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POINTWISE AND INTEGRAL ESTIMATES FOR THE FRACTIONAL INTEGRALS ON THE LAGUERRE HYPERGROUP

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Abstract. Let $\mathbb{K} = [0,\infty)\times\mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this paper, some pointwise and integral estimates for the fractional integrals in terms of the maximal and fractional maximal functions on the Laguerre hypergroup are obtained. Basing on these results, we prove interpolation theorems for the fractional maximal functions and fractional integrals, and the Sobolev theorem on the Laguerre hypergroup.

1. Introduction

In this paper, we define the fractional maximal function and fractional integral using harmonic analysis on Laguerre hypergroups which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [2], [5], [6], [9]–[12]). We obtain some pointwise and integral estimates that give a relation between the maximal and fractional maximal functions and fractional integrals on the Laguerre hypergroup and extend the available results to the objects of a more general nature. Based on these results, we prove interpolation theorems for the fractional maximal functions and fractional integrals, and the Sobolev theorem on the Laguerre hypergroup.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In section 3, we give polar coordinates in the Laguerre hypergroup. In the last section, we give the main results such as Sobolev's theorem, interpolation theorems for the fractional maximal function and fractional integrals on the Laguerre hypergroup.

Let m_α be the weighted Lebesgue measure on $\mathbb{K} = [0,\infty)\times\mathbb{R}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}, \quad \alpha \geqslant 0.$$

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We denote by $L_p(\mathbb{K}) = L_p(\mathbb{K}; dm_\alpha)$ the spaces of complex-valued functions f measurable on $\mathbb{K}$ such that

$$\|f\|_{L_p(\mathbb{K})} = \left(\int_{\mathbb{K}} |f(x,t)|^p \, dm_\alpha(x,t) \right)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{L_\infty(\mathbb{K})} = \underset{(x,t)\in\mathbb{K}}{ess\sup} |f(x,t)| \quad \text{if} \quad p = \infty.$$

For $1 \leqslant p < \infty$ we denote by $WL_p(\mathbb{K})$, the weak $L_p(\mathbb{K})$ spaces defined as the set of locally integrable functions f with the finite norm

$$\|f\|_{WL_p(\mathbb{K})} = \sup_{r>0} r \left(m_\alpha \left\{ (x,t) \in \mathbb{K} \, : \, |f(x,t)| > r \right\} \right)^{1/p}.$$

Let $|(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{1/4}$ be the homogeneous norm of $(x,t) \in \mathbb{K}$. For $r > 0$, we will denote by $\delta_r(x,t) = (rx, r^2 t)$ the dilation of $(x,t) \in \mathbb{K}$, and by $B_r(x,t)$ the ball centered at (x,t) with radius r, i.e., the set of $B_r(x,t) = \{(y,s) \in \mathbb{K} : |(x-y, t-s)|_{\mathbb{K}} < r\}$, ${}^{\complement}B_r(x,t) = \mathbb{K} \setminus B_r(x,t)$, and by B_r the ball $B_r(0,0)$. We denote by

$$f_r(x,t) = r^{-(2\alpha+4)} f\left(\delta_{\frac{1}{r}}(x,t) \right)$$

the dilated of the function f defined on $\mathbb{K}$ preserving the mean of f with respect to the measure dm_α, in the sense that

$$\int_{\mathbb{K}} f_r(x,t) dm_\alpha(x,t) = \int_{\mathbb{K}} f(x,t) dm_\alpha(x,t), \ \ \forall r > 0 \text{ and } \ f \in L_1(\mathbb{K}).$$

For $(x,t), (y,s) \in \mathbb{K}$ and $\theta \in [0, 2\pi[$, $r \in [0,1]$, let

$$((x,t),(y,s))_{\theta,r} = \left(\left(x^2 + y^2 + 2xyr\cos\theta \right)^{1/2}, t + s + xyr\sin\theta \right).$$

The generalized translation operator $T^{(\alpha)}_{(x,t)}$ defined on the Laguerre hypergroup is given for a suitable function f by

$$T^{(\alpha)}_{(x,t)} f(y,s) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f\left(((x,t),(y,s))_{\theta,1} \right) d\theta, & \alpha = 0; \\ \dfrac{\alpha}{\pi} \displaystyle\int_0^1 \left(\int_0^{2\pi} f\left(((x,t),(y,s))_{\theta,r} \right) d\theta \right) r(1-r^2)^{\alpha-1} dr, & \alpha > 0. \end{cases}$$

We define the fractional maximal function on the Laguerre hypergroup by

$$M_\beta f(x,t) = \sup_{r>0} \left(m_\alpha B_r \right)^{\frac{\beta}{2\alpha+4}-1} \int_{B_r} T^{(\alpha)}_{(x,t)}(|f|)(y,s) \, dm_\alpha(y,s), \quad 0 \leqslant \beta < 2\alpha+4$$

and the fractional integral by

$$I_\beta f(x,t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} f(y,s) dm_\alpha(y,s), \quad 0 < \beta < 2\alpha+4.$$

If $\beta = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator on the Laguerre hypergroup (see [5]).

The following theorem is proved in [5].

THEOREM 1. *1. If $f \in L_1(\mathbb{K})$, then $Mf \in WL_1(\mathbb{K})$ and*

$$\|Mf\|_{WL_1(\mathbb{K})} \leqslant A_1 \|f\|_{L_1(\mathbb{K})},$$

where $A_1 > 0$ is independent of f.

2. If $f \in L_p(\mathbb{K})$, $1 < p \leqslant \infty$, then $Mf \in L_p(\mathbb{K})$ and

$$\|Mf\|_{L_p(\mathbb{K})} \leqslant A_p \|f\|_{L_p(\mathbb{K})},$$

where $A_p > 0$ is independent of f.

COROLLARY 1. *If $f \in L_{loc}(\mathbb{K})$, then*

$$\lim_{r\to 0} \frac{1}{m_\alpha B_r} \int_{B_r} \left|T^{(\alpha)}_{(x,t)} f(y,s) - f(x,t)\right| dm_\alpha(y,s) = 0$$

for a.e. $(x,t) \in \mathbb{K}$.

2. Preliminaries

Consider the following partial differential operators system:

$$\begin{cases} D_1 = \dfrac{\partial}{\partial t}, \\ D_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2 \dfrac{\partial^2}{\partial t^2}, \\ (x,t) \in]0,\infty[\times\mathbb{R} \ \text{ and } \ \alpha \in [0,\infty[. \end{cases}$$

For $\alpha = n-1$, $n \in \mathbb{N} \setminus \{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the initial problem

$$\begin{cases} D_1 u = i\lambda u, \\ D_2 u = -4|\lambda|\left(m + \dfrac{\alpha+1}{2}\right)u, \\ u(0,0) = 1, \ \ \dfrac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all} \quad t \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t} \mathscr{L}^{(\alpha)}_m \left(|\lambda| x^2\right), \quad (x,t) \in \mathbb{K},$$

where $\mathscr{L}_m^{(\alpha)}$ is the Laguerre functions defined on $\mathbb{R}_+$ by

$$\mathscr{L}_m^{(\alpha)}(x) = e^{-x/2} L_m^{(\alpha)}(x) / L_m^{(\alpha)}(0)$$

and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree m and order α (see [2]).

For $f \in L_1(\mathbb{K})$, the Fourier-Laguerre transform $\mathscr{F}$ is defined by

$$\mathscr{F}(f)(\lambda, m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t) f(x,t) dm_\alpha(x,t)$$

such that

$$\|\mathscr{F}(f)\|_{L_\infty(\mathbb{K})} \leqslant \|f\|_{L_1(\mathbb{K})}$$

(see [2, 10]).

The generalized translation operators $T_{(x,t)}^{(\alpha)}$ on the Laguerre hypergroup satisfies the following properties (see [2, 10])

$$T_{(x,t)}^{(\alpha)} f(y,s) = T_{(y,s)}^{(\alpha)} f(x,t), T_{(0,0)}^{(\alpha)} f(y,s) = f(y,s),$$

$$\|T_{(x,t)}^{(\alpha)} f\|_{L_p(\mathbb{K})} \leqslant \|f\|_{L_p(\mathbb{K})} \quad \text{for all} \quad f \in L_p(\mathbb{K}),\ 1 \leqslant p \leqslant \infty, \tag{1}$$

$$\mathscr{F}\left(T_{(x,t)}^{(\alpha)} f\right)(\lambda, m) = \mathscr{F}(f)(\lambda, m)\ \varphi_{\lambda,m}(x,t).$$

The translation operator $T_{(x,t)}^{(\alpha)}$ is defined by

$$T_{(x,t)}^{(\alpha)} f(y,s) = \int_{\mathbb{K}} f(z,v) W_\alpha((x,t),(y,s),(z,v)) z^{2\alpha+1} dz dv,$$

where $dzdv$ is the Lebesgue measure on $\mathbb{K}$, and W_α is an appropriate kernel satisfying

$$\int_{\mathbb{K}} W_\alpha((x,t),(y,s),(z,v)) z^{2\alpha+1} dz dv = 1$$

(see [9]). For all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the function $\varphi_{\lambda,m}(x,t)$ satisfies the following product formula

$$\varphi_{\lambda,m}(x,t)\, \varphi_{\lambda,m}(y,s) = T_{(x,t)}^{(\alpha)} \varphi_{\lambda,m}(y,s).$$

By using the generalized translation operators $T_{(x,t)}^{(\alpha)}$, $(x,t) \in \mathbb{K}$, we define a generalized convolution product $*$ on $\mathbb{K}$ by

$$\left(\delta_{(x,t)} * \delta_{(y,s)}\right)(f) = T_{(x,t)}^{(\alpha)} f(y,s),$$

where $\delta_{(x,t)}$ is the Dirac measure at (x,t).

We define the convolution product on the space $M_b(\mathbb{K})$ of bounded Radon measures on $\mathbb{K}$ by

$$(\mu * \nu)(f) = \int_{\mathbb{K}\times\mathbb{K}} T_{(x,t)}^{(\alpha)} f(y,s)\, d\mu(x,t)\, d\nu(y,s).$$

If $\mu = h \cdot m_\alpha$ and $\nu = g \cdot m_\alpha$, then we have

$$\mu * \nu = (h * \check{g}) \cdot m_\alpha, \quad \text{with} \quad \check{g}(y,s) = g(y,-s),$$

where, h and g belong to the space $L_1(\mathbb{K})$ of the integrable functions on $\mathbb{K}$ with respect to the measure $dm_\alpha(x,t)$, and $h * g$ is the convolution product defined by

$$(h*g)(x,t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} h(y,s)\, g(y,-s)\, dm_\alpha(y,s), \quad \text{for all} \quad (x,t) \in \mathbb{K}.$$

Note that, for the convolution operators the Young inequality is valid: If $1 \leqslant p, r \leqslant q \leqslant \infty$, $1/p' + 1/q = 1/r$, $f \in L_p(\mathbb{K})$, and $g \in L_r(\mathbb{K})$, then $f * g \in L_q(\mathbb{K})$ and

$$\|f * g\|_{L_q(\mathbb{K})} \leqslant \|f\|_{L_p(\mathbb{K})} \|g\|_{L_r(\mathbb{K})}, \tag{2}$$

where $p' = p/(p-1)$.

$(M_b(\mathbb{K}), *, i)$ is an involutive Banach algebra, where i is the involution on $\mathbb{K}$ given by $i(x,t) = (x,-t)$ and the convolution product $*$ satisfies all the conditions of Jewett (see [3], [8]). Hence $(\mathbb{K}, *, i)$ is a hypergroup in the sense of Jewett and the functions $\varphi_{\lambda,m}$ are characters of $\mathbb{K}$. If $\beta = n-1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathscr{H}_n$.

3. Polar coordinates in Laguerre hypergroup

Let $\Sigma = \Sigma_2$ be the unit sphere in $\mathbb{K}$. We denote by ω_2 the surface area of Σ and by Ω_2 its volume (see [4, 5]). For $\xi = (x,t) \in \mathbb{K}$, consider the transformation given by

$$x = r(\cos\varphi)^{1/2}, \; t = r^2 \sin\varphi,$$

where $-\pi/2 \leqslant \varphi \leqslant \pi/2$, $r = |\xi|_{\mathbb{K}}$ and $\xi' = ((\cos\varphi)^{1/2}, \sin\varphi) \in \Sigma$.

The Jacobian of the above transformation is $r^{2\alpha+3}(\cos\varphi)^\alpha$. If f is integrable in $\mathbb{K}$, then

$$\int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t) \;=\; c_\alpha \int_{-\pi/2}^{\pi/2} \int_0^\infty f\big(r(\cos\varphi)^{1/2}, r^2 \sin\varphi\big)\, r^{2\alpha+3} (\cos\varphi)^\alpha\, dr\, d\varphi.$$

where $c_\alpha = \dfrac{1}{2\pi\Gamma(\alpha+1)}$. Since $c_\alpha \int_{-\pi/2}^{\pi/2} (\cos\varphi)^\alpha d\varphi = \int_\Sigma d\xi'$, we get

$$\int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t) = \int_\Sigma \int_0^\infty r^{2\alpha+3} f(\delta_r \xi')\, dr\, d\xi'. \tag{3}$$

Here $d\xi'$ is the surface area element on Σ.

LEMMA 1. [4, 5] *The following equalities are valid*

$$\omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\,\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}, \qquad \Omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

Note that for any $x \in \mathbb{K}$ and $r > 0$, the area of the sphere $S_r(x,t)$ is $r^{2\alpha+3}\omega_2$ and its volume is $r^{2\alpha+4}\Omega_2 = r^{2\alpha+4}\dfrac{\omega_2}{2\alpha+4}$.

4. Estimates of fractional integrals on the Laguerre hypergroup

We first prove a lemma in the following which is being pointwise estimate for fractional integrals $I_\beta f(x,t)$. Such type estimates are given in [1].

LEMMA 2. *Let $0 < \beta < 2\alpha+4$, $1 \leqslant p < \frac{\lambda}{\beta}$. Then for any locally summable function f, and for every $r > 0$ and $(x,t) \in \mathbb{K}$ the following inequality is valid*

$$I_\beta |f|(x,t) \leqslant C_1\, r^\beta (Mf)(x,t) + C_2\, r^{\beta - \frac{\lambda}{p}} (M_{\frac{\lambda}{p}} f)(x,t), \tag{4}$$

where $C_1 = \frac{\Omega_2 2^{2\alpha+4}}{2^\beta - 1}$, $C_2 = \frac{\Omega_2^{1-\frac{\lambda}{p(2\alpha+4)}}\, 2^{2\alpha+4-\frac{\lambda}{p}}}{1-2^{\beta-\frac{\lambda}{p}}}$.

Proof. For any $r > 0$, we have

$$\begin{aligned} I_\beta |f|(x,t) &= \left(\int_{B_r} + \int_{\complement_{B_r}}\right) T^{(\alpha)}_{(x,t)} |f(y,s)|\, |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} dm_\alpha(y,s) \\ &:= J_1(x,t,r) + J_2(x,t,r). \end{aligned}$$

Firstly, we estimate $J_1(x,r)$. Summarizing on all $k > 0$, we get

$$\begin{aligned} J_1(x,t,r) &\leqslant \int_{B_r} T^{(\alpha)}_{(x,t)} |f(y,s)|\, |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} dm_\alpha(y,s) \\ &= \sum_{k=1}^{\infty} \int_{B_{2^{-k+1}r} \setminus B_{2^{-k}r}} T^{(\alpha)}_{(x,t)} |f(y,s)|\, |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} dm_\alpha(y,s) \\ &\leqslant \sum_{k=1}^{\infty} \left(2^{-k} r\right)^{\beta-2\alpha-4} \int_{B_{2^{-k+1}r} \setminus B_{2^{-k}r}} T^{(\alpha)}_{(x,t)} |f(y,s)|\, dm_\alpha(y,s) \\ &\leqslant \Omega_2 r^\beta Mf(x,t) \sum_{k=1}^{\infty} \left(2^{-k}\right)^{\beta-2\alpha-4} \left(2^{-k+1}\right)^{2\alpha+4} \\ &= \Omega_2 2^{2\alpha+4} r^\beta Mf(x,t) \sum_{k=1}^{\infty} 2^{-k\beta} \leqslant C_1 r^\beta Mf(x,t). \end{aligned} \tag{5}$$

Therefore

$$J_1(x,t,r) \leqslant C_1 r^\beta Mf(x,t). \tag{6}$$

Secondly, we estimate $J_2(x,t,r)$.

$$\begin{aligned}
J_2(x,t,r) &= \int_{\complement B_r} T^{(\alpha)}_{(x,t)}|f(y,s)|\,|(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4}dm_\alpha(y,s)\\
&\leqslant \sum_{k=0}^{\infty}\int_{B_{2^{k+1}r}\setminus B_{2^k r}} T^{(\alpha)}_{(x,t)}|f(y,s)|\,|(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4}dm_\alpha(y,s)\\
&\leqslant \sum_{k=0}^{\infty}(2^k r)^{\beta-2\alpha-4}\int_{B_{2^{k+1}r}\setminus B_{2^k r}} T^{(\alpha)}_{(x,t)}|f(y,s)|dm_\alpha(y,s)\\
&\leqslant \Omega_2^{1-\frac{\lambda}{p(2\alpha+4)}} M_{\frac{\lambda}{p}} f(x,t)\sum_{k=0}^{\infty}(2^k r)^{\beta-2\alpha-4}(2^{k+1}r)^{2\alpha+4-\frac{\lambda}{p}}\\
&\leqslant C_2\, r^{\beta-\frac{\lambda}{p}}\, M_{\frac{\lambda}{p}} f(x,t),
\end{aligned}$$

where $\beta-\frac{\lambda}{p}<0$. Therefore

$$J_2(x,t,r)\leqslant C_2 r^{\beta-\frac{\lambda}{p}} M_{\frac{\lambda}{p}} f(x,t). \tag{7}$$

Then from (6) and (7) we get the inequality (4). Therefore the proof of Lemma 2 is completed.

THEOREM 2. *Let $0<\beta<\lambda$, $1<p<\frac{\lambda}{\beta}$, $1\leqslant r\leqslant\infty$, and $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{\lambda}+\frac{\beta p}{\lambda r}$. Then for any $f\in L_p(\mathbb{K})$ and $M_{\frac{\lambda}{p}} f\in L_r(\mathbb{K})$ the following estimation is valid:*

$$\|I_\beta f\|_{L_q(\mathbb{K})}\leqslant (C_1+C_2)A_p^{1-\frac{\beta p}{\lambda}}\ \|M_{\frac{\lambda}{p}} f\|_{L_r(\mathbb{K})}^{\frac{\beta p}{\lambda}}\ \|f\|_{L_p(\mathbb{K})}^{1-\frac{\beta p}{\lambda}}. \tag{8}$$

Proof. Taking

$$r=r(x,t)=\left(\frac{M_{\frac{\lambda}{p}} f(x,t)}{Mf(x,t)}\right)^{\frac{p}{\lambda}},$$

in (4) for every $(x,t)\in\mathbb{K}$ we have

$$I_\beta|f|(x,t)\leqslant (C_1+C_2)\left(M_{\frac{\lambda}{p}} f(x,t)\right)^{\frac{\beta p}{\lambda}}(Mf(x,t))^{1-\frac{\beta p}{\lambda}}. \tag{9}$$

Integrating on $\mathbb{K}$ and applying Hölder's inequality to inequality (9) we get

$$\begin{aligned}
\int_{\mathbb{K}} I_\beta|f|(x,t)^q dm_\alpha(x,t) &\leqslant (C_1+C_2)^q\int_{\mathbb{K}}\left(M_{\frac{\lambda}{p}} f(x,t)\right)^{\frac{\beta pq}{\lambda}}(Mf(x,t))^{q-\frac{\beta pq}{\lambda}}dm_\alpha(x,t)\\
&\leqslant (C_1+C_2)^q\left(\int_{\mathbb{K}}\left(M_{\frac{\lambda}{p}} f(x,t)\right)^{\frac{\beta pqs'}{\lambda}}dm_\alpha(x,t)\right)^{1/s'}\\
&\quad\times\left(\int_{\mathbb{K}}(Mf(x,t))^{(q-\frac{\beta pq}{\lambda})s}dm_\alpha(x,t)\right)^{1/s},
\end{aligned}$$

where $(q-\frac{\beta pq}{\lambda})s=p,\ s'=\frac{s}{s-1}=\frac{\lambda r}{\beta pq},\ \frac{1}{q}=\frac{1}{p}-\frac{\beta}{\lambda}+\frac{\beta p}{\lambda r}$. Then we have

$$\left(\int_{\mathbb{K}}\left|I_\beta f(x,t)\right|^q dm_\alpha(x,t)\right)^{1/q}\leqslant (C_1+C_2)\left(\int_{\mathbb{K}}(Mf(x,t))^p\,dm_\alpha(x,t)\right)^{1/sq}$$
$$\times\left(\int_{\mathbb{K}}\left(M_{\frac{\lambda}{p}}f(x,t)\right)^r dm_\alpha(x,t)\right)^{\frac{\beta p}{\lambda r}}$$
$$\leqslant (C_1+C_2)A_p^{\frac{p}{sq}}\left(\int_{\mathbb{K}}|f(x,t)|^p dm_\alpha(x,t)\right)^{1/sq}$$
$$\times\left(\int_{\mathbb{K}}\left(M_{\frac{\lambda}{p}}f(x,t)\right)^r dm_\alpha(x,t)\right)^{\frac{\beta p}{\lambda r}}$$

and therefore

$$\left\|I_\beta f\right\|_{L_q(\mathbb{K})}\leqslant (C_1+C_2)A_p^{\frac{p}{sq}}\ \|f\|_{L_p(\mathbb{K})}^{\frac{p}{sq}}\ \left\|M_{\frac{\lambda}{p}}f\right\|_{L_r(\mathbb{K})}^{\frac{\beta p}{\lambda}}$$
$$\leqslant (C_1+C_2)A_p{}^{1-\frac{\beta p}{\lambda}}\ \|f\|_{L_p(\mathbb{K})}^{1-\frac{\beta p}{\lambda}}\left\|M_{\frac{\lambda}{p}}f\right\|_{L_r(\mathbb{K})}^{\frac{\beta p}{\lambda}}.$$

Thus the proof of Theorem 2 is completed. □

THEOREM 3. *Let* $0<\beta<2\alpha+4,\ 1\leqslant p<\frac{2\alpha+4}{\beta}$ *and* $f\in L_p(\mathbb{K})$. *Then for any* $(x,t)\in\mathbb{K}$ *the following estimation is valid*

$$I_\beta|f|(x,t)\leqslant (C_1+C_3)\|f\|_{L_p(\mathbb{K})}^{\frac{\beta p}{2\alpha+4}}\ (Mf(x,t))^{1-\frac{\beta p}{2\alpha+4}}, \tag{10}$$

where $C_3=\left(\frac{\Omega_2}{p'(\frac{1}{p}-\frac{\beta}{2\alpha+2})}\right)^{\frac{1}{p'}}$.

Proof. As in the proof of Lemma 2, we write

$$I_\beta|f|(x,t)=J_1(x,t,r)+J_2(x,t,r).$$

For estimating $J_2(x,t,r)$, we use Hölder's inequality, and inequality (1). Then we get

$$J_2(x,t,r)\leqslant\left(\int_{\complement_{B_r}}\left(T_{(x,t)}^{(\alpha)}|f(y,s)|\right)^p dm_\alpha(y,s)\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\complement_{B_r}}|(y,s)|_{\mathbb{K}}^{(\beta-2\alpha-4)p'}\ dm_\alpha(y,s)\right)^{\frac{1}{p'}}$$
$$\leqslant\left\|T_{(x,t)}^{(\alpha)}f\right\|_{L_p(\mathbb{K})}\ \left(\int_{\complement_{B_r}}|(y,s)|_{\mathbb{K}}^{(\beta-2\alpha-4)p'}\ dm_\alpha(y,s)\right)^{\frac{1}{p'}}$$
$$\leqslant\|f\|_{L_p(\mathbb{K})}\ \left(\int_{\complement_{B_r}}|(y,s)|_{\mathbb{K}}^{(\beta-2\alpha-4)p'}\ dm_\alpha(y,s)\right)^{\frac{1}{p'}}.$$

Passing to spherical coordinates, we have

$$\left(\int_{\complement B_r} |(y,s)|_{\mathbb{K}}^{(\beta-2\alpha-4)p'}\, dm_\alpha(y,s)\right)^{\frac{1}{p'}} = \left(\int_\Sigma \int_r^\infty t^{2\alpha+3+(\beta-2\alpha-4)p'}\, dt d\xi'\right)^{\frac{1}{p'}} = C_3 r^{\beta-\frac{2\alpha+4}{p}}.$$

Hence

$$J_2(x,t,r) \leqslant C_3 r^{\beta-\frac{2\alpha+4}{p}} \|f\|_{L_p(\mathbb{K})}. \tag{11}$$

Thus from (5) and (11), we get

$$|I_\beta f|(x,t) \leqslant C_1 r^\beta Mf(x,t) + C_3 r^{\beta-\frac{2\alpha+4}{p}} \|f\|_{L_p(\mathbb{K})}.$$

Minimizing at $r = \left[(Mf(x,t))^{-1} \|f\|_{L_p(\mathbb{K})}\right]^{\frac{p}{2\alpha+4}}$, we have

$$|I_\beta f|(x,t) \leqslant (C_1 + C_3) \|f\|_{L_p(\mathbb{K})}^{\frac{\beta p}{2\alpha+4}} \, (Mf(x,t))^{1-\frac{\beta p}{2\alpha+4}}.$$

Theorem 3 is proved. $\square$

THEOREM 4. *Let* $0<\beta<2\alpha+4$. *Then for any measurable functions* $f \geqslant 0$ *and* $0<\theta<1$ *for any* $(x,t)\in\mathbb{K}$ *the following estimates*

$$I_{\beta\theta} f(x,t) \leqslant (C_4+1) \left(I_\beta f(x,t)\right)^\theta (Mf(x,t))^{1-\theta}, \tag{12}$$

and

$$I_{\beta\theta} f(x,t) \leqslant (C_4+C_5) \left(M_\beta f(x,t)\right)^\theta (Mf(x,t))^{1-\theta} \tag{13}$$

are valid, where $C_4 = \frac{\Omega_2 2^{2\alpha+4}}{2^{\beta\theta}-1}$, $C_5 = \frac{\Omega_2^{1-\frac{\beta}{2\alpha+4}} 2^{2\alpha+4-\beta}}{1-2^{\beta\theta-\beta}}$.

Proof. We have

$$\begin{aligned} I_{\beta\theta} f(x,t) &= \int_{\mathbb{K}} T_{(x,t)}^{(\alpha)} f(y,s)|(y,s)|_{\mathbb{K}}^{\beta\theta-2\alpha-4} dm_\alpha(y,s) \\ &= \left(\int_{B_r} + \int_{\complement B_r}\right) T_{(x,t)}^{(\alpha)} f(y,s)|(y,s)|_{\mathbb{K}}^{\beta\theta-2\alpha-4} dm_\alpha(y,s) \\ &= I_1(x,t,r) + I_2(x,t,r). \end{aligned}$$

Firstly, we prove the inequality (12). Consider $I_2(x,t,r)$. Since $0<\theta<1$, then $\beta\theta-\beta<0$ and $|(y,s)|_{\mathbb{K}}^{\beta\theta-\beta} \leqslant r^{\beta\theta-\beta}$ for any $(y,s)\in {}^{\complement}B_r$. Hence

$$\begin{aligned} I_2(x,t,r) &= \int_{\complement B_r} T_{(x,t)}^{(\alpha)} f(y,s)|(y,s)|_{\mathbb{K}}^{\beta\theta-2\alpha-4} dm_\alpha(y,s) \\ &\leqslant r^{\beta\theta-\beta} \int_{\complement B_r} T_{(x,t)}^{(\alpha)} f(y,s)|(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} dm_\alpha(y,s) \\ &\leqslant r^{\beta\theta-\beta} I_\beta f(x,t). \end{aligned} \tag{14}$$

Taking into account (5) and (14) we have

$$I_1(x,t,r) \leqslant C_4 r^{\beta\theta} Mf(x,t), \tag{15}$$

$$I_2(x,t,r) \leqslant r^{\beta\theta-\beta} I_\beta f(x,t). \tag{16}$$

Thus from (15) and (16), we get

$$I_{\beta\theta} f(x,t) \leqslant C_4 r^{\beta\theta}\, Mf(x,t) + r^{\beta\theta-\beta}\, I_\beta f(x,t). \tag{17}$$

Minimizing at $r = \left[(Mf(x,t))^{-1} I_\beta f(x,t)\right]^{\frac{1}{\beta}}$ we have

$$I_{\beta\theta} f(x,t) \leqslant (C_4+1)\left(I_\beta f(x,t)\right)^\theta \ (Mf(x,t))^{1-\theta}.$$

Secondly, we prove inequality (13). Consider $I_2(x,t,r)$. Summing over every $j>0$, we get

$$\begin{aligned} I_2(x,t,r) &\leqslant \sum_{j=0}^{\infty} \int_{B_{2^{j+1}r}\setminus B_{2^j r}} T^{(\alpha)}_{(x,t)} f(y,s) |(y,s)|_{\mathbb{K}}^{\beta\theta-2\alpha-4} dm_\alpha(y,s) \\ &\leqslant \sum_{j=0}^{\infty} (2^j r)^{\beta\theta-2\alpha-4} \int_{B_{2^{j+1}r}} T^{(\alpha)}_{(x,t)} f(y,s) dm_\alpha(y,s) \\ &\leqslant 2^{2\alpha+4-\beta} \Omega_2^{1-\frac{\beta}{2\alpha+4}}\ r^{\beta\theta-\beta}\ M_\beta f(x,t) \sum_{j=0}^{\infty} 2^{(\beta\theta-\beta)j} \\ &\leqslant C_5\ r^{\beta\theta-\beta}\ M_\beta f(x,t). \end{aligned}$$

Hence

$$I_2(x,t,r) \leqslant C_5\ r^{\beta\theta-\beta}\ M_\beta f(x,t). \tag{18}$$

Taking into account (15) and (18) we have

$$I_{\beta\theta} f(x,t) \leqslant C_4\ r^{\beta\theta}\ Mf(x,t) + C_5\ r^{\beta\theta-\beta}\ M_\beta f(x,t).$$

Minimizing at $r = \left[(Mf(x,t))^{-1}\ M_\beta f(x,t)\right]^{\frac{1}{\beta}}$, we get

$$I_{\beta\theta} f(x,t) \leqslant (C_4+C_5)\ \left(M_\beta f(x,t)\right)^\theta\ (Mf(x,t))^{1-\theta}.$$

Thus Theorem 4 is proved. □

THEOREM 5. *Let* $0<\beta<2\alpha+4$, $f \in L_p(\mathbb{K})$, $1<p<\frac{2\alpha+4}{\beta}$. *Then*

$$\|I_{\beta\theta} f\|_{Lr(\mathbb{K})} \leqslant (C_4+1) A_p^{1-\theta}\ \|I_\beta |f|\|^\theta_{L_q(\mathbb{K})}\ \|f\|^{1-\theta}_{L_p(\mathbb{K})} \tag{19}$$

and

$$\|I_{\beta\theta} f\|_{L_r(\mathbb{K})} \leqslant (C_4+C_5) A_p^{1-\theta}\ \|M_\beta f\|^\theta_{L_q(\mathbb{K})}\ \|f\|^{1-\theta}_{L_p(\mathbb{K})}, \tag{20}$$

where $0<\theta<1$, $0<q\leqslant\infty$, $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}$.

Proof. We prove inequality (19). Inequality (20) can be proved similarly. From (12) and Hölder's inequality we have

$$\|I_{\beta\theta}f\|_{L_r(\mathbb{K})} \leqslant (C_4+1)\| \left(I_\beta|f|\right)^{\theta} (Mf)^{1-\theta} \|_{L_r(\mathbb{K})}$$
$$\leqslant (C_4+1)\| \left(I_\beta|f|\right)^{\theta} \|_{L_{r\tau'}(\mathbb{K})}\| (Mf)^{1-\theta} \|_{L_{r\tau}(\mathbb{K})}.$$

Enter the following designation: $p=(1-\theta)r\tau$, $q=\theta r\tau'$, where $\tau'=\frac{\tau}{\tau-1}$. Then obviously, $\frac{1}{r\tau}=\frac{1-\theta}{p}$, and $\frac{1}{r\tau'}=\frac{\theta}{q}$. Hence we obtain

$$\|I_{\beta\theta}f\|_{L_r(\mathbb{K})} \leqslant (C_4+1)\|I_\beta|f|\|^{\theta}_{L_q(\mathbb{K})}\|Mf\|^{1-\theta}_{L_p(\mathbb{K})}.$$

From the last inequality and Theorem 1, we have

$$\|I_{\beta\theta}f\|_{L_r(\mathbb{K})} \leqslant (C_4+1)A_p^{1-\theta} \parallel I_\beta|f| \parallel^{\theta}_{L_q(\mathbb{K})} \|f\|^{1-\theta}_{L_p(\mathbb{K})}.$$

Theorem 5 is proved. □

By using Lemma 2 and Theorems 1 and 4, it can be easily proved that the following Hardy-Littlewood-Sobolev theorem for fractional integrals on the Laguerre hypergroup is valid (see [6]).

THEOREM 6. *Let* $0<\beta<2\alpha+4$ *and* $1\leqslant p<\frac{2\alpha+4}{\beta}$.

1) If $1<p<\frac{2\alpha+4}{\beta}$, $f\in L_p(\mathbb{K})$ *and* $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{2\alpha+4}$, *then* $I_\beta f\in L_q(\mathbb{K})$ *and*

$$\left\|I_\beta f\right\|_{L_q(\mathbb{K})} \leqslant (C_1+C_2)A_p^{\frac{p}{q}} \|f\|_{L_p(\mathbb{K})}.$$

2) If $f\in L_1(\mathbb{K})$ *and* $1-\frac{1}{q}=\frac{\beta}{2\alpha+4}$, *then* $I_\beta f\in WL_q(\mathbb{K})$ *and*

$$\left\|I_\beta f\right\|_{WL_q(\mathbb{K})} \leqslant q(q-1)^{1/q-1}C_1^{1/q}(C_2)^{1-1/q}A_1\|f\|_{L_1(\mathbb{K})}.$$

Proof. i) For $r=\infty$, $\lambda=2\alpha+4$ from (1) and (8), we have

$$\|I_\beta f\|_{L_{q,\alpha}} \leqslant (C_1+C_2)A_p{}^{\frac{p}{q}} \|M_{\frac{2\alpha+4}{p}}f\|^{1-\frac{p}{q}}_{L_\infty(\mathbb{K})} \|f\|^{\frac{p}{q}}_{L_p(\mathbb{K})}$$
$$\leqslant (C_1+C_2)A_p{}^{\frac{p}{q}} \operatorname*{ess\,sup}_{(x,t)\in\mathbb{K}}\|T^{(\alpha)}_{(x,t)}|f|\|^{1-\frac{p}{q}}_{L_p(\mathbb{K})} \|f\|^{\frac{p}{q}}_{L_p(\mathbb{K})}$$
$$\leqslant (C_1+C_2)A_p^{\frac{p}{q}} \|f\|_{L_p(\mathbb{K})}.$$

ii) From (4) for $p=1$ and $\lambda=2\alpha+4$, we obtain

$$I_\beta|f|(x,t) \leqslant C_1r^{\beta}Mf(x,t)+C_2r^{\beta-2\alpha-4}M_{2\alpha+4}f(x,t)$$
$$\leqslant q(q-1)^{1/q-1}C_1^{1/q}C_2^{1-1/q}\,(Mf(x,t))^{\frac{1}{q}} \,(M_{2\alpha+2}f(x,t))^{1-\frac{1}{q}}$$
$$\leqslant q(q-1)^{1/q-1}C_1^{1/q}(C_2)^{1-1/q}\,(Mf(x,t))^{\frac{1}{q}} \|f\|^{1-\frac{1}{q}}_{L_1(\mathbb{K})}.$$

Then applying Theorem 1, we get

$$\mu_\alpha\left\{(x,t)\in\mathbb{K}: I_\beta|f|(x,t)>t\right\}^{1/q}$$
$$\leqslant \mu_\alpha\left\{(x,t)\in\mathbb{K}: Mf(x,t)>q^{-q}(q-1)^{q-1}C_1^{-1}(C_2)^{1-q}t^q\|f\|_{L_1(\mathbb{K})}^{1-q}\right\}^{1/q}$$
$$\leqslant q(q-1)^{1/q-1}C_1^{1/q}(C_2)^{1-1/q}\frac{A_1}{t}\|f\|_{L_1(\mathbb{K})}.$$

Therefore the proof of the theorem is completed. □

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