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A NEW GENERAL BOAS–TYPE INEQUALITY AND RELATED CAUCHY–TYPE MEANS

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Abstract. We prove a new Boas-type inequality in a context of topological spaces and general σ-finite Borel measures. This enables us to introduce an one-parameter class of non-negative Boas differences and examine their properties, such as continuity and log-convexity. By proving the related Galvani's theorem and mean-value theorems of the Lagrange and Cauchy type we establish a new class of two-parameter Cauch-type means.

1. Introduction

R. P. Boas, in [3], proved that the inequality

$$\int_0^\infty \Phi\left(\frac{1}{M}\int_0^\infty f(tx)\,dm(t)\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\,\frac{dx}{x} \tag{1}$$

holds for all continuous convex functions $\Phi\colon [0,\infty\rangle \to \mathbb{R}$, measurable non–negative functions $f\colon \mathbb{R}_+ \to \mathbb{R}$, and non–decreasing bounded functions $m\colon [0,\infty\rangle \to \mathbb{R}$, where $M = m(\infty) - m(0) > 0$ and the inner integral on the left-hand side of (1) is the Lebesgue–Stieltjes integral with respect to m. After its author, the relation (1) was named the Boas inequality. In the case of a concave function Φ, (1) holds with the sign of inequality reversed.

Independently, S. Kaijser et al. [8] (see also the paper [9] of S. Levinson) established the so-called general Hardy-Knopp-type inequality for positive measurable functions $f\colon \mathbb{R}_+ \to \mathbb{R}$,

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\,\frac{dx}{x}, \tag{2}$$

where Φ is a real convex function on $\mathbb{R}_+$. Later on, A. Čižmešija et al. [6] generalized the relation (2) to the so-called strengthened Hardy-Knopp-type inequality by inserting a weight function and integrating over intervals of non-negative real numbers. Further, in [5] A. Čižmešija et al. considered a general Borel measure λ on $\mathbb{R}_+$, such that $L = \lambda(\mathbb{R}_+) = \int_0^\infty d\lambda(t) < \infty$, and for a convex function Φ on an interval $I \subseteq \mathbb{R}$ and for a weight function u on $\mathbb{R}_+$ proved that the inequality

$$\int_0^\infty u(x)\Phi(Af(x))\,\frac{dx}{x} \leqslant \frac{1}{L}\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x} \tag{3}$$

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holds for all measurable functions $f\colon \mathbb{R}_+ \to \mathbb{R}$, such that $f(x) \in I$ for all $x \in \mathbb{R}_+$, where $Af(x) = \frac{1}{L}\int_0^\infty f(tx)\,d\lambda(t)$ and $w(x) = \int_0^\infty u\left(\frac{x}{t}\right)d\lambda(t) < \infty$, $x \in \mathbb{R}_+$. They also gave the following refinement of (3):

$$\begin{aligned}
&\frac{1}{L}\int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x} - \int_0^\infty u(x)\Phi(Af(x))\,\frac{dx}{x} \\
&\quad \geqslant \frac{1}{L}\left|\int_0^\infty\int_0^\infty u(x)|\Phi(f(tx)) - \Phi(Af(x))|\,d\lambda(t)\,\frac{dx}{x}\right. \\
&\qquad \left. - \int_0^\infty\int_0^\infty u(x)|\varphi(Af(x))||f(tx) - Af(x)|\,d\lambda(t)\,\frac{dx}{x}\right|,
\end{aligned}$$

where φ denotes any function with values in the subdifferential of Φ.

Observe that the non-decreasing and bounded function $m\colon [0,\infty\rangle \to \mathbb{R}$, such that $M = m(\infty) - m(0) > 0$, induces a finite Borel measure λ on $\mathbb{R}_+$ and vice versa. For such function and measure, related Lebesgue and Lebesgue-Stieltjes integrals are equivalent. Thus, all the above results can be interpreted as for $Af(x)$ defined by

$$Af(x) = \frac{1}{M}\int_0^\infty f(tx)\,dm(t),\, x \in \mathbb{R}_+,$$

so they refine and generalize inequality (1).

The Boas inequality (1) has been generalized in some other ways. One of them is by using the weighted Hardy-Littlewood average $U_\psi f$ defined by

$$U_\psi f(x) = \int_0^1 f(tx)\psi(t)\,dt,$$

where ψ is a non-negative function on $[0,1]$. J. Xiao [12] characterized functions ψ for which U_ψ is bounded on either $L^p(\mathbb{R}^n)$, $p \in [1,\infty]$, or $BMO(\mathbb{R}^n)$. Recall that the space $BMO(\mathbb{R}^n)$ consists of all measurables functions $f \in L^1_{loc}(\mathbb{R}^n)$ with bounded mean oscillation

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_{Q\subset\mathbb{R}^n}\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ of sides parallel to the axes, $f_Q = |Q|^{-1}\int_Q f(x)dx$ stands for the average of f over Q, and $|Q|$ denotes the measure of Q.

Another generalization of (1) was given by D. Luor [10] in a setting with σ-finite Borel measures μ and ν on a topological space X and a Borel probability measure λ on $\mathbb{R}_+$. For a λ-balanced Borel set E in X and the measure μ_t defined for all Borel sets $D \subseteq X$ and $t \in \mathbb{R}_+$ by $\mu_t(D) = \mu(t^{-1}D)$, he proved the inequality

$$\int_E \phi(Hf(x))\,d\mu(x) \leqslant \int_E \phi(f(x))\left(\int_0^\infty \frac{d\mu_t}{d\nu}(x)\,d\lambda(t)\right)d\nu(x),$$

where ϕ is a non-negative convex function, $\mu_t \ll \nu$, $t \in \operatorname{supp}\lambda$, and Hf is the Hardy–Littlewood average of a non-negative Borel function f on X, defined by

$$Hf(x) = \int_0^\infty f(tx)\,d\lambda(t),\ x \in X.$$

Our first goal in this paper is to obtain the weighted version of the mentioned Luor's result. Further, by exploring non-negativity of the difference between the right-hand side and the left-hand side of our inequality obtained, we introduce an isotonic linear functional, the so-called Boas difference. We examine its properties and give some Lyapunov-type inequalities which provide us with an upper bound for its values. We review some one-dimensional special cases of these differences as well. Also, we give some related results for balls in $\mathbb{R}^n$. Our Boas-type inequality and the notion of the Boas differences allow us to state and prove new mean value theorems of the Lagrange and Cauchy-type and to define a new class of two-parameter Cauchy-type means.

This paper is a continuition of a previous work of J. Pečarić with S. Hussain and M. Anwar [2], [7]. It is organized in the following way: after the Introduction, in Section 2 we prove the Boas inequality in a setting with general weighted topological spaces and σ-finite measures. Further, in Section 3, we explore logarithmic and exponential convexity of the Boas differences and derive the related Lyapunov-type inequality. Finally, in the last Section 4, we introduce a new class of the Cauchy-type means related to the Boas-type differences and examine their intermediacy, continuity, symmetry and monotonicity properties.

Conventions. Throughout this paper, all measures are assumed to be positive, all functions are assumed to be measurable on their respective domains and expressions of the form $0\cdot\infty$, $\frac{0}{0}$, $\frac{a}{\infty}(a \in \mathbb{R})$ and $\frac{\infty}{\infty}$ are taken to be equal to zero. As usual, by dx and $d\mathbf{x}$ we denote the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$ $(n \in \mathbb{N}, n \geqslant 2)$ respectively, while by a weight function we mean a non–negative measurable function on the actual set. An interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$, while $\mathbb{R}_+ = \langle 0,\infty\rangle$. For $b > 0$, by $B(b)$ we denote a ball in $\mathbb{R}^n$ centered at the origin and of radius b, that is, $B(b) = \{\mathbf{x} \in \mathbb{R}^n\colon |\mathbf{x}| \leqslant b\}$, where $|\mathbf{x}|$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$. Finally, its dual set is $\mathbb{R}^n \setminus B(b) = \{\mathbf{x} \in \mathbb{R}^n\colon |\mathbf{x}| > b\}$.

2. A new weighted general Boas-type inequality

After introducing some necessary notation, in this section we state and prove a new weighted general Boas-type inequality in a setting with a topological space and σ-finite Borel measures. Utilizing the inequality obtained, we further define a related isotonic linear functional, the so-called Boas difference.

Let λ be a finite Borel measure on $\mathbb{R}_+$. By $\operatorname{supp}\lambda$ we denote its support, that is, the set of all $t \in \mathbb{R}_+$ such that $\lambda(N_t) > 0$ holds for all open neighbourhoods N_t of t. Hence,

$$L = \int_{\operatorname{supp}\lambda} d\lambda(t) = \int_0^\infty d\lambda(t) = \lambda(\mathbb{R}_+) < \infty. \tag{4}$$

On the other hand, let X be a topological space equipped with a continuous scalar multiplication $(a,\mathbf{x}) \mapsto a\mathbf{x} \in X$, for $a \in \mathbb{R}_+$, $\mathbf{x} \in X$, such that

$$1\mathbf{x} = \mathbf{x}, \;\; a(b\mathbf{x}) = (ab)\mathbf{x}, \;\; \mathbf{x} \in X, \;\; a,b \in \mathbb{R}_+.$$

Further, let the Borel set $\Omega \subseteq X$ be λ-balanced, that is, $t\Omega = \{t\mathbf{x}\colon \mathbf{x} \in \Omega\} \subseteq \Omega$, for all $t \in \operatorname{supp} \lambda$. For a Borel measurable function $f\colon \Omega \to \mathbb{R}$, we define its Hardy–Littlewood average Af as

$$Af(\mathbf{x}) = \frac{1}{L}\int_0^\infty f(t\mathbf{x})\, d\lambda(t), \; \mathbf{x} \in \Omega. \tag{5}$$

Finally, suppose that μ and ν are σ–finite Borel measures on X. For $t > 0$ and a Borel set $S \subseteq X$ we define

$$\mu_t(S) = \mu\left(\frac{1}{t}S\right). \tag{6}$$

Obviously, μ_t is a σ-finite Borel measure on X for each $t \in \mathbb{R}_+$. Throughout this paper, we suppose that the measures μ_t are absolutely continuous with respect to the measure ν, that is, $\mu_t \ll \nu$ for each $t \in \operatorname{supp} \lambda$. As usual, by $\frac{d\mu_t}{d\nu}$ we denote the related Radon–Nikodym derivative.

We start with a generalization of the main theorem in [10], that is, we state and prove a new weighted general Boas-type inequality.

THEOREM 1. *Let λ be a finite Borel measure on $\mathbb{R}_+$ and L be defined by (4). Let μ and ν be σ-finite Borel measures on a topological space X, μ_t be defined by (6) and such that $\mu_t \ll \nu$ for all $t \in \operatorname{supp} \lambda$. Further, let $\Omega \subseteq X$ be a λ-balanced set and u be a non-negative function on X, such that*

$$v(\mathbf{x}) = \int_0^\infty u\left(\frac{1}{t}\mathbf{x}\right)\frac{d\mu_t}{d\nu}(\mathbf{x})\, d\lambda(t) < \infty, \; \mathbf{x} \in \Omega. \tag{7}$$

Suppose $\Phi\colon I \to \mathbb{R}$ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If a function $f\colon \Omega \to \mathbb{R}$ is Borel measurable, such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$, and Af is defined by (5), then $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$ and the inequality

$$\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\, d\mu(\mathbf{x}) \leqslant \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\, d\nu(\mathbf{x}) \tag{8}$$

holds. For a non-positive concave function Φ, the sign of inequality in (8) is reversed.

Proof. For a fixed $\mathbf{x} \in \Omega$, we define the function $h_\mathbf{x}\colon \mathbb{R}_+ \to \mathbb{R}$ as $h_\mathbf{x}(t) = f(t\mathbf{x}) - Af(\mathbf{x})$. Then (4) and (5) imply

$$\int_0^\infty h_\mathbf{x}(t)\, d\lambda(t) = \int_0^\infty f(t\mathbf{x})\, d\lambda(t) - Af(\mathbf{x})\int_0^\infty d\lambda(t) = 0. \tag{9}$$

Since the set Ω is λ-balanced and $f(\Omega) \subseteq I$, it follows that $f(t\mathbf{x}) \in I$ for all $t \in \operatorname{supp}\lambda$ and each $\mathbf{x} \in \Omega$. Suppose that there exists $\mathbf{x}_0 \in \Omega$ such that $Af(\mathbf{x}_0) \notin I$. Then we have either $Af(\mathbf{x}_0) < f(t\mathbf{x}_0)$ for all $t \in \operatorname{supp} \lambda$, or $Af(\mathbf{x}_0) > f(t\mathbf{x}_0)$ for all $t \in \operatorname{supp} \lambda$, so the function $h_{\mathbf{x}_0}$ is either strictly positive or strictly negative on $\mathbb{R}_+$. This contradicts (9), so we proved that $Af(\mathbf{x}) \in I$ for all $\mathbf{x} \in \Omega$.

Finally, we prove (8). By using Jensen's inequality, Fubini's theorem, the substitution $\mathbf{y} = t\mathbf{x}$, the fact that Ω is λ-balanced and Φ is non-negative, and the Radon-Nikodym theorem, we obtain

$$\begin{aligned}
\int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\,d\mu(\mathbf{x}) &\leqslant \frac{1}{L}\int_\Omega u(\mathbf{x})\int_0^\infty \Phi(f(t\mathbf{x}))\,d\lambda(t)\,d\mu(\mathbf{x}) \\
&= \frac{1}{L}\int_0^\infty \int_\Omega u(\mathbf{x})\Phi(f(t\mathbf{x}))\,d\mu(\mathbf{x})\,d\lambda(t) \\
&= \frac{1}{L}\int_0^\infty \int_{t\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\Phi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t) \\
&\leqslant \frac{1}{L}\int_0^\infty \int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\Phi(f(\mathbf{y}))\,d\mu_t(\mathbf{y})\,d\lambda(t) \\
&= \frac{1}{L}\int_0^\infty \int_{\Omega} u\left(\frac{1}{t}\mathbf{y}\right)\Phi(f(\mathbf{y}))\frac{d\mu_t}{d\nu}(\mathbf{y})\,d\nu(\mathbf{y})\,d\lambda(t) \\
&= \frac{1}{L}\int_\Omega \left(\int_0^\infty u\left(\frac{1}{t}\mathbf{y}\right)\frac{d\mu_t}{d\nu}(\mathbf{y})\,d\lambda(t)\right)\Phi(f(\mathbf{y}))\,d\nu(\mathbf{y}) \\
&= \frac{1}{L}\int_\Omega v(\mathbf{y})\Phi(f(\mathbf{y}))\,d\nu(\mathbf{y}),
\end{aligned}$$

so the proof is completed. □

Notice that the condition on non-negativity of the convex function Φ in Theorem 1 can be omitted only in a particular setting with cones in X. More precisely, the following corollary holds.

COROLLARY 1. *If in Theorem 1 we have $t\Omega = \Omega$ for λ-a.e. $t \in \operatorname{supp} \lambda$, then (8) holds for all convex functions Φ on an interval $I \subseteq \mathbb{R}$. In that case, for all concave functions Φ relation (8) holds with the sign of inequality reversed.*

In Theorem 1 we considered general measures μ, ν, and λ, a set Ω, and a function Φ. Now, we give an overview of results obtained by specializing inequality (8) to some interesting particular settings. First, we consider the classical one-dimensional cases.

COROLLARY 2. *Let λ be a finite Borel measure on $\mathbb{R}_+$ and L be defined by (4). Suppose that $\Omega \subseteq \mathbb{R}_+$ is a λ-balanced set and that u is a non-negative function on $\mathbb{R}_+$, such that*

$$w(x) = \int_0^\infty u\left(\frac{x}{t}\right)\,d\lambda(t) < \infty, x \in \Omega. \tag{10}$$

Let $\Phi\colon I \to \mathbb{R}$ be a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If $f\colon \Omega \to \mathbb{R}$ is a Borel measurable function, such that $f(x) \in I$ for all $x \in \Omega$, and Af is defined by

(5)*, then the inequality*

$$\int_\Omega u(x)\Phi(Af(x))\,\frac{dx}{x} \leqslant \frac{1}{L}\int_\Omega w(x)\Phi(f(x))\,\frac{dx}{x} \tag{11}$$

holds. If the function Φ *is non-positive and concave, the sign of inequality in* (11) *is reversed.*

Proof. It follows directly from Theorem 1 if we set $X=\mathbb{R}_+$, the measures μ and ν to be the Lebesgue measures and replace the weight function u with $x\mapsto \frac{u(x)}{x}$. For such measures we get $\frac{d\mu_t}{d\nu}(x)=\frac{1}{t}$, $t\in\mathbb{R}_+$. In this setting, we have

$$v(x)=\int_0^\infty u\left(\frac{x}{t}\right)\cdot\frac{t}{x}\cdot\frac{1}{t}\,d\lambda(t)=\frac{1}{x}\int_0^\infty u\left(\frac{x}{t}\right)\,d\lambda(t)=\frac{w(x)}{x}\ ,\ x\in\Omega,$$

where the function v is defined by (7). $\square$

Notice that the inequality (11) obviously generalizes (3).

COROLLARY 3. *Let* $0<b\leqslant\infty$, u *be a non-negative function on* $\langle 0,b\rangle$, *such that the function* $t\mapsto\frac{u(t)}{t^2}$ *is locally integrable in* $\langle 0,b\rangle$, *and let*

$$w(x)=x\int_x^b u(t)\,\frac{dt}{t^2}\ ,\ x\in\langle 0,b\rangle.$$

If Φ *is a convex function on an interval* $I\subseteq\mathbb{R}$, *then the inequality*

$$\int_0^b u(x)\Phi(Hf(x))\,\frac{dx}{x}\leqslant\int_0^b w(x)\Phi(f(x))\,\frac{dx}{x} \tag{12}$$

holds for all functions f *on* $\langle 0,b\rangle$ *with values in* I *and for* Hf *defined by*

$$Hf(x)=\frac{1}{x}\int_0^x f(t)\,dt,\ x\in\langle 0,b\rangle.$$

Proof. Rewrite Theorem 1 with $d\lambda(t)=\chi_{\langle 0,1\rangle}(t)\,dt$, $X=\Omega=\mathbb{R}_+$, $d\mu(x)=\chi_{\langle 0,b\rangle}(x)\,dx$, and $\nu(x)=dx$, as well as with the function $x\mapsto\frac{u(x)}{x}\chi_{\langle 0,b\rangle}(x)$ instead of the weight u. Then $\operatorname{supp}\lambda=\langle 0,1]$, $L=1$, $\frac{d\mu_t}{d\nu}(x)=\frac{1}{t}\chi_{\langle 0,tb\rangle}(x)$,

$$Af(x)=\int_0^1 f(tx)\,dt=Hf(x),$$

and

$$v(x)=\int_0^1\frac{u\left(\frac{1}{t}x\right)}{\frac{1}{t}x}\cdot\frac{1}{t}\chi_{\langle 0,tb\rangle}(x)\,dt=\frac{1}{x}\int_{\frac{x}{b}}^1 u\left(\frac{x}{t}\right)\,dt=\int_x^b u(y)\,\frac{dy}{y^2}=\frac{w(x)}{x},$$

for $x \in \langle 0,b \rangle$, so (12) holds. Since the conditions of Corollary 1 are fulfilled, the function Φ does not have to be non-negative. $\square$

The result of Corollary 3 can be found in [5], [6], and [8], so Theorem 1 can be regarded as its generalization. On the other hand, considering $d\lambda(t) = \chi_{[1,\infty\rangle}(t)\dfrac{dt}{t^2}$, as in the proof of Corollary 3 we get a dual result to (12) (see also [5, 6, 8]).

COROLLARY 4. *For $0 \leqslant b < \infty$, suppose $u\colon \langle b,\infty\rangle \to \mathbb{R}$ is a non-negative function, locally integrable in $\langle b,\infty\rangle$, and w is defined on $\langle b,\infty\rangle$ by*

$$w(x) = \frac{1}{x}\int_b^x u(t)\,dt.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_0^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x} \leqslant \int_0^\infty w(x)\Phi(f(x))\,\frac{dx}{x}$$

holds for all functions f on $\langle b,\infty\rangle$ with values in I and for $\tilde{H}f$ defined by

$$\tilde{H}f(x) = x\int_x^\infty f(t)\,\frac{dt}{t^2},\ x \in \langle b,\infty\rangle.$$

Further corollaries are related to a multidimensional setting with balls in $\mathbb{R}^n$ centered at the origin.

COROLLARY 5. *Suppose that $0 < b \leqslant \infty$ and that a positive function ψ on $[0,1]$ and a non-negative function u on $\mathbb{R}^n$ are such that*

$$v(\mathbf{x}) = \int_{\frac{|\mathbf{x}|}{b}}^1 u\left(\frac{1}{t}\mathbf{x}\right) t^{-n}\psi(t)\,dt < \infty,\ \ \mathbf{x} \in B(b) \tag{13}$$

and

$$P_1 = \int_0^1 \psi(t)\ dt < \infty. \tag{14}$$

Suppose Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If $f\colon B(b) \to \mathbb{R}$ is a Borel–measurable function, such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in B(b)$, then the inequality

$$\int_{B(b)} u(\mathbf{x})\Phi\left(\frac{1}{P_1}\int_0^1 \psi(t)f(t\mathbf{x})\,dt\right)\,d\mathbf{x} \leqslant \frac{1}{P_1}\int_{B(b)} v(\mathbf{x})\Phi(f(\mathbf{x}))\,d\mathbf{x} \tag{15}$$

holds.

Proof. Follows from Theorem 1 and Corollary 1 rewritten with $X = \mathbb{R}^n$, $\Omega = B(b)$, $d\lambda(t) = \psi(t)\chi_{\langle 0,1\rangle}(t)\,dt$, $d\mu(\mathbf{x}) = \chi_{B(b)}(\mathbf{x})\,d\mathbf{x}$, and $d\nu(\mathbf{x}) = d\mathbf{x}$. Here we have $\operatorname{supp}\lambda = \langle 0,1]$, $\dfrac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{B(tb)}(\mathbf{x})$, and $Af(\mathbf{x}) = \dfrac{1}{P_1}\displaystyle\int_0^1 \psi(t)f(t\mathbf{x})\,dt$. It is easy to see that in this setting (13) reduces to (7), and (8) becomes (15). $\square$

A similar unweighted n-dimensional result can be found in [10]. Applying Corollary 5 to some particular u and Φ we get the following result.

COROLLARY 6. *Let $0 < b \leqslant \infty$, let the positive function ψ on $[0,1]$ be such that*

$$v(\mathbf{x}) = \int_{\frac{|\mathbf{x}|}{b}}^{1} t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in B(b),$$

and let P_1 be defined by (14). If $f\colon B(b) \to \mathbb{R}$ is a non-negative Borel-measurable function, then the inequality

$$\int_{B(b)} \left(\int_0^1 \psi(t) f(t\mathbf{x})\,dt \right)^p d\mathbf{x} \leqslant P_1^{p-1} \int_{B(b)} v(\mathbf{x}) f^p(\mathbf{x}) d\mathbf{x}$$
$$\leqslant P_1^{p-1} \left(\int_0^1 t^{-n}\psi(t)\,dt \right) \int_{B(b)} f^p(\mathbf{x})\,d\mathbf{x} \qquad (16)$$

holds for all $p \in \mathbb{R} \setminus [0,1\rangle$. If $p \in \langle 0,1\rangle$, then the first inequality in (16) holds with reversed sign of inequality.

Proof. *The first inequality in (16) is equivalent with inequality (15), rewritten with $u(\mathbf{x}) \equiv 1$ and with the convex function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$, $\Phi(x) = x^p$, $p \in \mathbb{R} \setminus [0,1\rangle$. For $p \in \langle 0,1\rangle$, the function Φ is concave.* □

Analogously, we get the following result.

COROLLARY 7. *Suppose that $0 \leqslant b < \infty$ and that the positive function ψ on $[1,\infty\rangle$ and the non-negative function u on $\mathbb{R}^n$ are such that*

$$v(\mathbf{x}) = \int_1^{\frac{|\mathbf{x}|}{b}} u\left(\frac{1}{t}\mathbf{x}\right) t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in \mathbb{R}^n \setminus B(b). \qquad (17)$$

and

$$P_\infty = \int_1^\infty \psi(t)\; dt < \infty. \qquad (18)$$

Suppose that Φ is a non-negative convex function on an interval $I \subseteq \mathbb{R}$. If a function $f\colon \mathbb{R}^n \setminus B(b) \to \mathbb{R}$ is Borel measurable, such that $f(\mathbf{x}) \in I$ for all $\mathbf{x} \in \mathbb{R}^n \setminus B(b)$, then the inequality

$$\int_{\mathbb{R}^n \setminus B(b)} u(\mathbf{x}) \Phi\left(\frac{1}{P_\infty} \int_1^\infty \psi(t) f(t\mathbf{x}) dt \right) d\mathbf{x} \leqslant \frac{1}{P_\infty} \int_{\mathbb{R}^n \setminus B(b)} v(\mathbf{x}) \Phi(f(\mathbf{x})) d\mathbf{x} \qquad (19)$$

holds.

Proof. The proof follows from Theorem 1 and Corollary 1 if we set $d\lambda(t) = \psi(t)\chi_{\langle 1,\infty\rangle}(t)\,dt$, $X = \mathbb{R}^n$, $\Omega = \mathbb{R}^n \setminus B(b)$, $d\mu(\mathbf{x}) = \chi_{\mathbb{R}^n \setminus B(b)}(\mathbf{x})\,d\mathbf{x}$, and $d\nu(\mathbf{x}) = d\mathbf{x}$. Then we get $\operatorname{supp} \lambda = [1,\infty\rangle$, $\dfrac{d\mu_t}{d\nu}(\mathbf{x}) = t^{-n}\chi_{\mathbb{R}^n \setminus B(tb)}(\mathbf{x})$, and

$$Af(\mathbf{x}) = \frac{1}{P_\infty} \int_1^\infty \psi(t) f(t\mathbf{x})\,dt,$$

so (7) and (8) become (17) and (19), respectively. □

An unweighted form of this result can be found in [10].

COROLLARY 8. *Let* $0 \leqslant b < \infty$ *and let the function* $\psi\colon [1,\infty\rangle \to [0,\infty\rangle$ *be such that*

$$v(\mathbf{x}) = \int_1^{\frac{|\mathbf{x}|}{b}} t^{-n}\psi(t)\,dt < \infty, \quad \mathbf{x} \in \mathbb{R}^n \setminus B(b).$$

If $f\colon \mathbb{R}^n \setminus B(b) \to \mathbb{R}$ *is a Borel-measurable function and* P_∞ *is defined by* (18)*, then the inequality*

$$\int_{\mathbb{R}^n\setminus B(b)} \left(\int_1^\infty \psi(t) f(t\mathbf{x})\,dt\right)^p d\mathbf{x} \leqslant P_\infty^{p-1} \int_{\mathbb{R}^n\setminus B(b)} v(\mathbf{x}) f^p(\mathbf{x})\,d\mathbf{x} \tag{20}$$

holds for all $p \in \mathbb{R}\setminus[0,1\rangle$. *For* $p \in \langle 0,1\rangle$, *the sign of inequality in* (20) *is reversed.*

Proof. Again, like in Corollary 6, we take $u(\mathbf{x}) \equiv 1$ and the convex function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$, $\Phi(\mathbf{x}) = x^p$, $p \in \mathbb{R}\setminus[0,1\rangle$. Notice that Φ is concave for $p \in \langle 0,1\rangle$. ☐

After giving the examples based on applications of Theorem 1 to specific measures and sets, we continue our analysis by introducing another convex function. The following lemma obviously holds.

LEMMA 1. *For* $s \in \mathbb{R}$, *let the functions* $\varphi_s\colon \mathbb{R}_+ \to \mathbb{R}$ *be defined by*

$$\varphi_s(x) = \begin{cases} -\log x, & s = 0 \\ x\log x, & s = 1 \\ \dfrac{x^s}{s(s-1)}, & \text{otherwise.} \end{cases} \tag{21}$$

Then $\varphi_s''(x) = x^{s-2}$ *and* φ_s *is a convex function for all* $s \in \mathbb{R}$.

Since φ_s are convex, we can apply Corollary 1 to these particular functions.

COROLLARY 9. *Let the conditions of Corollary 1 be fulfilled with a positive function* f *and let* φ_s *be defined by* (21)*. Then*

$$\int_\Omega u(\mathbf{x})\varphi_s(Af(\mathbf{x}))\,d\mu(\mathbf{x}) \leqslant \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi_s(f(\mathbf{x}))\,d\nu(\mathbf{x}) \tag{22}$$

holds for all $s \in \mathbb{R}$.

REMARK 1. Observe that not the all functions φ_s are non-negative. Therefore, Corollary 1 does not assure inequality (22) to hold if there exists a set $S \subseteq \operatorname{supp}\lambda$, $\lambda(S) > 0$, such that $t\Omega \subsetneqq \Omega$, $t \in S$.

Corollary 9 enables us to define the Boas difference, that is, the non-negative function $\xi\colon \mathbb{R} \to [0,\infty\rangle$,

$$\xi(s) = \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi_s(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\varphi_s(Af(\mathbf{x}))\,d\mu(\mathbf{x}). \tag{23}$$

In particular, under the conditions of Corollary 2, Corollary 3 and Corollary 4, we respectively define the following Boas differences:

$$\xi_1(s) = \frac{1}{L}\int_\Omega w(x)\varphi_s(f(x))\,\frac{dx}{x} - \int_\Omega u(x)\varphi_s(Af(x))\,\frac{dx}{x}, \tag{24}$$

$$\xi_2(s) = \int_0^b w(x)\varphi_s(f(x))\,\frac{dx}{x} - \int_0^b u(x)\varphi_s(Hf(x))\,\frac{dx}{x}, \tag{25}$$

$$\xi_3(s) = \int_b^\infty w(x)\varphi_s(f(x))\,\frac{dx}{x} - \int_b^\infty u(x)\varphi_s(\tilde{H}f(x))\,\frac{dx}{x}. \tag{26}$$

The same can be done also with Corollary 5 and Corollary 7, so we shall omit it.

REMARK 2. For $u(x)\equiv 1$, in Corollary 3 we have $w(x) = x\int_x^b \frac{dt}{t^2} = 1-\frac{x}{b}$, so (25) becomes

$$\xi_2(s) = \int_0^b \left(1-\frac{x}{b}\right)\varphi_s(f(x))\,\frac{dx}{x} - \int_0^b \varphi_s(Hf(x))\,\frac{dx}{x}.$$

Inequality $\xi_2(s)\geqslant 0$ was obtained in [7].

REMARK 3. For $u(x)\equiv 1$, in Corollary 4 we have $w(x) = \frac{1}{x}\int_b^x dt = 1-\frac{b}{x}$, so (26) reduces to

$$\xi_3(s) = \int_b^\infty \left(1-\frac{b}{x}\right)\varphi_s(f(x))\,\frac{dx}{x} - \int_b^\infty \varphi_s(\tilde{H}f(x))\,\frac{dx}{x}.$$

In that case, inequality $\xi_3(s)\geqslant 0$ was obtained in [7].

3. Exponential Convexity of the Boas differences

In this section, we explore the Boas differences defined by (23). For this purpose, we recall some notion and facts about logarithmically convex and exponentially convex functions.

Let $I\subseteq\mathbb{R}$ be an interval. A positive function $\xi : I\to\mathbb{R}$ is said to be logarithmically convex, or log-convex, if the function $\log\xi$ is convex. It is well-known that each log-convex function is convex and that the relation

$$\xi(s_2)^{s_3-s_1} \leqslant \xi(s_1)^{s_3-s_2}\xi(s_3)^{s_2-s_1} \tag{27}$$

holds for all such functions ξ and all s_1, s_2, $s_3\in I$, such that $s_1<s_2<s_3$. We also recall Galvani's theorem for log-convex functions $\xi : I\to\mathbb{R}$. It claims that the inequality

$$\left[\frac{\xi(s_2)}{\xi(s_1)}\right]^{\frac{1}{s_2-s_1}} \leqslant \left[\frac{\xi(t_2)}{\xi(t_1)}\right]^{\frac{1}{t_2-t_1}} \tag{28}$$

holds for all $s_1, s_2, t_1, t_2 \in I$, such that $s_1 \leqslant t_1$, $s_2 \leqslant t_2$ and $s_1 \neq s_2$, $t_1 \neq t_2$.

On the other hand, an exponentially convex function on I is any continuous function $\xi : I \to \mathbb{R}$ satisfying

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i \alpha_j \xi(s_i + s_j) \geqslant 0 \tag{29}$$

for all $k \in \mathbb{N}$ and all sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ of real numbers, such that $s_i + s_j \in I$, $i, j \in \mathbb{N}$. It can be proved that every exponentially convex function is log-convex and thus convex. Moreover, the condition (29) can be replaced with a more suitable condition

$$\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i \alpha_j \xi\left(\frac{s_i + s_j}{2}\right) \geqslant 0, \tag{30}$$

which has to hold for all $k \in \mathbb{N}$, all sequences $(\alpha_n)_{n\in\mathbb{N}}$ of real numbers, and all sequences $(s_n)_{n\in\mathbb{N}}$ in I. More precisely, a function $\xi : I \to \mathbb{R}$ is exponentially convex if and only if it is continuous and fulfills (30). Further information on log-convex and exponentially convex functions can be found in [1] and [11], as well as in the references given in those monographs.

Using the concept of exponential convexity, we get the following Lyapunov-type inequality related to the Boas differences (23).

THEOREM 2. *Let the conditions of Corollary 1 be fulfilled with a positive function f and let φ_s be defined by (21). Then the function $\xi : \mathbb{R} \to [0,\infty\rangle$ defined by (23) is continuous, exponentially convex and the inequality*

$$[\xi(r)]^{q-p} \leqslant [\xi(p)]^{q-r} \cdot [\xi(q)]^{r-p} \tag{31}$$

holds for all $p, q, r \in \mathbb{R}$, such that $p < r < q$.

Proof. First, we prove that ξ is continuous on $\mathbb{R}$. Since the mapping $s \mapsto \dfrac{x^s}{s(s-1)}$ is continuous on $\mathbb{R} \setminus \{0,1\}$ for all $x \in \mathbb{R}_+$, we only need to prove the continuity of ξ in $s = 0$ and $s = 1$. Since under assumptions of Corollary 1 we have

$$\frac{1}{L}\int_\Omega v(\mathbf{x})\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\, d\mu(\mathbf{x}) = 0, \tag{32}$$

the L'Hospital rule [13] implies

$$\begin{aligned}
\lim_{s\to 0} \xi(s) &= \lim_{s\to 0}\left[\frac{1}{L}\int_\Omega v(\mathbf{x})\frac{f^s(\mathbf{x})}{s(s-1)}\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\frac{(Af(\mathbf{x}))^s}{s(s-1)}\, d\mu(\mathbf{x})\right] \\
&= \lim_{s\to 0}\frac{1}{s(s-1)}\left[\frac{1}{L}\int_\Omega v(\mathbf{x}) f^s(\mathbf{x})\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^s\, d\mu(\mathbf{x})\right] \\
&= \lim_{s\to 0}\frac{1}{2s-1}\left[\frac{1}{L}\int_\Omega v(\mathbf{x}) f^s(\mathbf{x}) \log f(\mathbf{x}) d\nu(\mathbf{x})\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^s \log Af(\mathbf{x}) d\mu(\mathbf{x})\Big] \\
&= -\frac{1}{L}\int_\Omega v(\mathbf{x}) \log f(\mathbf{x})\, d\nu(\mathbf{x}) + \int_\Omega u(\mathbf{x}) \log Af(\mathbf{x})\, d\mu(\mathbf{x}) \\
&= \frac{1}{L}\int_\Omega v(\mathbf{x})\varphi_0(f(\mathbf{x}))\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\varphi_0(Af(\mathbf{x}))\, d\mu(\mathbf{x}) = \xi(0).
\end{aligned}
$$

Similary, for $s = 1$, the identity

$$
\frac{1}{L}\int_\Omega v(\mathbf{x}) f(\mathbf{x})\; d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x}) Af(\mathbf{x})\; d\mu(\mathbf{x}) = 0 \tag{33}
$$

yields

$$
\begin{aligned}
\lim_{s\to 1} \xi(s) &= \frac{1}{L}\int_\Omega v(\mathbf{x}) f(\mathbf{x}) \log f(\mathbf{x})\, d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x}) Af(\mathbf{x}) \log Af(\mathbf{x})\, d\mu(\mathbf{x}) \\
&= \xi(1),
\end{aligned}
$$

so ξ is continuous on the entire real line. To prove that it is exponentially convex, it suffices to check condition (30). Fix $k \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, and $s_i \in \mathbb{R}$, for $i \in \{1,\ldots,k\}$. Denote $s_{ij} = \dfrac{s_i + s_j}{2}$ and define the function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$ by $\Phi(x) = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \varphi_{s_{ij}}(x)$. By using Lemma 1, we easily get

$$
\Phi''(x) = \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j x^{s_{ij}-2} = \left(\sum_{i=1}^{k} \alpha_i x^{\frac{s_i}{2}-1}\right)^2 \geqslant 0, x \in \mathbb{R}_+,
$$

so the function Φ is convex. Thus, applying Corollary 1 to this function Φ, we finally get

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \xi(s_{ij}) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \left[\frac{1}{L}\int_\Omega v(\mathbf{x})\varphi_{s_{ij}}(f(\mathbf{x}))\; d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\varphi_{s_{ij}}(Af(\mathbf{x}))\; d\mu(\mathbf{x})\right] \\
&= \frac{1}{L}\int_\Omega v(\mathbf{x}) \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \varphi_{s_{ij}}(f(\mathbf{x}))\; d\nu(\mathbf{x}) \\
&\qquad - \int_\Omega u(\mathbf{x}) \sum_{i=1}^{k}\sum_{j=1}^{k} \alpha_i\alpha_j \varphi_{s_{ij}}(Af(\mathbf{x}))\; d\mu(\mathbf{x}) \\
&= \frac{1}{L}\int_\Omega v(\mathbf{x})\Phi(f(\mathbf{x}))\; d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})\Phi(Af(\mathbf{x}))\; d\mu(\mathbf{x}) \geqslant 0.
\end{aligned}
$$

Therefore, (30) holds and ξ is exponentially convex. Since every exponentially convex function is log-convex, (31) follows directly from (27). □

REMARK 4. Theorem 2 does not hold without assuming that $t\Omega = \Omega$ for λ-a.e. $t \in \operatorname{supp} \lambda$. This condition was crucial in proving identities (32) and (33).

As a direct consequence of Theorem 2 we get an upper bound for the Boas difference ξ.

COROLLARY 10. *Let the conditions of Theorem 2 be fulfilled. Then*

$$\xi(r) \leqslant [\xi(p)]^{\frac{q-r}{q-p}} \cdot [\xi(q)]^{\frac{r-p}{q-p}} \tag{34}$$

holds for all $p,q,r \in \mathbb{R}$*, such that* $p < r < q$.

REMARK 5. Relation (34) can be written as

$$\xi(r) \leqslant \inf_{\substack{p,q\in\mathbb{R} \\ p<r<q}} [\xi(p)]^{\frac{q-r}{q-p}} \cdot [\xi(q)]^{\frac{r-p}{q-p}}, \ r \in \mathbb{R}.$$

As a consequence of Theorem 2, we get the following modified Galvani's theorem generated by the Boas difference ξ.

COROLLARY 11. *Under the conditions of Theorem 2, the inequality*

$$\left(\frac{\xi(p)}{\xi(r)}\right)^{\frac{1}{p-r}} \leqslant \left(\frac{\xi(t)}{\xi(s)}\right)^{\frac{1}{t-s}}. \tag{35}$$

holds for all $p,r,s,t \in \mathbb{R}$*, such that* $r \leqslant s$*,* $p \leqslant t$*,* $r \neq p$*, and* $s \neq t$.

Proof. Since the function ξ is exponentially convex, thus log-convex, inequality (35) follows from (28). $\square$

REMARK 6. The results obtained in Theorem 2, Corollary 10 and Corollary 11 can be rewritten with ξ_i, $i = 1,2,3$, defined by (24), (25) and (26), respectively.

4. Cauchy-type means related to the Boas inequality

Notice that each side of relation (35) has a form of a mean, while (35) as a whole looks like an inequality between two means of the same type. Here, we justify this conjecture by proving that the expressions mentioned above are means of the Cauchy type. For more information about means and their inequalities see e.g. [4].

Let the measures λ, μ, ν, the number L, the λ-balanced set Ω, the interval I, and the functions u, v, f, and Af be as in Theorem 1 and Corollary 1. First, we define the linear functional $F : C^2(I) \to \mathbb{R}$ by

$$F(h) = \frac{1}{L}\int_\Omega v(\mathbf{x})h(f(\mathbf{x}))\,d\nu(\mathbf{x}) - \int_\Omega u(\mathbf{x})h(Af(\mathbf{x}))\,d\mu(\mathbf{x}). \tag{36}$$

Its properties will enable us to introduce a new class of the Cauchy-type means related to the Boas difference (23).

Observe that $F(\varphi_p) = \xi(p)$, $p \in \mathbb{R}$, where the functions φ_p are defined by (21) and ξ denotes the Boas difference introduced by (23). Hence, F can be considered as a generalized Boas difference. Moreover, according to Theorem 2, the mapping $p \mapsto F(\varphi_p)$ is continuous on $\mathbb{R}$.

Next, we have to adjust some known mean value theorems to our context. The first result in this direction is the following Lagrange-type mean value theorem.

THEOREM 3. *Under the conditions of Corollary 1, suppose that I is a compact interval in $\mathbb{R}$. If $h \in C^2(I)$, then there exists $c \in I$ such that the identity*

$$F(h) = h''(c) \cdot F(\varphi_2) \tag{37}$$

holds, where F is defined by (36) and $\varphi_2 \colon I \to \mathbb{R}$, $\varphi_2(x) = \dfrac{x^2}{2}$.

Proof. Since h'' is continuous on the compact set I, there exist $m = \min_{x\in I} h''(x)$ and $M = \max_{x\in I} h''(x)$. Define $h_m, h_M \colon I \to \mathbb{R}$ by

$$h_m(x) = h(x) - \frac{m}{2}x^2 = h(x) - m\varphi_2(x),$$
$$h_M(x) = \frac{M}{2}x^2 - h(x) = M\varphi_2(x) - h(x).$$

Since $h_m, h_M \in C^2(I)$ and $h_m''(x) = h''(x) - m \geqslant 0$, $h_M''(x) = M - h''(x) \geqslant 0$, for all $x \in I$, we conclude that h_m and h_M are convex functions on I. Therefore, applying Corollary 1 to these functions as Φ, we get $F(h_m) \geqslant 0$ and $F(h_M) \geqslant 0$. Obviously, $F(h_m) = F(h) - mF(\varphi_2)$ and $F(h_M) = MF(\varphi_2) - F(h)$, so therefrom we obtain

$$mF(\varphi_2) \leqslant F(h) \leqslant MF(\varphi_2). \tag{38}$$

Notice that function φ_2 is convex, so $F(\varphi_2) \geqslant 0$ holds by Corollary 1. In particular, if $F(\varphi_2) = 0$, then from (38) we get $F(h) = 0$, so (37) holds for all $c \in I$. On the other hand, if $F(\varphi_2) > 0$, then (38) yields $m \leqslant \dfrac{F(h)}{F(\varphi_2)} \leqslant M$. Since h'' takes all values from $[m, M]$, there exists $c \in I$ such that

$$h''(c) = \frac{F(h)}{F(\varphi_2)},$$

so the proof is completed. □

Now, we state and prove a new Cauchy-type mean value theorem.

THEOREM 4. *Let I be a compact interval in $\mathbb{R}$ and $\varphi_2\colon I \to \mathbb{R}$ be defined by $\varphi_2(x) = \frac{x^2}{2}$. Under the conditions of Corollary 1, let F be defined by (36) and let $F(\varphi_2) > 0$. If the functions $h_1, h_2 \in C^2(I)$ are such that $F(h_1), F(h_2) \neq 0$, and $h_2''(x) \neq 0$, for all $x \in I$, then there exists $c \in I$ such that*

$$\frac{h_1''(c)}{h_2''(c)} = \frac{F(h_1)}{F(h_2)}. \tag{39}$$

Proof. Define the function $h_0 = F(h_2)h_1 - F(h_1)h_2$. Then $h_0 \in C^2(I)$ and we have $F(h_0) = F(h_2)F(h_1) - F(h_1)F(h_2) = 0$. On the other hand, from Theorem 3 we know that there exists $c \in I$ such that $F(h_0) = h_0''(c)F(\varphi_2)$. Since $F(\varphi_2) \neq 0$, we get $h_0''(c) = 0$, that is, $F(h_2)h_1''(c) = F(h_1)h_2''(c)$, which is equivalent to (39). □

A special case of Theorem 4 related to power functions defined on a compact interval $I \subseteq \mathbb{R}_+$ will be of our special interest. Namely, let $h_1, h_2\colon I \to \mathbb{R}$ be defined by $h_1(x) = x^p$ and $h_2(x) = x^r$, where $p, r \in \mathbb{R} \setminus \{0,1\}$, $p \neq r$. Then $h_1(x) = p(p-1)\varphi_p(x)$, $h_2(x) = r(r-1)\varphi_r(x)$, $h_1''(x) = p(p-1)x^{p-2}$ and $h_2''(x) = r(r-1)x^{r-2}$, where φ_p and φ_r are given by (21). Hence, we obtain the following result.

COROLLARY 12. *Let the conditions of Corollary 1 be fulfilled with a positive function f with values in a compact interval $I \subseteq \mathbb{R}_+$ and let $F(\varphi_s) > 0$, $s \in \mathbb{R} \setminus \{0,1\}$, where φ_s and F are defined by (21) and (36), respectively. Then*

$$\left(\frac{F(\varphi_p)}{F(\varphi_r)}\right)^{\frac{1}{p-r}} \in I, \tag{40}$$

for all $p, r \in \mathbb{R}$, $(p-r)p(p-1)r(r-1) \neq 0$.

Proof. Fix $p, r \in \mathbb{R}$, such that $(p-r)p(p-1)r(r-1) \neq 0$. Observe that the power functions h_1 and h_2 defined before the statement of Corollary 12 fulfill the conditions of Theorem 4. Hence, there exists $c \in I$ such that

$$\frac{p(p-1)c^{p-2}}{r(r-1)c^{r-2}} = \frac{F(h_1)}{F(h_2)}. \tag{41}$$

According to the definition (21) of the functions φ_s, $s \in \mathbb{R}$, identity (41) reads

$$c^{p-r} = \frac{F(\varphi_p)}{F(\varphi_r)},$$

so we get (40). □

Notice that expression (40) can be written in the form

$$\left(\frac{F(\varphi_p)}{F(\varphi_r)}\right)^{\frac{1}{p-r}} = \left(\frac{\xi(p)}{\xi(r)}\right)^{\frac{1}{p-r}}$$

according to the definition (23) of the Boas difference ξ. As announced, under the conditions of Corollary 12, we introduce a new two-variable function M with values in I, defined by

$$M(p,r) = \left(\frac{\xi(p)}{\xi(r)}\right)^{\frac{1}{p-r}}, \; p,r \in \mathbb{R}\setminus\{0,1\}, p \neq r. \tag{42}$$

Evidently, M is symmetric, that is, $M(p,r) = M(r,p)$ holds for all $p,r \in \mathbb{R}\setminus\{0,1\}$, $p \neq r$. Moreover, by Theorem 2, M is also continuous in both arguments.

Now, we would like to extend this function to $\mathbb{R}^2$. Fix $r \in \mathbb{R}\setminus\{0,1\}$. Applying continuity of the mapping ξ on $\mathbb{R}$, we obtain

$$\begin{aligned}
\lim_{p\to 0} M(r,p) &= \lim_{p\to 0} M(p,r) = \lim_{p\to 0} \exp\left(\frac{(\log\xi(p) - \log\xi(r))}{p-r}\right) \\
&= \exp\left(\lim_{p\to 0}\frac{\log\xi(p) - \log\xi(r)}{p-r}\right) = \exp\left(\frac{\log\xi(r) - \log\xi(0)}{r}\right) \\
&= \left(\frac{\xi(r)}{\xi(0)}\right)^{\frac{1}{r}}
\end{aligned}$$

and, analogously,

$$\lim_{p\to 1} M(r,p) = \lim_{p\to 1} M(p,r) = \left(\frac{\xi(1)}{\xi(r)}\right)^{\frac{1}{1-r}}.$$

Thus, in order to keep continuity of M, we define

$$M(0,r) = M(r,0) = \left(\frac{\xi(r)}{\xi(0)}\right)^{\frac{1}{r}} \text{ and } M(1,r) = M(r,1) = \left(\frac{\xi(r)}{\xi(1)}\right)^{\frac{1}{r-1}}, \tag{43}$$

$r \in \mathbb{R}\setminus\{0,1\}$, as in formula (42).

Observe that ξ is derivable for $r \in \mathbb{R}\setminus\{0,1\}$ and

$$\begin{aligned}
\xi'(r) = \frac{1}{r(r-1)}\Bigg[&(1-2r)\xi(r) + \frac{1}{L}\int_\Omega v(\mathbf{x}) f^r(\mathbf{x})\log f(\mathbf{x})\, d\nu(\mathbf{x}) \\
&- \int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^r \log Af(\mathbf{x})\, d\mu(\mathbf{x})\Bigg].
\end{aligned}$$

Therefore, applying L'Hospital rule [13], for $r \in \mathbb{R}\setminus\{0,1\}$ we have

$$\begin{aligned}\lim_{p\to r} M(p,r) &= \lim_{p\to r} M(r,p) = \exp\left(\lim_{p\to r}\frac{\log\xi(p)-\log\xi(r)}{p-r}\right)\\ &= \exp\left\{\frac{1}{r(r-1)}\left[1-2r+\frac{1}{\xi(r)}\left(\frac{1}{L}\int_\Omega v(\mathbf{x})f^r(\mathbf{x})\log f(\mathbf{x})d\nu(\mathbf{x})\right.\right.\right.\\ &\qquad \left.\left.\left. -\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^r\log Af(\mathbf{x})\,d\mu(\mathbf{x})\right)\right]\right\}, \end{aligned} \tag{44}$$

which enables us to set $M(r,r)=\lim_{p\to r} M(p,r),\ r\in\mathbb{R}\setminus\{0,1\}$. Finally, to define $M(0,0)$ and $M(1,1)$, notice that for $p\in\{0,1\}$ we get

$$\begin{aligned}\lim_{r\to p}\xi'(r) &= \lim_{r\to p}\frac{1}{2r-1}\left[-2\xi(r)+(1-2r)\xi'(r)\right.\\ &\quad +\frac{1}{L}\int_\Omega v(\mathbf{x})f^r(\mathbf{x})\log^2 f(\mathbf{x})\,d\nu(\mathbf{x})\\ &\quad \left.-\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^r\log^2 Af(\mathbf{x})\,d\mu(\mathbf{x})\right]\\ &= 2(-1)^p\xi(p)-\lim_{r\to p}\xi'(r)+(-1)^p\left[\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^p\log^2 Af(\mathbf{x})\,d\mu(\mathbf{x})\right.\\ &\quad \left.-\frac{1}{L}\int_\Omega v(\mathbf{x})f^p(\mathbf{x})\log^2 f(\mathbf{x})\,d\nu(\mathbf{x})\right],\end{aligned}$$

so

$$\begin{aligned}\xi'(p) = \lim_{r\to p}\xi'(r) &= (-1)^p\xi(p)+\frac{(-1)^p}{2}\left[\int_\Omega u(\mathbf{x})(Af(\mathbf{x}))^p\log^2 Af(\mathbf{x})\,d\mu(\mathbf{x})\right.\\ &\quad \left.-\frac{1}{L}\int_\Omega v(\mathbf{x})f^p(\mathbf{x})\log^2 f(\mathbf{x})\,d\nu(\mathbf{x})\right],\ p\in\{0,1\}.\end{aligned}$$

Hence, we set

$$\begin{aligned}M(0,0) &= \lim_{r\to 0} M(r,r) = \lim_{r\to 0} M(r,0)\\ &= \exp\left(\lim_{r\to 0}\frac{\log\xi(r)-\log\xi(0)}{r}\right) = \exp\frac{\xi'(0)}{\xi(0)}\\ &= \exp\left\{1+\frac{1}{2\xi(0)}\left[\int_\Omega u(\mathbf{x})\log^2 Af(\mathbf{x})\,d\mu(\mathbf{x})\right.\right.\\ &\qquad \left.\left.-\frac{1}{L}\int_\Omega v(\mathbf{x})\log^2 f(\mathbf{x})\,d\nu(\mathbf{x})\right]\right\}\end{aligned} \tag{45}$$

and

$$
\begin{aligned}
M(1,1) &= \lim_{r\to 1} M(r,r) = \lim_{r\to 1} M(r,1) \\
&= \exp\left(\lim_{r\to 1} \frac{\log\xi(r) - \log\xi(1)}{r-1}\right) = \exp\frac{\xi'(1)}{\xi(1)} \\
&= \exp\left\{-1 + \frac{1}{2\xi(1)}\left[\frac{1}{L}\int_{\Omega} v(\mathbf{x}) f(\mathbf{x}) \log^2 f(\mathbf{x})\, d\nu(\mathbf{x})\right.\right. \\
&\qquad \left.\left. - \int_{\Omega} u(\mathbf{x}) Af(\mathbf{x}) \log^2 Af(\mathbf{x})\, d\mu(\mathbf{x})\right]\right\}. \qquad (46)
\end{aligned}
$$

By the above construction, we have obviously defined a continuous function $M : \mathbb{R}^2 \to \mathbb{R}$, with values in the compact interval I. Considering its other properties, in fact, we obtained a new class of two-parametic means of the Cauchy-type. Namely, the following theorem holds.

THEOREM 5. *Under the conditions of Corollary 12, let the function $M : \mathbb{R}^2 \to \mathbb{R}$ be defined by relations (42) - (46). Then M is a continuous and symmetric function with values in the compact interval I, such that the inequality*

$$
M(p,r) \leqslant M(q,s) \qquad (47)
$$

holds for all $p,q,r,s \in \mathbb{R}$, $p \leqslant q, r \leqslant s$.

Proof. Taking into account the previous construction and analysis, it is only left to prove the monotonicity property (47) of M. However, it follows immediately from Corollary 11 and from continuity of M. □

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