$
$$\leqslant p\int_{B(1)} \left(a_1|\nabla k|^{p-1} + \overline{a}u^{p-1}\right)G'(u)\eta^{p-1}|\nabla\eta| dx, \tag{13}$$

and

$$\int_{B(1)} B(x,u,\nabla u)\, G(u)\eta^p dx \leqslant \int_{B(1)} \left(b_1 |\nabla u|^{p-1} + \overline{b}\overline{u}^{p-1}\right) G(u)\eta^p dx. \tag{14}$$

From (11) - (14) we obtain

$$\begin{aligned}\int_{B(1)} \left|G'(u)\right| |\nabla u|^p \eta^p dx \leqslant &\int_{B(1)} \overline{u}^p \left|G'(u)\right| \eta^p \overline{b} dx \\ &+ \int_{B(1)} \overline{u}^{p-1} G(u)\eta^{p-1} \left(\overline{b}\eta + p\overline{a}|\nabla \eta|\right) dx \\ &+ \int_{B(1)} |\nabla u|^{p-1} G(u)\eta^{p-1} (pa_1 |\nabla \eta| + b_1 \eta)\, dx.\end{aligned} \tag{15}$$

Set

$$G(u) = \overline{u}^{\beta}, \quad \beta < 0,$$

and obtain

$$\begin{aligned}|\beta| \int_{B(1)} |\nabla u|^p \overline{u}^{\beta-1} u \eta^p dx \leqslant |\beta| &\int_{B(1)} \overline{b}\overline{u}^{\beta-1} u \eta^p dx \\ &+ \int_{B(1)} \overline{u}^{\beta-1+p} u \eta^{p-1} (pa_1 |\nabla \eta| + b_1 \eta)\, dx.\end{aligned}$$

Now define

$$v = \begin{cases} \bar{u}^q & \text{where} \quad pq = p+\beta-1, \beta \neq 1-p \\ \log \bar{u} & \text{if} \quad \beta = 1-p. \end{cases}$$

If $\beta \neq 1-p$, we use Young's inequality to obtain

$$\int_{B(1)} \eta^p |\nabla v|^p dx \leqslant C|q|^p \left(1+|\beta|^{-1}\right)^p \left[\int_{B(1)} |\nabla \eta|^p v^p dx + \int_{B(1)} f \eta^p v^p dx\right]$$

where $f = b^{p/p-1} + b_0^p + d$.

We apply Corollary 1 to obtain

$$\|\eta \nabla \upsilon\|_p \leqslant |p|^{p/\varepsilon} (1+|\beta|^{-p})^{\frac{1}{\varepsilon}} \|(\eta + |\nabla rp)\upsilon\|_p$$

Also, when $\beta = 1-p$ we have

$$\int_{B(1)} \eta^p |\nabla v|^p dx \leqslant C \int_{B(1)} (f\eta^p + |\nabla \eta|^p)\, dx$$

and by Corollary 1, this yields

$$\int_{B(1)} \eta^p |\nabla v|^p dx \leqslant C \int_{B(1)} (\eta^p + |\nabla \eta|^p)\, dx.$$

Thus, we have

$$\|\eta \nabla v\|_p \leqslant \begin{cases} C|q|^{p/\varepsilon} (1+|B|^{-p})^{1/\varepsilon} \|(\eta + |\nabla \eta)\upsilon\|_p & \text{for } \beta \neq 1-p \\ C\|\eta + |\nabla \eta|\|_p & \text{for } \beta = \eta - p, \end{cases} \tag{16}$$

where $C = C(p,n,\varepsilon,a_1,\Delta_{a_2}^{\frac{p}{r-1}},b_0M)$. If $\beta \neq 1-p, \beta < 0,$ Sobolev's inequality yields

$$\|\eta\upsilon\|_{\chi p} \leqslant C|q|^{p/\varepsilon}(1+|\beta|^{p})^{1/\varepsilon}\|(\eta+|\nabla\eta|)\upsilon\|_p$$

where $p^{*/p} \geqslant \chi > 1$ for $p < n$ and $\chi > 1$ for $p = n$. Also, we will need later that

$$\gamma/\chi < p-1 \tag{17}$$

which is a restriction on γ.

Let η be a cut off function such that $\eta \equiv 1$ on $\beta(h')$ and identically zero on the complement of $\beta(h)$, then

$$\|v\|_{\chi p,B(h')} \leqslant C|q|^{p/\varepsilon}(1+|\beta|^{-p})^{1/\varepsilon}(h-h')^{-1}\|v\|_{p,B(h)}.$$

with $\alpha = pq = p+\beta-1$ and $\alpha > 0,$ this can be written as

$$\|\bar{u}\|_{\chi\alpha,B(h')} \leqslant \left[C\alpha^{p/\varepsilon}(1+|\beta|^{-p})^{1/\varepsilon}(h-h')^{-1}\right]^{p/\alpha}\|\bar{u}\|_{\alpha,B(h)}. \tag{18}$$

while

$$\|\bar{u}\|_{\chi\alpha,B(h')} \geqslant \left[C(-\alpha)^{p/\varepsilon}(h-h')^{-1}\right]^{p/\alpha}\|\bar{u}\|_{\alpha,B(h)}. \tag{19}$$

If $\alpha < 0$. In this latter case note that $|\beta|^{-1} < (p-1)^{-1}$, and that, for notational convenience, we will extend the usual notation for $\|u\|_p$ to include negative values of p.

These inequalities will be iterated. Let $\rho \in (0,1)$ be any number such that $\max(\sigma,\tau) < \rho$ and let a positive integer j be fixed. We set $\alpha_i = \chi^{i-j-1}\gamma, i = 0,\ldots,j+1$. Then for any $i = 0,\ldots,j,$ the corresponding value of $|\beta|$ will be lower bounded by $p-1-\gamma$ providing by (17) an upper bound on the constant $(1+|\beta|^{-p})$ that appears in (18). Define

$$h_i = \sigma + 2^{-i}(\rho-\sigma), h_i' = h_{i+1}, i = 0,\ldots,j.$$

Hence, from (18) we have

$$\|\bar{u}\|_{\gamma,B(\sigma)} \leqslant C\|\bar{u}\|_{\alpha_0,B(\rho)}$$

for any $\alpha_0 \in \{\chi^{-j-1}\gamma : j = 1,2,\ldots\}$. An easy application of Hölder's inequality extends the estimate to any $\alpha_0 > 0$.

Interaction of (19) for $\alpha < 0$ will yield a lower bound for $\inf \bar{u}$. For this, let

$$h_i = \rho + 2^i(\tau-\rho),\ h_i' = h_{i+1},\ \ i = 0,1,2,...$$

and obtain

$$\|\bar{u}\|_{-\alpha_{0,B(\rho)}} \leqslant C\|\bar{u}\|_{-\alpha,B(\rho)}.$$

The proof is concluded by showing the existence of $\alpha_0 > 0$ such that

$$\|\bar{u}\|_{-\alpha_0,B(\rho)} \leqslant C\|\bar{u}\|_{-\alpha_0,B(\rho)}.$$

Referring to the case $\beta = 1-p, v = \ln u$ in (16) and to Lemma 2, there exists $\alpha_0 > 0$ such that

$$\int_{B(\rho)} e^{\alpha_0|v-v_0|}dx \leqslant C$$

where

$$v_0 = \frac{1}{|B(\rho)|}\int_{B(\rho)} vdx.$$

Thus

$$\int_{B(\rho)} e^{\alpha_0 v}dx \int_{B(\rho)} -e^{\alpha_0 v}dx \leqslant Ce^{\alpha_0 v_0}e^{\alpha_p v_0} = C.$$

The final result is obtained by rescaling with $x \to rr$. □

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