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ON SOME CHARACTERIZATIONS OF THE PARTIAL ORDERINGS FOR BOUNDED OPERATORS

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Abstract. In this paper, we investigate some common characterizations and various individual properties of the star ordering, the left star ordering, the right star ordering and the minus partial ordering of bounded operators on a Hilbert space. Some generalizations of results known in the literature and a number of new results for bounded operators are derived.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces over the same field. We denote the set of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$ by $B(\mathscr{H},\mathscr{K})$ and by $B(\mathscr{H})$ when $\mathscr{H}=\mathscr{K}$. For $A\in B(\mathscr{H},\mathscr{K})$, let A^*, $\mathscr{R}(A)$ and $\mathscr{K}(A)$ be the adjoint, the range and the null space of A, respectively. An operator $P\in B(\mathscr{H})$ is said to be idempotent if $P^2=P$. An idempotent P is called an orthogonal projection if $P^2=P=P^*$. The orthogonal projection onto closed subspace $\mathscr{M}\subseteq\mathscr{H}$ is denoted by $P_{\mathscr{M}}$. Let $P_{\mathscr{M},\mathscr{N}}$ denote the idempotent with $\mathscr{R}(P_{\mathscr{M},\mathscr{N}})=\mathscr{M}$ and $\mathscr{K}(P_{\mathscr{M},\mathscr{N}})=\mathscr{N}$. For closed subspaces $\mathscr{M}$ and $\mathscr{N}$, the direct sum and the orthogonal direct sum are denoted by $\mathscr{M}\oplus\mathscr{N}$ and $\mathscr{M}\oplus^{\perp}\mathscr{N}$, respectively. It is clear $\mathscr{R}(P_{\mathscr{M}})+\mathscr{K}(P_{\mathscr{M}})=\mathscr{M}\oplus^{\perp}\mathscr{M}^{\perp}=\mathscr{H}$ and $\mathscr{R}(P_{\mathscr{M},\mathscr{N}})+\mathscr{K}(P_{\mathscr{M},\mathscr{N}})=\mathscr{M}\oplus\mathscr{N}=\mathscr{H}$.

For $A,B\in B(\mathscr{H})$, we omit the trivial cases $A=0$ and $A=B$. The star ordering $A\overset{*}{\leqslant}B$, the left star ordering $A*\!\leqslant B$ and the right star ordering $A\leqslant\!*B$ are defined, respectively, by

$$\begin{array}{lll} A^*A=A^*B & \text{and} & AA^*=BA^*, \\ A^*A=A^*B & \text{and} & \mathscr{R}(A)\subseteq\mathscr{R}(B), \\ AA^*=BA^* & \text{and} & \mathscr{R}(A^*)\subseteq\mathscr{R}(B^*). \end{array} \tag{1.1}$$

The first ordering was introduced by Drazin [13] and Mitra [22]. The last two orderings were defined and characterized by Baksalary [1] (see also [1]–[4], [13]–[16], [18]–[24] for more details). And the minus partial ordering on $\mathbb{C}_n$ introduced by Hartwig [18] is defined by

$$A\overset{-}{\leqslant}B \Longleftrightarrow \mathrm{rank(B-A)}=\mathrm{rank(B)}-\mathrm{rank(A)}.$$

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Recently, Peter Šemrl [23, Definition 1] generalized this definition to bounded linear operators acting on an infinite-dimensional Hilbert space. For $A,B \in B(\mathscr{H})$, the minus partial ordering on $B(\mathscr{H})$ is defined by $A \overset{-}{\leqslant} B$ if and only if there exist idempotent operators $P,Q \in B(\mathscr{H})$ such that

$$\mathscr{R}(P) = \overline{\mathscr{R}(A)}, \quad \mathscr{K}(Q) = \mathscr{K}(A), \quad PA = PB \quad \text{and} \quad AQ = BQ.$$

The relation $A \overset{-}{\leqslant} B$ is indeed a partial ordering and $A \overset{-}{\leqslant} B \Longleftrightarrow A^* \overset{-}{\leqslant} B^*$ (see [23, Corollary 3]).

The Moore-Penrose inverse (for short, MP inverse) of T is denoted by T^+, and it is the unique solution to the following four operator equations ([5, 6, 7]),

$$TXT = T, \quad XTX = X, \quad TX = (TX)^*, \quad XT = (XT)^*.$$

Recall that any matrix is MP invertible. For an arbitrary Hilbert space, it is not true that every element in $\mathscr{B}(\mathscr{H})$ is MP invertible. But, if $\mathscr{R}(T)$ is closed, T has MP inverse and the MP inverse is unique with $(T^*)^+ = (T^+)^*$, $TT^+ = P_{\mathscr{R}(T)}$ and $T^+T = P_{\mathscr{R}(T^*)}$. And T, as an operator from $\mathscr{R}(T^*) \oplus^{\perp} \mathscr{K}(T)$ onto $\mathscr{R}(T) \oplus^{\perp} \mathscr{K}(T^*)$, can be written as $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$, where T_1 is invertible. So $T^+ = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} = T^*(TT^* + P_{\mathscr{K}(T^*)})^{-1}$ (see [5, 7]). An element $B \in B(\mathscr{H})$ is the Drazin inverse of $A \in B(\mathscr{H})$ provided that

$$A^{k+1}B = A^k, \quad BAB = B \quad \text{and} \quad AB = BA$$

hold for some nonnegative integer k. The smallest k in the previous definition is called the Drazin index of A, and will be denoted by $\operatorname{ind}(A)$. If A has the Drazin inverse with $\operatorname{ind}(A) = k$, then $\mathscr{R}(A^k)$ is closed and the Drazin inverse A^D is unique. It is well-known that, if $A \in B(\mathscr{H})$ has the Drazin inverse, then 0 is not the accumulation point of the spectrum $\sigma(A)$ (see [5, 7]). In the case where $\operatorname{ind}(A) \leqslant 1$, A^D is called the group inverse of A and denoted by $A^{\#}$. In particular, if A is group invertible and $\mathscr{R}(A) = \mathscr{R}(A^*)$, then A is called an EP operator. If A is Drazin invertible, then the spectral idempotent A^{π} of A corresponding to $\{0\}$ is given by $A^{\pi} = I - AA^D$. The operator matrix form of A with respect to the space decomposition $\mathscr{H} = \mathscr{K}(A^{\pi}) \oplus \mathscr{R}(A^{\pi})$ is characterized by $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_1 is invertible and A_2 is nilpotent (see [8]–[10], [17]).

The purpose of this paper is to investigate some common characterizations and various individual properties of the star ordering, the left star ordering, the right star ordering and the minus partial ordering of bounded operators on a Hilbert space. Section 2 is concerned with the problem of establishing matrix expressions and obtaining some relationships among these orderings, with the problem of MP invertibility and detailed formulae of MP inverses. Sections 3 deals with group inverses of these orderings. Since many of the usual techniques used in finite dimensional spaces (as pseudoinverses or singular value decompositions) are no longer available for general Hilbert spaces, we introduce new techniques which allow us to show that some known properties which hold for matrices can be generalized to operators acting on a Hilbert space, and to obtain simpler proofs. On the other hand, several generalizations of the results known in the literature and a number of new results are derived.

2. Characterizations referring to operator partial orderings

The relations and the representations of four operator partial orderings in $B(\mathscr{H})$ will be investigated in the present section. First, we state the well known criterium due to Douglas [12] (see also Lance [19, Theorem 3.1]) about ranges and factorizations of operators. For the sake of convenience, we state it without the proof.

LEMMA 2.1. *Let $A,B,C \in B(\mathscr{H})$. Then*

(i) $\mathscr{R}(A) \subseteq \mathscr{R}\left[(A,C)\right] = \mathscr{R}(A)+\mathscr{R}(C)$; $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ *if and only if* $A = BD$ *for some* $D \in B(\mathscr{H})$. *If* E *and* F *are invertible such that* $EAF = B$, *then* $\mathscr{R}(A)$ *is closed if and only if* $\mathscr{R}(B)$ *is closed.*

(ii) *The closeness of any one of the following sets implies the closeness of the remaining three sets:* $\mathscr{R}(A)$, $\mathscr{R}(A^*)$, $\mathscr{R}(AA^*)$ *and* $\mathscr{R}(A^*A)$. *If* $\mathscr{R}(A)$ *is closed, then* $\mathscr{R}(A) = \mathscr{R}(AA^*)$.

THEOREM 2.1. *Let $A,B \in B(\mathscr{H})$. Then A as an operator from $\mathscr{H} = \overline{\mathscr{R}(A^*)} \oplus^{\perp} \mathscr{K}(A)$ into $\mathscr{H} = \overline{\mathscr{R}(A)} \oplus^{\perp} \mathscr{K}(A^*)$ has the 2×2 operator matrix form $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$, where A_1 is injective and:*

(i) $A \overset{*}{\leqslant} B$ *if and only if* $B = \begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}$;

(ii) $A \ast\!\leqslant B$ *if and only if* $B = \begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}$;

(iii) $A \leqslant\!\ast B$ *if and only if* $B = \begin{pmatrix} A_1 & DB_2 \\ 0 & B_2 \end{pmatrix}$;

(iv) $A \overset{-}{\leqslant} B$ *if and only if* $B = \begin{pmatrix} A_1+DB_2S & DB_2 \\ B_2S & B_2 \end{pmatrix}$,

where $B_2 \in B\big(\mathscr{K}(A),\mathscr{K}(A^*)\big)$, $D \in B\big(\mathscr{K}(A^*),\overline{\mathscr{R}(A)}\big)$ *and* $S \in B\big(\overline{\mathscr{R}(A^*)},\mathscr{K}(A)\big)$.

Proof. Let $A \in B(\mathscr{H})$. Then A can be written as $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \overline{\mathscr{R}(A^*)} \\ \mathscr{K}(A) \end{pmatrix} \to \begin{pmatrix} \overline{\mathscr{R}(A)} \\ \mathscr{K}(A^*) \end{pmatrix}$, where A_1 is injective. Partition B conformably with A as

$$B = \begin{pmatrix} B_1 & B_3 \\ B_4 & B_2 \end{pmatrix} : \begin{pmatrix} \overline{\mathscr{R}(A^*)} \\ \mathscr{K}(A) \end{pmatrix} \to \begin{pmatrix} \overline{\mathscr{R}(A)} \\ \mathscr{K}(A^*) \end{pmatrix}. \quad \text{Thus} \quad B-A = \begin{pmatrix} B_1-A_1 & B_3 \\ B_4 & B_2 \end{pmatrix}.$$

(i) See the proof in [11, Lemma 3].

(ii) If $A \ast\!\leqslant B$, then $A^*(B-A) = 0$ and $\mathscr{R}(A) \subseteq \mathscr{R}(B)$. From $A^*(B-A) = 0$ we derive that $B_1 = A_1$ and $B_3 = 0$. Since $\mathscr{R}(A) \subseteq \mathscr{R}(B)$, by Lemma 2.1,

$$\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = A = BC = \begin{pmatrix} A_1 & 0 \\ B_4 & B_2 \end{pmatrix} \begin{pmatrix} C_1 & C_3 \\ -S & C_2 \end{pmatrix} = \begin{pmatrix} A_1C_1 & A_1C_3 \\ B_4C_1-B_2S & B_4C_3+B_2C_2 \end{pmatrix}$$

for some $C = \begin{pmatrix} C_1 & C_3 \\ -S & C_2 \end{pmatrix} : \begin{pmatrix} \overline{\mathscr{R}(A^*)} \\ \mathscr{K}(A) \end{pmatrix} \to \begin{pmatrix} \overline{\mathscr{R}(A^*)} \\ \mathscr{K}(A) \end{pmatrix}$. Comparing the two sides of the above equation, we get $C_1 = I$ and $B_4 = B_2S$. Hence, there exist $B_2 \in B\big(\mathscr{K}(A),\mathscr{K}(A^*)\big)$, and $S \in B\big(\overline{\mathscr{R}(A^*)},\mathscr{K}(A)\big)$ such that $B = \begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}$.

(iii) Similar to (ii), the details are omitted.

(iv) If $A \overset{-}{\leqslant} B$, then there exist idempotents P,Q such that $\mathscr{R}(P) = \overline{\mathscr{R}(A)}$ and $\mathscr{K}(Q) = \mathscr{K}(A)$. Thus P and Q can be written as

$$P = \begin{pmatrix} I & -D \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \overline{\mathscr{R}(A)} \\ \mathscr{K}(A^*) \end{pmatrix} \to \begin{pmatrix} \overline{\mathscr{R}(A)} \\ \mathscr{K}(A^*) \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} I & 0 \\ -S & 0 \end{pmatrix} : \begin{pmatrix} \overline{\mathscr{R}(A^*)} \\ \mathscr{K}(A) \end{pmatrix} \to \begin{pmatrix} \overline{\mathscr{R}(A^*)} \\ \mathscr{K}(A) \end{pmatrix},$$

where $D \in B\big(\mathscr{K}(A^*), \overline{\mathscr{R}(A)}\big)$ and $S \in B\big(\overline{\mathscr{R}(A^*)}, \mathscr{K}(A)\big)$. The condition $PA = PB$ implies that $\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1 - DB_4 & B_3 - DB_2 \\ 0 & 0 \end{pmatrix}$. We get $B_1 = A_1 + DB_4$ and $B_3 = DB_2$. Similarly, $AQ = BQ$ implies that $B_4 = B_2 S$. Hence B can be represented as $B = \begin{pmatrix} A_1 + DB_2 S & DB_2 \\ B_2 S & B_2 \end{pmatrix}$. □

Theorem 2.1 is supplemented with a number of observations (see [1]–[4], [18]–[22] for the matrix case), which provide further evidence of the usefulness of the matrix representations. The first of them is that $A \overset{*}{\leqslant} B$ implies

$$A^* \overset{*}{\leqslant} B^*, \quad AA^* \overset{*}{\leqslant} BB^*, \quad A^*A \overset{*}{\leqslant} B^*B, \quad |A| \overset{*}{\leqslant} |B| \quad \text{and} \quad (B-A) \overset{*}{\leqslant} B,$$

where $|A| = (A^*A)^{\frac{1}{2}}$ is the modulus of A. The next three illustrations are:

$$\begin{aligned} &\text{(i)} \quad A \overset{*}{\leqslant} B \Longrightarrow \mathscr{R}(A) \subseteq \mathscr{R}(B) \quad \text{and} \quad \mathscr{R}(A^*) \subseteq \mathscr{R}(B^*); \\ &\text{(ii)} \quad A \overset{*}{\leqslant} B \Longleftrightarrow A = P_{\overline{\mathscr{R}(A)}} B = B P_{\overline{\mathscr{R}(A^*)}}; \\ &\text{(iii)} \quad \text{If } A \text{ is normal,} \quad A \overset{*}{\leqslant} B \Longrightarrow AB^* = B^*A. \end{aligned}$$

Hence, if $B^2 = B$ and $A \overset{*}{\leqslant} B$, then

$$A^2 = P_{\overline{\mathscr{R}(A)}} BBP_{\overline{\mathscr{R}(A^*)}} = P_{\overline{\mathscr{R}(A)}} BP_{\overline{\mathscr{R}(A^*)}} = P_{\overline{\mathscr{R}(A)}} A = A.$$

This fact was noted by Hartwig and Styan [16, Theorem 3.1]. Let us now turn our attention to the relationship between the minus partial ordering and the star ordering as well as the left and the right star ordering. The more refined results are given in the following theorem.

THEOREM 2.2. *Let $A,B \in B(\mathscr{H})$. Then there exist invertible operators $E,F \in B(\mathscr{H})$ such that*

(i) $A \overset{-}{\leqslant} B \Longleftrightarrow EAF \overset{*}{\leqslant} EBF$;

(ii) $A \ast\!\leqslant B \Longleftrightarrow EAF \leqslant\!\ast EBF$.

Proof. Let $E = \begin{pmatrix} I & -D \\ 0 & I \end{pmatrix}$ and $F = \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}$. By Theorem 2.1,

$$A \overset{-}{\leqslant} B \Longleftrightarrow EAF = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } EBF = \begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix} \Longleftrightarrow EAF \overset{*}{\leqslant} EBF.$$

Similarly, we have $A \ast\!\leqslant B \Longleftrightarrow EAF \leqslant\!\ast EBF$. □

Šemrl in [23, Theorem 2] characterized structures of $B \in B(\mathscr{H})$ such that $A \overset{-}{\leqslant} B$. The theorem below provides an analogous characterization when A,B have the representations as in Theorem 2.1.

THEOREM 2.3. [23, Theorem 2] *Let* $A,B \in B(\mathscr{H})$. *Then*

$$A \overset{-}{\leqslant} B \Longleftrightarrow \overline{\mathscr{R}(B)} = \overline{\mathscr{R}(A)} \oplus^{\perp} \overline{\mathscr{R}(B-A)} \quad \text{and} \quad \overline{\mathscr{R}(B^*)} = \overline{\mathscr{R}(A^*)} \oplus^{\perp} \overline{\mathscr{R}(B^*-A^*)}.$$

Proof. See the proof in [23, Theorem 2]. □

Under the assumption that $A,B \in \mathbb{C}_{n,n}$, Baksalary and Mitra [1] (also see Image-Serving the International Linear Algebra Community, The bulletin of the International Linear Algebra Society, 31(2003), 30-32.) have got that

$$A \overset{*}{\leqslant} B \Longrightarrow A *\!\leqslant B \Longrightarrow A \overset{-}{\leqslant} B. \tag{2.2}$$

From Theorem 2.1, we observe that (2.2) still holds under the case that $A,B \in B(\mathscr{H})$. The converse does not hold even assuming that A and B are selfadjoint. For example, let A,B as operators on $\mathscr{H} \oplus \mathscr{H}$ have the forms as

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2I & I \\ I & I \end{pmatrix}.$$

Then A,B are selfadjoint and $A \overset{-}{\leqslant} B$. But $A *\!\leqslant B$ does not hold since $A^*A \neq A^*B$. The next results deal with situations in which A,B are idempotents or normal operators. We have the following results.

THEOREM 2.4. *Let* $A,B \in B(\mathscr{H})$.

(i) *If* $AB = BA$, *then* $A \overset{*}{\leqslant} B \Longrightarrow A^k \overset{*}{\leqslant} B^k$ *for an arbitrary integer* $k \geqslant 2$;

(ii) *If* A,B *are idempotent operators, then*

$$A \overset{*}{\leqslant} B \Longleftrightarrow A *\!\leqslant B \Longleftrightarrow A \leqslant\!* B \Longleftrightarrow A \overset{-}{\leqslant} B;$$

(iii) *If* A,B *are normal operators, then* $A *\!\leqslant B \Longleftrightarrow A \overset{*}{\leqslant} B$ *and*

$$A *\!\leqslant B \Longrightarrow A^k *\!\leqslant B^k \quad \text{for an arbitrary integer} \quad k \geqslant 2.$$

Proof. (i) By the definition (1.1), $A \overset{*}{\leqslant} B \Longleftrightarrow A^*(B-A) = 0$ and $(B-A)A^* = 0$. For an arbitrary integer $k \geqslant 2$, if $AB = BA$, then

$$(A^k)^*(B^k - A^k) = (A^{k-1})^*A^*(B-A)\left[B^{k-1} + B^{k-2}A + \cdots + BA^{k-2} + A^{k-1}\right] = 0$$

and

$$(B^k - A^k)(A^k)^* = \left[B^{k-1} + B^{k-2}A + \cdots + BA^{k-2} + A^{k-1}\right](B-A)A^*(A^{k-1})^* = 0.$$

Thus, $A^k \overset{*}{\leqslant} B^k$, as desired.

(ii) It is clear we only need to show that $A \overset{-}{\leqslant} B \Longrightarrow A \overset{*}{\leqslant} B$ and $A *\!\leqslant B \Longrightarrow A \overset{*}{\leqslant} B$. Since A is an idempotent, A has a simple form as $A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the space decomposition $\mathscr{H} = \mathscr{R}(A) \oplus \mathscr{K}(A)$. If $A \overset{-}{\leqslant} B$, then there exist idempotents P, Q such that $\mathscr{R}(P) = \mathscr{R}(A)$ and $\mathscr{K}(Q) = \mathscr{K}(A)$. Let $P = \begin{pmatrix} I & -D \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I & 0 \\ -S & 0 \end{pmatrix}$. Similar to the proof in Theorem 2.1, we derive that $B = \begin{pmatrix} I+DB_2S & DB_2 \\ B_2S & B_2 \end{pmatrix}$. Note that $PAB = \begin{pmatrix} I+DB_2S & DB_2 \\ 0 & 0 \end{pmatrix}$ and $PB^2 = PB = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Since $A \overset{-}{\leqslant} B$ and $B^2 = B$, we deduce that $PAB = PBB = PB$ and therefore $DB_2 = 0$. In the same way we get that $B_2S = 0$ and $B = \begin{pmatrix} I & 0 \\ 0 & B_2 \end{pmatrix}$. Hence $A \overset{*}{\leqslant} B$. Similarly, if $A *\!\leqslant B$, then B can be written as $B = \begin{pmatrix} I & 0 \\ B_2S & B_2 \end{pmatrix}$. Since

$$B^2 = \begin{pmatrix} I & 0 \\ B_2S+B_2^2S & B_2^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ B_2S & B_2 \end{pmatrix} = B,$$

we know that $B_2^2 = B_2$ and $B_2S = 0$. Hence $A \overset{*}{\leqslant} B$.

(iii) Since A is normal, $\overline{\mathscr{R}(A)} = \overline{\mathscr{R}(AA^*)} = \overline{\mathscr{R}(A^*A)} = \overline{\mathscr{R}(A^*)}$. If $A *\!\leqslant B$, by Theorem 2.1 (ii), A^k exists and $A_1A_1^* = A_1^*A_1$,

$$BB^* = \begin{pmatrix} A_1A_1^* & A_1(B_2S)^* \\ B_2SA_1^* & B_2S(B_2S)^*+B_2B_2^* \end{pmatrix} \quad \text{and} \quad B^*B = \begin{pmatrix} A_1^*A_1+(B_2S)^*B_2S & (B_2S)^*B_2 \\ B_2^*B_2S & B_2^*B_2 \end{pmatrix}.$$

Since $BB^* = B^*B$, we obtain $(B_2S)^*B_2S = 0$, which implies that $B_2S = 0$ and $B = \begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}$. Hence, $A \overset{*}{\leqslant} B$ and $A^k *\!\leqslant B^k$. $\square$

Let $A \overset{*}{\leqslant} B$ and A, B have the corresponding representations as in Theorem 2.1. In general, $AB \neq BA$ since A_1^2 is not defined. In [21, Corollary 2.2 and Theorem 3.1], Merikoski and Liu proved that, if A, B are normal finite matrices and $A \overset{*}{\leqslant} B$, then $AB = BA$ and $A^2 \overset{*}{\leqslant} B^2$. Theorem 2.4 generalizes the results to the case that $A *\!\leqslant B$. Moreover, if A, B are positive, then $A *\!\leqslant B \Longleftrightarrow A^k *\!\leqslant B^k$ and $A \leqslant\!* B \Longleftrightarrow A^k \leqslant\!* B^k$.

As we know, an operator is MP invertible if and only if its range is closed. Using the results in Theorems 2.1 and 2.4, some MP inverse results can be obtained easily.

THEOREM 2.5. *Let $A, B \in B(\mathscr{H})$ such that A is MP invertible. If B satisfies any ordering relation among $A \overset{*}{\leqslant} B$, $A *\!\leqslant B$, $A \leqslant\!* B$ and $A \overset{-}{\leqslant} B$, then B is MP invertible if and only if $\Gamma =: (I - AA^+)B(I - A^+A)$ (i.e., B_2 in Theorem 2.1) is MP invertible. Moreover, we have the following detailed statements:*

(i) *If $A \overset{*}{\leqslant} B$, then B is MP invertible if and only if $\Gamma = B - A$ is MP invertible. The detailed relations are $B^+ - A^+ = \Gamma^+$ and*

$$(A^*A)^+ = (A^*B)^+ = (B^*A)^+ = A^+(A^+)^* = B^+(A^+)^* = A^+(B^+)^*.$$

(ii) *If $A *\!\leqslant B$, then B is MP invertible if and only if $\Gamma = B(I - A^+A)$ is MP invertible. The detailed relation is*

$$B^+ - A^+ = \Gamma^+ - \Gamma^+BA^+.$$

In addition, if A and B are normal, then

$$B^{\#}-A^{\#}=(B-A)^{\#}, \quad (A^2)^{\#}=(AB)^{\#}=B^{\#}A^{\#}=A^{\#}B^{\#}.$$

(iii) *If $A \overset{-}{\leqslant} B$, then B is MP invertible if and only if $\Gamma=(I-AA^+)B(I-A^+A)$ is MP invertible. The detailed relations are*

$$B^+-A^+=\Gamma^+-\Gamma^+BA^+-A^+B\Gamma^++\Gamma^+BA^+B\Gamma^+.$$

Proof. By Theorem 2.1, if A is MP invertible, then A_1 is invertible with $A^+=\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. In the following, we always suppose that B has the corresponding matrix form as given in Theorem 2.1.

(i) The condition $A \overset{*}{\leqslant} B$ implies that $A^+(B-A)=0$ and $(B-A)A^+=0$. Note that $A=\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A^+=\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. Let B have the corresponding matrix representation as $B=\begin{pmatrix} B_1 & B_3 \\ B_4 & B_2 \end{pmatrix}$. Then $A^+(B-A)=0$ implies that $B_1=A_1$ and $B_3=0$. And $(B-A)A^+=0$ implies that $B_4=0$. So B is MP invertible if and only if $\Gamma=(I-AA^+)B(I-A^+A)=(B-A)(I-A^+A)=B-A=\begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix}$ is MP invertible with $B^+=\begin{pmatrix} A_1^{-1} & 0 \\ 0 & B_2^+ \end{pmatrix}=A^++(B-A)^+=A^++\Gamma^+$. The results follow immediately.

(ii) By Theorem 2.1, item (ii), if $A \ast\!\leqslant B$, then $B=\begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}$ and $A^+(B-A)=0$. Note that, by Lemma 2.1

$$\mathscr{R}\left[\begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}\right]=\mathscr{R}\left[\begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}\begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}\right]=\mathscr{R}\left[\begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}\right]=\mathscr{R}(A)\oplus^{\perp}\mathscr{R}(B_2)$$

is closed if and only if $\mathscr{R}(B_2)$ is closed. We get B is MP invertible if and only if $\Gamma=(I-AA^+)B(I-A^+A)=B(I-A^+A)=\begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix}$ is MP invertible. Now, B_2 and S can be written as $B_2=\begin{pmatrix} B_{22} & 0 \\ 0 & 0 \end{pmatrix}$ and $S=\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$, where B_{22} is invertible. Then B has the form

$$\begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}=\begin{pmatrix} A_1 & 0 & 0 \\ B_{22}S_1 & B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}:\begin{pmatrix} \mathscr{R}(A^*) \\ \mathscr{R}(B_2^*) \\ \mathscr{K}(B_2) \end{pmatrix}\rightarrow\begin{pmatrix} \mathscr{R}(A) \\ \mathscr{R}(B_2) \\ \mathscr{K}(B_2^*) \end{pmatrix},$$

where $\begin{pmatrix} A_1 & 0 \\ B_{22}S_1 & B_{22} \end{pmatrix}$ is invertible and $\begin{pmatrix} A_1 & 0 \\ B_{22}S_1 & B_{22} \end{pmatrix}^{-1}=\begin{pmatrix} A_1^{-1} & 0 \\ -S_1A_1^{-1} & B_{22}^{-1} \end{pmatrix}$. So we get

$$\begin{aligned} B^+ &= \begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}^+=\begin{pmatrix} A_1^{-1} & 0 \\ -S_1A_1^{-1} & B_{22}^{-1} \end{pmatrix}\oplus 0=\begin{pmatrix} A_1^{-1} & 0 \\ -B_2^+B_2SA_1^{-1} & B_2^+ \end{pmatrix} \\ &= A^++\Gamma^+-\Gamma^+BA^+. \end{aligned}$$

In addition, if A is normal, $\mathscr{R}(A)=\mathscr{R}(AA^*)=\mathscr{R}(A^*A)=\mathscr{R}(A^*)$. So A is an EP operator, i.e., $A^+=A^{\#}$. Since B is normal, $B=\begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}$ by Theorem 2.4. Thus B is also an EP operator with $B^+=B^{\#}$ and the results follow directly by item (i).

(iii) If $A \overset{-}{\leqslant} B$, then there exist invertible operators $E=\begin{pmatrix} I & -D \\ 0 & I \end{pmatrix}$ and $F=\begin{pmatrix} I & 0 \\ -S & I \end{pmatrix}$ such that $EBF=\begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}$. Thus B is MP invertible if and only if $\Gamma=(I-AA^+)B(I-$

$A^+A) = \begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix}$ is MP invertible by Lemma 2.1. Let $D = \begin{pmatrix} D_1, D_2 \end{pmatrix}$, $B_2 = \begin{pmatrix} B_{22} & 0 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$. Then B has the form

$$\begin{pmatrix} A_1+DB_2S & DB_2 \\ B_2S & B_2 \end{pmatrix} = \begin{pmatrix} A_1+D_1B_{22}S_1 & D_1B_{22} & 0 \\ B_{22}S_1 & B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(A^*) \\ \mathscr{R}(B_2^*) \\ \mathscr{K}(B_2) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathscr{R}(A) \\ \mathscr{R}(B_2) \\ \mathscr{K}(B_2^*) \end{pmatrix}.$$

Since B_{22} is invertible and the Schur complement

$$S_{\text{Schur}} = (A_1 + D_1B_{22}S_1) - D_1B_{22}B_{22}^{-1}B_{22}S_1 = A_1$$

is invertible, we may conclude that $\begin{pmatrix} A_1+D_1B_{22}S_1 & D_1B_{22} \\ B_{22}S_1 & B_{22} \end{pmatrix}$ is invertible and

$$\begin{aligned} B^+ &= \begin{pmatrix} A_1+DB_2S & DB_2 \\ B_2S & B_2 \end{pmatrix}^+ = \begin{pmatrix} \begin{pmatrix} A_1+D_1B_{22}S_1 & D_1B_{22} \\ B_{22}S_1 & B_{22} \end{pmatrix}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A_1^{-1} & -A_1^{-1}D_1 & 0 \\ -S_1A_1^{-1} & B_{22}^{-1}+S_1A_1^{-1}D_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= A^+ + \Gamma^+ - \Gamma^+BA^+ - A^+B\Gamma^+ + \Gamma^+BA^+B\Gamma^+. \quad \square \end{aligned}$$

3. The group inverse

The representations in Theorem 2.1, on the one hand, have their obvious advantages for convenient MP inverse calculations. On the other hand, their weaknesses lie in the impossibility of having A_1^2 and B_2^2 unless $\overline{\mathscr{R}(A)} = \overline{\mathscr{R}(A^*)}$. Let us now consider the case of A being an EP operator. A further observation is that, if A is an EP operator, $\mathscr{R}(A) = \mathscr{R}(A^*)$ is closed and $\mathscr{R}(A)^\perp = \mathscr{R}(A^*)^\perp = \mathscr{K}(A)$. Hilbert space $\mathscr{H}$ has direct sum decomposition $\mathscr{H} = \mathscr{R}(A) \oplus \mathscr{K}(A)$. So, we can modify the representations in Theorem 2.1 such that A_1^2 and B_2^2 exist.

THEOREM 3.1. *Let $A, B \in B(\mathscr{H})$ such that A is an EP operator. If A has 2×2 operator matrix form $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to space decomposition $\mathscr{H} = \mathscr{R}(A) \oplus \mathscr{K}(A)$, where $A_1 \in B\big(\mathscr{R}(A)\big)$ is invertible. Then*

(i) $B = \begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}$ *if and only if* $A \overset{*}{\leqslant} B$;

(ii) $B = \begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}$ *if and only if* $A \,{*}\!\leqslant B$;

(iii) $B = \begin{pmatrix} A_1 & DB_2 \\ 0 & B_2 \end{pmatrix}$ *if and only if* $A \leqslant\!{*}\, B$;

(iv) $B = \begin{pmatrix} A_1+DB_2S & DB_2 \\ B_2S & B_2 \end{pmatrix}$ *if and only if* $A \overset{-}{\leqslant} B$,

where $B_2 \in B\big(\mathscr{K}(A)\big)$, $D \in B\big(\mathscr{K}(A), \mathscr{R}(A)\big)$ and $S \in B\big(\mathscr{R}(A), \mathscr{K}(A)\big)$.

Proof. Similar to the proof of Theorem 2.1, the details are omitted. □

If A is an EP operator and $A \overset{*}{\leqslant} B$, it follows immediately from Theorem 3.1 that $A^k \overset{*}{\leqslant} B^k$ for any integer $k \geqslant 2$ and $AB = BA = A^2$. From Theorems 2.1 and 3.1, we observe that the left star and right star orderings are located between the star and minus orderings in the sense that

$$A \overset{*}{\leqslant} B \Longrightarrow A *\!\leqslant B \quad \text{and} \quad A \leqslant\!* B,$$
$$A *\!\leqslant B \Longrightarrow A \overset{-}{\leqslant} B \quad \text{and} \quad A \leqslant\!* B \Longrightarrow A \overset{-}{\leqslant} B.$$

The above properties have been given by J. K. Baksalary and S.K. Mitra [1, Theorem 2.1] when A and B are finite matrices.

THEOREM 3.2. *Let $A, B \in B(\mathscr{H})$ such that A is an EP operator.*

(i) *If $AB = BA$, then $A \overset{*}{\leqslant} B \Longleftrightarrow A *\!\leqslant B \Longleftrightarrow A \leqslant\!* B \Longleftrightarrow A \overset{-}{\leqslant} B$.*

(ii) *$A *\!\leqslant B \Longleftrightarrow A^2 = AB$ and $\mathscr{R}(A) \subseteq \mathscr{R}(B)$.*

(iii) *$A \leqslant\!* B \Longleftrightarrow A^2 = BA$ and $\mathscr{R}(A^*) \subseteq \mathscr{R}(B^*)$.*

Proof. (i) We only prove that $A \overset{-}{\leqslant} B \Longrightarrow A \overset{*}{\leqslant} B$. The remaining parts can be proved in the same way. If A is an EP operator such that $A \overset{-}{\leqslant} B$, by Theorem 3.1, $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} A_1 + DB_2S & DB_2 \\ B_2S & B_2 \end{pmatrix}$. Since A_1 is invertible and $AB = BA$, we have $DB_2 = 0$ and $B_2S = 0$. So $B = \begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}$ and $A \overset{*}{\leqslant} B$.

(ii) If A is an EP operator and $A *\!\leqslant B$, by item (ii) of Theorem 3.1, it is clear that $A^2 = AB$ and $\mathscr{R}(A) \subseteq \mathscr{R}(B)$. On the other hand, let $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$. If $A^2 = AB$, then B has the form as $\begin{pmatrix} A_1 & 0 \\ B_4 & B_2 \end{pmatrix}$. Similar to the proof in Theorem 2.1 (ii), if $\mathscr{R}(A) \subseteq \mathscr{R}(B)$, then there exists a linear bounded operator $S \in B\big(\mathscr{R}(A), \mathscr{K}(A)\big)$ such that $B_4 = B_2S$. Hence $B = \begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}$, i.e., $A *\!\leqslant B$.

(iii) Similar to (ii), the details are omitted. □

For a triangular matrix, the following result, which is proved in [17] for matrices, has been extended to a bounded linear operator [10] and to arbitrary elements in a Banach algebra [8].

LEMMA 3.1. ([10, Theorem 5.1]) *If $A \in B(\mathscr{H})$ and $D \in B(\mathscr{K})$ are Drazin invertible with $ind(A) = l$ and $ind(D) = s$, $C \in B(\mathscr{H}, \mathscr{K})$, then $M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ is Drazin invertible and*

$$M^D = \begin{pmatrix} A^D & 0 \\ X & D^D \end{pmatrix},$$

where $X = \sum_{n=0}^{l-1} (D^D)^{n+2} C A^n A^\pi + D^\pi \sum_{n=0}^{s-1} D^n C (A^D)^{n+2} - D^D C A^D$.

THEOREM 3.3. *Let $A,B \in B(\mathscr{H})$ such that A is an EP operator. If $\Gamma =: A^{\pi}BA^{\pi}$ (i.e., B_2 in Theorem 3.1) is group invertible and B satisfies any ordering relation among $A \overset{*}{\leqslant} B$, $A *\!\leqslant B$, $A \leqslant\!* B$ and $A \overset{-}{\leqslant} B$, then B is group invertible. Moreover, we have the following detailed statements:*

(i) *If $A \overset{*}{\leqslant} B$, then B is group invertible if and only if $\Gamma = B - A$ is group invertible. The detailed relations are $B^{\#} - A^{\#} = \Gamma^{\#}$ and*

$$(A^2)^{\#} = (AB)^{\#} = (BA)^{\#} = B^{\#}A^{\#} = A^{\#}B^{\#}.$$

(ii) *If $A *\!\leqslant B$, then B is Drazin invertible if and only if $\Gamma = BA^{\pi}$ is Drazin invertible and*

$$B^D - A^{\#} = \Gamma^D - \Gamma^D BA^{\#} + \Gamma^{\pi} \sum_{n=0}^{l-1} B^n A^{\pi} B (A^{\#})^{n+2},$$

where $l = ind(\Gamma)$. In particular, if $ind(\Gamma) = 1$, then $B^{\#} - A^{\#} = \Gamma^{\#} - \Gamma^{\#}BA^{\#}$.

(iii) *If $A \overset{-}{\leqslant} B$ and $\Gamma = A^{\pi}BA^{\pi}$ is group invertible, then B is group invertible and*

$$B^{\#} - A^{\#} = \Gamma^{\#} - \Gamma^{\#}BA^{\#} - A^{\#}B\Gamma^{\#} + \Gamma^{\#}BA^{\#}B\Gamma^{\#}.$$

Proof. (i) By Theorem 3.1, if A is an EP operator and $A \overset{*}{\leqslant} B$, then A_1 is invertible with $A^{\#} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, $A^{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ and $B = \begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix}$. The results follow immediately.

(ii) By Theorem 3.1, item (ii), if $A *\!\leqslant B$, then $B = \begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}$. Since A_1 is invertible, B is Drazin invertible if and only if B_2 is Drazin invertible, i,e, $\Gamma = A^{\pi}BA^{\pi} = BA^{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix}$ is Drazin invertible. If $\operatorname{ind}(\Gamma) = l$, by Lemma 3.1, we get

$$\begin{aligned} B^D &= \begin{pmatrix} A_1 & 0 \\ B_2S & B_2 \end{pmatrix}^D = \begin{pmatrix} A_1^{-1} & 0 \\ (I - B_2B_2^D)\sum\limits_{n=0}^{l-1} B_2^n B_2 S A_1^{-(n+2)} - B_2^D B_2 S A_1^{-1} & B_2^D \end{pmatrix} \\ &= A^{\#} + \Gamma^D - \Gamma^D BA^{\#} + \Gamma^{\pi} \sum_{n=0}^{l-1} B^n A^{\pi} B (A^{\#})^{n+2}. \end{aligned}$$

If B is group invertible, then $BB^DB = B$. It implies that $B_2B_2^DB_2 = B_2$ and $(I - B_2B_2^D)\sum_{n=0}^{l-1} B_2^n B_2 S A_1^{-(n+2)} = 0$. Hence, Γ is group invertible and

$$B^{\#} = \begin{pmatrix} A_1^{-1} & 0 \\ -B_2^{\#}B_2SA_1^{-1} & B_2^{\#} \end{pmatrix} = A^{\#} + \Gamma^{\#} - \Gamma^{\#}BA^{\#}.$$

(iii) If $A \overset{-}{\leqslant} B$ and $\Gamma = A^{\pi}BA^{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix}$ is group invertible, then

$$B = \begin{pmatrix} A_1 + DB_2S & DB_2 \\ B_2S & B_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -B_2^{\#}B_2S & I \end{pmatrix} \begin{pmatrix} A_1 & DB_2 \\ B_2^{\#}B_2SA_1 & B_2 + B_2^{\#}B_2SDB_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ B_2^{\#}B_2S & I \end{pmatrix}.$$

The group invertibility of B_2 implies that B_2 can be written as $B_2 = \begin{pmatrix} B_{22} & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the space decomposition $\mathscr{H} = \mathscr{K}(B_2^{\pi}) \oplus \mathscr{R}(B_2^{\pi})$, where B_{22} is invertible. Let $D = \begin{pmatrix} D_1, D_2 \end{pmatrix}$ and $S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$. Then

$$\begin{pmatrix} A_1 & DB_2 \\ B_2^{\#}B_2SA_1 & B_2+B_2^{\#}B_2SDB_2 \end{pmatrix} = \begin{pmatrix} A_1 & D_1B_{22} & 0 \\ S_1A_1 & B_{22}+S_1D_1B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(A) \\ \mathscr{K}(B_2^{\pi}) \\ \mathscr{R}(B_2^{\pi}) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(A) \\ \mathscr{K}(B_2^{\pi}) \\ \mathscr{R}(B_2^{\pi}) \end{pmatrix}.$$

Since B_{22} is invertible and the Schur complement

$$S_{\text{Schur}} = (B_{22}+S_1D_1B_{22}) - S_1A_1A_1^{-1}D_1B_{22} = B_{22}$$

is invertible, $\begin{pmatrix} A_1 & D_1B_{22} \\ S_1A_1 & B_{22}+S_1D_1B_{22} \end{pmatrix}$ is invertible and

$$\begin{aligned} B^D &= \begin{pmatrix} I & 0 \\ -B_2^{\#}B_2S & I \end{pmatrix} \begin{pmatrix} A_1 & DB_2 \\ B_2^{\#}B_2SA_1 & B_2+B_2^{\#}B_2SDB_2 \end{pmatrix}^D \begin{pmatrix} I & 0 \\ B_2^{\#}B_2S & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -B_2^{\#}B_2S & I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} A_1 & D_1B_{22} \\ S_1A_1 & B_{22}+S_1D_1B_{22} \end{pmatrix}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ B_2^{\#}B_2S & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -B_2^{\#}B_2S & I \end{pmatrix} \begin{pmatrix} A_1^{-1}+A_1^{-1}D_1S_1 & -A_1^{-1}D_1 & 0 \\ -B_{22}^{-1}S_1 & B_{22}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ B_2^{\#}B_2S & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ -B_2^{\#}B_2S & I \end{pmatrix} \begin{pmatrix} A_1^{-1}+A_1^{-1}DB_2^{\#}B_2S & -A_1^{-1}DB_2^{\#}B_2 \\ -B_2^{\#}S & B_2^{\#} \end{pmatrix} \begin{pmatrix} I & 0 \\ B_2^{\#}B_2S & I \end{pmatrix} \\ &= \begin{pmatrix} A_1^{-1} & -A_1^{-1}DB_2B_2^{\#} \\ -B_2^{\#}B_2SA_1^{-1} & B_2^{\#}+B_2^{\#}B_2SA_1^{-1}DB_2B_2^{\#} \end{pmatrix}. \end{aligned}$$

We get $BB^D = B^DB = \begin{pmatrix} I & 0 \\ 0 & B_2^{\#}B_2 \end{pmatrix}$ and $BB^DB = B$. So $B^D = B^{\#}$ is the group inverse of B. Since $A^{\#} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ and $\Gamma^{\#} = \begin{pmatrix} 0 & 0 \\ 0 & B_2^{\#} \end{pmatrix}$, we get

$$\begin{aligned} B^{\#} &= \begin{pmatrix} A_1^{-1} & -A_1^{-1}DB_2B_2^{\#} \\ -B_2^{\#}B_2SA_1^{-1} & B_2^{\#}+B_2^{\#}B_2SA_1^{-1}DB_2B_2^{\#} \end{pmatrix} \\ &= A^{\#}+\Gamma^{\#}-\Gamma^{\#}BA^{\#}-A^{\#}B\Gamma^{\#}+\Gamma^{\#}BA^{\#}B\Gamma^{\#}. \quad \square \end{aligned}$$

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