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A NOTE ON STOLARSKY, ARITHMETIC AND LOGARITHMIC MEANS

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Abstract. We present a way to study differences of some Stolarsky means as a way to discover new inequalities, or place known inequalities in a wider context. In particular, as an application we prove a very sharp upper bound for the difference between the arithmetic and the logarithmic means of two positive numbers.

1. Introduction and main results

Alomari, Darus and Dragomir [1], as an application of Hadamard-Hermite type inequalities, give some estimates for the difference between some generalized logarithmic means and power means (see Propositions 1, 2, and 3 in [1]). For example, they prove

$$\left|\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}-\frac{a^n+b^n}{2}\right| \leqslant \frac{n(n-1)}{12}(b-a)^2 \max\{|a|^{n-2},|b|^{n-2}\},$$

where $a<b$ are real and $n \geqslant 2$ is an integer. This inequality is very sharp, though the authors do not specify if constants such as 12 are best possible. In our Theorem 2 below (see part (c) in reference to the example we just quoted) we extend these inequalities to the full range of powers $p \in \mathbb{R}$, thus showing interesting reversals and changes that occur when p falls in certain intervals, besides putting this kind of result in a broader, perhaps more natural context (our proofs also show that the inequalities given here are optimal within their natural category).

The broader context we suggest is the one provided by the large family of Stolarsky means (or difference means, as they are sometimes called — see below for the definitions). With the notation to be introduced in Definition 1, the general idea is to study differences of powers

$$S^p_{(a,b)} - S^q_{(c,d)}$$

of Stolarsky means for various parameters a,b,c,d,p,q, and derive inequalities for them. To carry out the general program at once seems prohibitive because of the nature of the calculations involved, but a large variety of special cases appear feasible and can lead to new discoveries.

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On top of the example mentioned at the start, as an illustration of this method we will prove Theorem 5, which in turn will produce a very sharp and unusual looking upper bound for the difference between the arithmetic and the logarithmic means of two positive numbers (see Theorem 8).

DEFINITION 1. For $p,q \in \mathbb{R}$, $q \neq -1$ and $x \neq y$ in $\mathbb{R}^+$ we define the *Stolarsky mean* of x and y by

$$S_{(p,q)}(x,y) := \left(\frac{q(x^p - y^p)}{p(x^q - y^q)}\right)^{1/(p-q)}$$

(see [9, 10]). This notion offers the chance to unify an astonishing array of previously known averages and means at the price of the introduction of more parameters, and thus losing some transparency. In applications we are often interested in several special cases. For example, taking the limit $q \to 0$ we set

$$S_{(p,0)}(x,y) := \left(\frac{x^p - y^p}{\ln(x^p) - \ln(y^p)}\right)^{\frac{1}{p}} = L_0(x^p, y^p)^{\frac{1}{p}}$$

(see further below for the definition of L_0). Also, taking the limit $q \to p$ we set

$$S_{(p,p)}(x,y) := \exp\left(\frac{x^p \ln x - y^p \ln y}{x^p - y^p} - \frac{1}{p}\right),$$

Note that $\lim_{p\to 0} S_{(p,p)}(x,y) = \sqrt{xy}$, which is the familiar *geometric mean* of x and y.

Often the special case $S_{(p,1)}$ (where we set $q = 1$) is considered, and is referred to as the generalized logarithmic mean of x and y. The specialized notation then becomes

$$L_p(x,y) := \left(\frac{x^p - y^p}{p(x-y)}\right)^{1/(p-1)}$$

for $p \notin \{0,1\}$, and (taking the limit $p \to 1$)

$$L_1(x,y) := \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}$$

(the latter being the so-called *identric mean* of x and y). Taking the limit $p \to 0$ we also set

$$L_0(x,y) := \frac{x-y}{\ln x - \ln y},$$

which is the familiar *logarithmic mean* of x and y. Note that $L_{-1}(x,y)$ is again the geometric mean of x and y. Other special cases of interest are

$$L_2(x^p, y^p) = M_p(x,y)^p,$$

$$L_{-\infty}(x,y) = \min\{x,y\} = M_{-\infty}(x,y),$$

$$L_{\infty}(x,y) = \max\{x,y\} = M_{\infty}(x,y),$$

where of course the $M_p(x,y)$ are the very familiar *power means* of x and y. Further, we note that we can also obtain these power means with a different choice of parameters, namely

$$S_{(2p,p)}(x,y) = \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} = M_p(x,y),$$

Theorem 2 below generalizes the inequalities stated in Propositions 1,2, and 3 of [1], in that it gives estimates for the difference

$$S_{(2p,p)}(x,y)^p - S_{(p+1,1)}(x,y)^p$$

for all values of the parameter p. The attentive reader will certainly notice that in Theorem 2 and Lemma 3 we fail to mention any upper estimate in the case when $p < 0$. The reason for this is that when p is negative then the difference we are estimating can grow without bounds as either x or y is small enough, and the order of growth to infinity obviously depends on the value of parameter p: thus, it seems like a losing proposition to pursue an estimate in this case, since estimates are generally supposed to be algebraically simpler than the original expression.

THEOREM 2. *If $x,y > 0$ and $p \in \mathbb{R}$, then we have the inequalities*

(a) If $p < 0$:

$$\frac{p(p-1)}{12}(x-y)^2 \min\{x,y\}^{p-2} \leqslant M_p(x,y)^p - L_{p+1}(x,y)^p.$$

(b) If $0 < p < 1$:

$$\begin{aligned}\frac{p-1}{2(p+1)}(x-y)^2 \min\{x,y\}^{p-2} &\leqslant M_p(x,y)^p - L_{p+1}(x,y)^p \\ &\leqslant \frac{p(p-1)}{12}(x-y)^2 \min\{x,y\}^{p-2} \\ &\leqslant 0.\end{aligned}$$

(c) If $2 < p$:

$$\begin{aligned}\frac{p-1}{2(p+1)}(x-y)^2 \max\{x,y\}^{p-2} &\leqslant M_p(x,y)^p - L_{p+1}(x,y)^p \\ &\leqslant \frac{p(p-1)}{12}(x-y)^2 \max\{x,y\}^{p-2}.\end{aligned}$$

(d) If $1 < p < 2$:

$$\begin{aligned}\frac{p(p-1)}{12}(x-y)^2 \min\{x,y\}^{p-2} &\leqslant M_p(x,y)^p - L_{p+1}(x,y)^p \\ &\leqslant \frac{p-1}{2(p+1)}(x-y)^2 \min\{x,y\}^{p-2}.\end{aligned}$$

Note that if $p \in \{0,1\}$ *we trivially have* $M_p(x,y)^p = L_{p+1}(x,y)^p$*, and if* $p = 2$ *the identity* $M_p(x,y)^p - L_{p+1}(x,y)^p = \frac{1}{6}(x-y)^2$ *is also immediate. In every other case identity holds if and only if* $x = y$.

Theorem 2 will follow easily from the following lemma, which we state separately as it could be of independent interest.

LEMMA 3. *If* $0 \leqslant x \leqslant 1$*, and* $p \in \mathbb{R}$*, then we have the inequalities*

(a) If $p < 0$*:*

$$\frac{p(p-1)}{12}(x-1)^2 \leqslant M_p(x,1)^p - L_{p+1}(x,1)^p.$$

(b) If $0 < p < 1$ *or* $2 < p$*:*

$$\frac{p-1}{2(p+1)}(x-1)^2 \leqslant M_p(x,1)^p - L_{p+1}(x,1)^p \leqslant \frac{p(p-1)}{12}(x-1)^2.$$

(c) If $1 < p < 2$*:*

$$\frac{p(p-1)}{12}(x-1)^2 \leqslant M_p(x,1)^p - L_{p+1}(x,1)^p \leqslant \frac{p-1}{2(p+1)}(x-1)^2.$$

Note that if $p \in \{0,1\}$ *we trivially have* $M_p(x,1)^p = L_{p+1}(x,1)^p$*, and if* $p = 2$ *the identity* $M_p(x,1)^p - L_{p+1}(x,1)^p = \frac{1}{6}(x-1)^2$ *is also immediate. In every other case identity holds if and only if* $x = 0$ *or* $x = 1$.

The proof of Lemma 3 is delayed to the next section.

Proof of Theorem 2. Let $x, y > 0$, and assume wlog that $x < y$. Applying Lemma 3 to $x/y \in (0,1)$ gives inequalities for

$$M_p\left(\frac{x}{y},1\right)^p - L_{p+1}\left(\frac{x}{y},1\right)^p = y^{-p}\left(M_p(x,y)^p - L_{p+1}(x,y)^p\right),$$

and this easily leads to the theorem. □

We would like to extend these results to general differences of Stolarsky means, though this program often meets with considerable technical problems. Very helpful in this context is the following comparison result by Leach and Sholander [5], which we quote in a version later proved by Páles:

THEOREM 4. (see Páles [7]) *Let* $a,b,c,d \in \mathbb{R}$*. Then the comparison inequality*

$$S_{(a,b)}(x.y) \leqslant S_{(c,d)}(x,y)$$

holds true for all $x,y>0$ *if and only if* $a+b\leqslant c+d$ *and*

$$\begin{array}{ll} L(a,b)\leqslant L(c,d) & \text{if } 0\leqslant \min\{a,b,c,d\} \\ \mu(a,b)\leqslant \mu(c,d) & \text{if } \min\{a,b,c,d\}<0<\max\{a,b,c,d\} \\ -L(-a,-b)\leqslant -L(-c,-d) & \text{if } \max\{a,b,c,d\}\leqslant 0, \end{array}$$

where

$$L(x,y):=\begin{cases} \frac{x-y}{\ln x-\ln y} & x,y>0,\ x\neq y \\ 0 & x\cdot y=0 \end{cases}$$

and

$$\mu(x,y):=\begin{cases} \frac{|x|-|y|}{x-y} & x\neq y \\ \operatorname{sign}(x) & x=y. \end{cases}$$

If $a+b\leqslant c+d$, we have

$$S_{(c,d)}(x,y)^{c-d}-S_{(a,b)}(x,y)^{a-b}=\frac{d(x^c-y^c)}{c(x^d-y^d)}-\frac{b(x^a-y^a)}{a(x^b-y^b)}.$$

For example, if $p\in\mathbb{R}$ is fixed then

$$S_{(p,0)}(x.y)\leqslant S_{(t+p,t)}(x,y)$$

for all $x,y>0$ provided that $t\geqslant 0$. This pair of Stolarsky means is of interest if we look at the special case $p=t$, in which case the inequality between the two means takes on the more familiar form

$$L_0(x^t,y^t)^{\frac{1}{t}}\leqslant M_t(x,y)$$

(see our introduction for the notation) or, replacing x^t,y^t with x,y,

$$L_0(x,y)\leqslant M_1(x,y),$$

which of course is the well-known inequality between the arithmetic and the logarithmic mean of x and y. With the target of finding an upper bound for the difference between these two means, we now present our next theorem. A more interesting lower bound was not pursued here since the focus of our analysis revolves around the behavior of the expression for $y=1$ and around $x=1$, and this technique doesn't seem to yield useful information when x is small.

THEOREM 5. *Let* $p,t>0$ *and* $x,y\in\mathbb{R}$. *Then*

$$0\leqslant S_{(p+t,t)}(x,y)^p-S_{(p,0)}(x,y)^p\leqslant \max\left\{\frac{t}{t+p},\frac{pt}{12}\right\}(x-y)^2\max\{x,y\}^{p-2}.$$

The proof of Theorem 5 is delayed to the next section.

Letting $y=1$ in Theorem 5 we obtain:

COROLLARY 6. *For every $p,t>0$ and $x\in\mathbb{R}$ we have*

$$0\leqslant \frac{t}{t+p}\frac{x^{t+p}-1}{x^t-1}-\frac{x^p-1}{p\ln x}\leqslant \max\left\{\frac{t}{t+p},\frac{pt}{12}\right\}(x-1)^2\max\{x,1\}^{p-2}. \qquad \square$$

Letting $p=t$ in Theorem 5 and replacing x^t,y^t with x,y we then have the announced estimate for the difference between the arithmetic and logarithmic means:

COROLLARY 7. *For every $x,y,t>0$ we have*

$$0\leqslant \frac{x+y}{2}-\frac{x-y}{\ln x-\ln y}\leqslant \frac{1}{2}\max\left\{1,\frac{t^2}{6}\right\}(x^{1/t}-y^{1/t})^2\max\{x,y\}^{1-2/t}. \qquad \square \tag{1}$$

Finally, we improve on this corollary by showing that $t=\sqrt{6}$ yields the best right hand side value:

THEOREM 8. *For every $x,y>0$ we have*

$$0\leqslant \frac{x+y}{2}-\frac{x-y}{\ln x-\ln y}\leqslant \frac{1}{2}\left(x^{1/\sqrt{6}}-y^{1/\sqrt{6}}\right)^2\max\{x,y\}^{1-2/\sqrt{6}}. \tag{2}$$

Proof. We need to show that the right hand side in (2) is the infimum (and the limit for $t\to\infty$) of the right hand side in (1). Wlog, assume that $x<y$ and rewrite the right hand side of (1) as

$$\frac{1}{2}y\max\left\{1,\frac{t^2}{6}\right\}\left(\left(\frac{x}{y}\right)^{1/t}-1\right)^2. \tag{3}$$

Trivially, $(x/y)^{1/t}$ increases as t increases from 0 to $\sqrt{6}$ and thus (3) decreases. When $t\geqslant\sqrt{6}$, (3) simplifies to

$$\frac{1}{12}yt^2\left(\left(\frac{x}{y}\right)^{1/t}-1\right)^2.$$

Now, the function $f(t)=t(1-u^{1/t})$ is easily shown to be increasing for $t>0$ (and positive for $u\in(0,1)$) and thus (3) must have reached a minimum when $t=\sqrt{6}$. The theorem is proved. $\square$

Inequality (2) compares very nicely with an analogously very tight bound on the ratio between the two means proved by Carlson [3] (see Theorems 2.6-2 and 2.6-3). While Carlson's inequalities are hard to beat when x and y are relatively close, our result appears to be better than his when the ratio x/y is far enough from 1. The details are extremely cumbersome, however (though they can be easily visualized in *Mathematica*, for example), so we omit further discussion.

As a final curiosity, a slightly weaker but perhaps more "readable" inequality can be quickly obtained from the proof of Theorem 8:

COROLLARY 9. *For every $x,y>0$ we have*

$$0 \leqslant \frac{x+y}{2} - \frac{x-y}{\ln x - \ln y} \leqslant \frac{1}{12}\Big(\ln x - \ln y\Big)^2 \max\{x,y\}. \tag{4}$$

Proof. To see this, note the the function $f(t)$ defined at the end of the proof of Theorem 8 satisfies $\lim_{t\to\infty} f(t) = -\ln u$. Since this function is increasing, as we move further up and away from $t=\sqrt{6}$ the upper bound in our inequality becomes larger, and the corollary follows. □

2. Proofs of Lemma 3 and Theorem 5

Proof. [Proof of Lemma 3] For $p \in \mathbb{R}\setminus\{-1\}$ define

$$f(x,p) := \frac{x^p+1}{2} - \frac{x^{p+1}-1}{(p+1)(x-1)},$$

and set

$$f(x,-1) := \frac{\frac{1}{x}+1}{2} - \frac{\ln x}{x-1}$$

(the latter function being $\lim_{p\to-1} f(x,p)$: the claim for $p=-1$ will simply follow from the case $p\neq -1$ and taking limits, and this is why in the rest of the proof we will always assume $p\neq -1$). Then $f(x,p) = M_p(x,1)^p - L_{p+1}(x,1)^p$ for all $p\in\mathbb{R}$. For all p, expressing $f(x,p)$ as a power series centered at $x=1$ shows that

$$f(x,p) = \frac{p(p-1)}{12}(x-1)^2 + O\big((x-1)^3\big) \tag{5}$$

(we omit the standard details), and this is where we find the motivation for the main bound of $f(x,p)$. To simplify the calculations, define the function

$$g(x,p) := \frac{2(p+1)}{(x-1)^2}\Big(\frac{p(p-1)}{12}(x-1)^2 - f(x,p)\Big) \tag{6}$$

Our next target is to prove that $g(x,p)$ is strictly monotone on the interval $(0,1)$. To this end, observe that

$$\frac{\partial}{\partial x} g(x,p)(x) = -\frac{g_1(x,p)}{(x-1)^4}, \tag{7}$$

where $g_1(x,p)$ is the function

$$\begin{aligned} g_1(x,p) := {} & (p-2)(p-1)x^{p+1} - 2(p-2)(p+1)x^p + p(p+1)x^{p-1} \\ & -2(p+1)x + 2(p-2). \end{aligned}$$

In terms of $g_1(x,p)$, our claim now translates into saying that $g_1(x,p)$ keeps the same sign through the interval $(0,1)$. We now proceed to verify this.

A calculation reveals first that

$$\frac{\partial^2}{\partial x^2} g_1(x,p) = (p-2)(p-1)p(p+1)(x-1)^2 x^{p-3}, \tag{8}$$

and thus that $\frac{\partial^2}{\partial x^2} g_1(x,p)$ keeps the same sign (and is non-zero) through $(0,1)$ (for whoever noticed: the exception $p \neq 1$ has already been considered, and similarly $p \in \{0,1,2\}$ will be recovered as limit cases). In turn, this tells us that $\frac{\partial}{\partial x} g_1(x,p)$ is strictly monotone on $[0,1]$, and from the formula

$$\frac{\partial}{\partial x} g_1(x,p) = (p+1)\Big((p-2)(p-1)x^p + (p-1)px^{p-2} - 2(p-2)px^{p-1} - 2\Big)$$

we conclude that $\frac{\partial}{\partial x} g_1(x,p)$ changes monotonically from its value or limit at $x=0$ to 0 (which is its value at $x=1$). It follows that $g_1(x,p)$ is strictly monotone on $(0,1)$. Since the limit of $g_1(x,p)$ for $p \to 1$ is zero, $g_1(x,p)$ (and thus $\frac{\partial}{\partial x} g(x,p)(x)$ by (7)) keeps the same sign over the whole interval $(0,1)$ and the claim is proved: we now know that $g(x,p)$ is strictly monotone.

Moving on now, let us note that if $p < -1$, or $0 < p < 1$, or $p > 2$ then by (8) $\frac{\partial^2}{\partial x^2} g_1(x,p)$ is positive on $(0,1)$, therefore $\frac{\partial}{\partial x} g_1(x,p)$ is strictly increasing and negative. Thus, $g_1(x,p)$ is strictly decreasing on $(0,1)$, which implies that it's always positive for $x \in (0,1)$. A look at equation (7) tells us that $g(x,p)$ decreases to 0 (its value at $x=1$) in all these cases.

If instead we assume that $-1 < p < 0$ or $1 < p < 2$, then by (8) $\frac{\partial^2}{\partial x^2} g_1(x,p)$ is negative on $(0,1)$, therefore $\frac{\partial}{\partial x} g_1(x,p)$ is strictly decreasing and positive. Thus, $g_1(x,p)$ is strictly increasing on $(0,1)$, which implies that it's always negative for $x \in (0,1)$. A look at equation (7) tells us that $g(x,p)$ increases to 0 (its value at $x=1$).

Considering that

$$\lim_{x \downarrow 0} g(x,p) = \begin{cases} \frac{(p+3)(p-1)(p-2)}{6} & : p > 0 \\ -\infty & : p < 0 \end{cases}$$

we have the inequalities

$$\begin{aligned} 0 \leqslant g(x,p) &< \infty, \quad p < -1, \\ -\infty < g(x,p) &\leqslant 0, \quad -1 < p < 0, \\ 0 \leqslant g(x,p) &\leqslant (p+3)(p-1)(p-2)/6, \quad 0 < p < 1 \text{ or } 2 < p, \\ (p+3)(p-1)(p-2)/6 \leqslant g(x,p) &\leqslant 0, \quad 1 < p < 2. \end{aligned}$$

Using equation (6) these inequalities immediately translate into

$$\frac{p(p-1)}{12}(x-1)^2 \leqslant f(x,p),$$

in the case when $p < 0$;

$$\frac{p-1}{2(p+1)}(x-1)^2 \leqslant f(x,p) \leqslant \frac{p(p-1)}{12}(x-1)^2$$

if $0<p<1$ or $2<p$ and, finally,

$$\frac{p(p-1)}{12}(x-1)^2 \leqslant f(x,p) \leqslant \frac{p-1}{2(p+1)}(x-1)^2$$

if $1<p<2$. Since $f(x,p)=M_p(x,1)^p-L_{p+1}(x,1)^p$, the lemma is thus proved. □

Proof of Theorem 5. Define the function

$$\begin{aligned} f(x,t,p) &:= \frac{1}{(x-1)^2}\big(S_{(p+t,t)}(x,1)^p - S_{(p,0)}(x,1)^p\big) \\ &= \frac{1}{(x-1)^2}\Big(\frac{t}{t+p}\frac{x^{t+p}-1}{x^t-1} - \frac{x^p-1}{p\ln x}\Big). \end{aligned}$$

Let us first assume that $x\in(0,1)$, and note that

$$\lim_{x\downarrow 0} f(x,t,p) = \frac{t}{t+p} \quad \text{and} \quad \lim_{x\uparrow 1} f(x,t,p) = \frac{pt}{12}.$$

The theorem (in the case $x\in(0,1)$) is then proved if we show that the derivative $\frac{\partial}{\partial x}f(x,t,p)$ never vanishes for $x\in(0,1)$. A calculation gives

$$\frac{\partial}{\partial x}f(x,t,p) = \frac{x}{p(t+p)(x-1)^2(x^t-1)(\ln x)^2}\,g(x,t,p),$$

where we set

$$g(x,t,p) := -(t+p)(\ln x)(1-x^t)(1-x^p) - pt(\ln x)^2(1+x^{t+p}).$$

If we let $a:=x^{-t}$ and $b:=x^{-p}$ (note that $a,b>1$ by our assumptions) we then obtain

$$g(x,t,p) = \frac{1}{ab}\Big((\ln a+\ln b)(a-1)(b-1) - \ln a\ln b\,(ab+1)\Big),$$

We claim that $g(x,t,p)<0$ for $x\in(0,1)$. In terms of a and b this inequality is equivalent to

$$\frac{ab+1}{(a-1)(b-1)} > \frac{1}{\ln a} + \frac{1}{\ln b}.$$

Now, we can rewrite the left hand side as

$$\frac{ab+1}{(a-1)(b-1)} = \frac{a+1}{2(a-1)} + \frac{b+1}{2(b-1)} + \frac{2}{(a-1)(b-1)}$$

and, since

$$\frac{u+1}{2(u-1)} > \frac{1}{\ln u}$$

when $u \neq 1$ (a calculus exercise, or see [6, Lemma 2.2] for a simple proof), the claim is proved. Hence the inequality

$$0 \leqslant S_{(p+t,t)}(x,1)^p - S_{(p,0)}(x,1)^p \leqslant \max\left\{\frac{t}{t+p}, \frac{pt}{12}\right\}(x-1)^2 \tag{9}$$

follows, where identity only holds when $x \in \{0,1\}$.

Now assume that $x, y > 0$ are given with $x < y$. Applying (9) to $x/y \in (0,1)$ we have

$$\begin{aligned} y^{-p}\left(S_{(p+t,t)}(x,y)^p - S_{(p,0)}(x,y)^p\right) &= S_{(p+t,t)}(x/y,1)^p - S_{(p,0)}(x/y,1)^p \\ &\leqslant \max\left\{\frac{t}{t+p}, \frac{pt}{12}\right\}\left(\frac{x}{y}-1\right)^2. \end{aligned}$$

and we are done. □

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