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LYAPUNOV TYPE INEQUALITY FOR THE EQUATION INCLUDING 1-dim p-LAPLACIAN

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Abstract. Lyapunov type inequality, for the existence of the solution of the equation including (generalized) p-Laplacian:

$$(-1)^{(m)}\left(|u^{(m)}(x)|^{p-2}u^{(m)}(x)\right)^{(m)} = r(x)|u(x)|^{p-2}u(x) \quad (a \leqslant x \leqslant b)$$

under clamped boundary condition is obtained. The usage of the best constant of L^p Sobolev inequality clarifies the process for obtaining such inequality.

1. Introduction

Let us consider the second-order linear differential equation:

$$\begin{cases} u''(x) + r(x)u(x) = 0 \quad (a \leqslant x \leqslant b) \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

where $r \in C([a,b],[0,\infty))$. It is well known that the Lyapunov inequality:

$$\frac{4}{b-a} < \int_a^b r(x)dx \tag{2}$$

gives a necessary condition for the existence of non-trivial classical solution of (1). Various extensions and improvements for the above result have been attempted; see for example, Ha [2], Yang [6] and their references. We would like to note that all such results consider linear equations which are extensions of (1). On the other hand, J. Pinasco recently obtained the necessary condition for the existence of the non-trivial solution of the nonlinear equation including p-Laplacian;

$$-\left(|u'(x)|^{p-2}u'(x)\right)' = r(x)|u(x)|^{p-2}u(x) \quad (a \leqslant x \leqslant b) \tag{3}$$

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under Dirichlet boundary condition [3] and Dirichlet-Neumann boundary condition [4]. This paper studies the necessary condition for the existence of the non-trivial solution of the following clamped boundary value problem including (generalized) p-Laplacian:

$$\begin{cases} (-1)^{(m)}\left(|u^{(m)}(x)|^{p-2}u^{(m)}(x)\right)^{(m)} = r(x)|u(x)|^{p-2}u(x) \quad (a \leqslant x \leqslant b) \\ u^{(i)}(a) = u^{(i)}(b) = 0 \quad (i = 0,1,\cdots m-1), \end{cases} \tag{4}$$

for $m = 1,2,3$ and Dirichlet-Neumann boundary value problem:

$$\begin{cases} -\left(|u'(x)|^{p-2}u'(x)\right)' = r(x)|u(x)|^{p-2}u(x) \quad (a \leqslant x \leqslant b) \\ u'(a) = u(b) = 0. \end{cases} \tag{5}$$

For the case $m = 1$ of problem (4) and problem (5), although the results have already been known by Pinasco [3, 4], alternative proofs which use the best constant of Sobolev inequality are presented. To introduce the result precisely, let us fix some notations. Let $W^{m,p}(a,b)$ be a Sobolev space defined on the interval $[a,b]$; $W^{m,p}(a,b) := \{u \,|\, u^{(i)} \in L^p(a,b) \quad (i = 0,\ldots,m)\}$, $W_0^{m,p}(a,b)$ a sub-space of $W^{m,p}(a,b)$ whose derivatives up to $m-1$ vanish at $x = a,b$, where $u^{(i)}$ denotes i-th derivative of u in a distributional sense. We say $u \in W_0^{m,p}(a,b)$ is a solution of (4), for arbitrary $v \in W_0^{m,p}(a,b)$ it holds that

$$\int_a^b |u^{(m)}(x)|^{p-2}u^{(m)}(x)v^{(m)}(x)dx = \int_a^b r(x)|u(x)|^{p-2}u(x)v(x)dx \tag{6}$$

and $u \in W^{1,p}(a,b)$, $u(b) = 0$ is a solution of (5), for arbitrary $v \in W^{1,p}(a,b)$, $v(b) = 0$ it holds that

$$\int_a^b |u'(x)|^{p-2}u'(x)v'(x)dx = \int_a^b r(x)|u(x)|^{p-2}u(x)v(x)dx. \tag{7}$$

Note that from (7), $u'(a) = 0$ holds. With these settings we have the following results:

THEOREM 1. *Let $m = 1,2,3$, $p > 1$, q be a conjugate exponent of p, i.e. $1/p + 1/q = 1$ and $u \in W_0^{m,p}(a,b)$ a non-trivial solution of (4). Then*

$$\frac{1}{C(m,p)^p} < \int_a^b r(x)dx \tag{8}$$

holds, where $C(m,p)$ is a best constant of L^p Sobolev inequality:

$$\Big(\sup_{a \leqslant x \leqslant b} |v(x)|\Big) \leqslant C\left(\int_a^b |v^{(m)}(x)|^p dx\right)^{\frac{1}{p}} \qquad (v \in W_0^{m,p}(a,b)). \tag{9}$$

Concrete form of $C(m,p)$ *is given by*

$$C(1,p)=\frac{(b-a)^{1/q}}{2},\quad C(2,p)=\frac{(b-a)^{(q+1)/q}}{2^3(q+1)^{1/q}},$$

$$C(3,p)=2^{\frac{1-2q}{q}}\Big(\int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\alpha}(x-\frac{a+b}{2})^q(\frac{a+b}{2}+\alpha-x)^q dx$$

$$+\int_{\frac{a+b}{2}+\alpha}^{b}(x-\frac{a+b}{2})^q(x-\frac{a+b}{2}-\alpha)^q dx\Big)^{\frac{1}{q}}$$

where α *in* $C(3,p)$ *is the unique solution of the equation*

$$\int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\alpha}(x-\frac{a+b}{2})^q(\frac{a+b}{2}+\alpha-x)^{q-1}dx=\int_{\frac{a+b}{2}+\alpha}^{b}(x-\frac{a+b}{2})^q(x-\frac{a+b}{2}-\alpha)^{q-1}dx$$

satisfying $0<\alpha<(b-a)/2$. *Moreover,* (8) *is sharp in the sense, that there exists* u *and* r *such that the right-hand-side of* (8) *can be arbitrarily closed to left-hand-side.*

THEOREM 2. *Let* $p>1$ *and* $u\in W^{1,p}(a,b)$ *be a solution of* (5)*, then it holds that*

$$\frac{1}{(b-a)^{p-1}}<\int_a^b r(x)dx. \tag{10}$$

Moreover, (10) *is sharp.*

2. Proof of Theorem 1 and 2

To prove Theorem 1 and 2, we make use of the best constant of Sobolev inequality as (9). It should be noted that the idea to use the best constant of Sobolev inequality for obtaining Lyapunov type inequality owes to Brown-Hinton [1, Theorem 2.1].

LEMMA 1. *Let* $m=1,2,3,\ p>1$ *and let* $u\in W_0^{m,p}(a,b)$, *then the best constant of Sobolev inequality* (9) *is given by* $C(m,p)$ *in Theorem 1. Moreover, the functions* u^* *which attain the best constants take their absolute maximums at* $x=(a+b)/2$ *for* $m=1,2,3$.

Proof. See [5, Theorem 1.1]. □

LEMMA 2. *Let* $1<p$ *and* $S:=\{\phi\in W^{1,p}(a,b)|\quad \phi(b)=0\}$. *Then, for* $u\in S$, *the best constant of Sobolev inequality*

$$\sup_{a\leqslant x\leqslant b}|u(x)|\leqslant C\cdot\Big(\int_a^b|u'(x)|^p dx\Big)^{1/p} \tag{11}$$

is $(b-a)^{(p-1)/p}$. *Moreover, the function* u^* *which attains the best constant takes its absolute maximum at* $x=a$.

Proof. Let us define

$$H(x,y) := \begin{cases} 0 & (a \leqslant x \leqslant y) \\ -1 & (y \leqslant x \leqslant b). \end{cases}$$

Then, for $u \in S$, we have

$$u(y) = \int_a^b u'(x)H(x,y)dx \qquad (a \leqslant y \leqslant b). \tag{12}$$

So, by Hölder's inequality, we have

$$\begin{aligned} \sup_{a\leqslant y\leqslant b} |u(y)| &\leqslant \Big(\sup_{a\leqslant y\leqslant b} \int_a^b |H(x,y)|^q dx\Big)^{1/q} \|u'\|_{L^p(a,b)} \\ &= \sup_{a\leqslant y\leqslant b} (b-y)^{1/q} \|u'\|_{L^p(a,b)} \\ &= (b-a)^{1/q}\|u'\|_{L^p(a,b)} = (b-a)^{(p-1)/p}\|u'\|_{L^p(a,b)}. \end{aligned} \tag{13}$$

Thus, if it exists, the best constant is smaller than $(b-a)^{(p-1)/p}$. However, the equality holds in (13) for $u(x) = -x+b$. Hence, we have proved the lemma. □

Proof of Theorem 1. Substituting $v = u$ in (6) and using Lemma 1, we have

$$\begin{aligned} \int_a^b |u^{(m)}|^p dx = \int_a^b r|u|^p dx &< \|u\|^p_{L^\infty(a,b)} \int_a^b r(x)dx \\ &\leqslant C(m,p)^p \int_a^b |u^{(m)}|^p dx \int_a^b r(x)dx \end{aligned} \tag{14}$$

Thus, we have (8). To see that (8) is sharp, let us define the functional

$$J(\phi) := \frac{\int_a^b |\phi^{(m)}|^p dx}{\int_a^b \tilde{r}|\phi|^p dx} \qquad (\phi \in W_0^{m,p}(a,b), \phi \not\equiv 0),$$

where $\tilde{r} \in C([a,b],[0,\infty))$. By the standard argument (see Appendix), J has the minimizer $u \in W_0^{m,p}(a,b)$, i.e.

$$\lambda_1 := \min_{\phi\in W_0^{m,p}(a,b),\phi\not\equiv 0} J(\phi) = J(u).$$

Hence it satisfies Euler-Lagrange equation (as non-trivial solution):

$$(-1)^{(m)}\Big(|u^{(m)}(x)|^{p-2}u^{(m)}(x)\Big)^{(m)} = \lambda_1 \tilde{r}(x)|u(x)|^{p-2}u(x) \quad (a \leqslant x \leqslant b) \tag{15}$$

Further, it holds that

$$\begin{aligned} \lambda_1 &= \min_{\phi\in W_0^{m,p}(a,b),\phi\not\equiv 0} \frac{\int_a^b |\phi^{(m)}|^p dx}{\int_a^b \tilde{r}|\phi|^p dx} \\ &> \min_{\phi\in W_0^{m,p}(a,b),\phi\not\equiv 0} \frac{\int_a^b |\phi^{(m)}|^p dx}{(\sup_{a\leqslant x\leqslant b}|\phi|)^p \int_a^b \tilde{r}dx} \geqslant \frac{1}{C(m,p)^p \int_a^b \tilde{r}dx} \end{aligned} \tag{16}$$

Here, let us fix $\tilde{r}$ as

$$\tilde{r}(x) := \begin{cases} \frac{x-\frac{a+b}{2}}{\delta}+1 & (\frac{a+b}{2}-\delta < x \leqslant \frac{a+b}{2}) \\ -\frac{x-\frac{a+b}{2}}{\delta}+1 & (\frac{a+b}{2} < x < \frac{a+b}{2}+\delta) \\ 0 & else. \end{cases}$$

For such $\tilde{r}$, let us substitute $\phi = u^*$ in (16). Since u^* takes its maximum at $x = (a+b)/2$, taking δ sufficiently small, we see that the right-hand-side of (16) can be arbitrarily closed to the left-hand-side, i.e. for small positive ε_1, λ_1 can be written as

$$\lambda_1 = \frac{1}{C(m,p)^p \int_a^b \tilde{r}dx} + \varepsilon_1 \tag{17}$$

Putting $r = \lambda_1 \tilde{r}$, we see from (15), solution u of

$$(-1)^{(m)} \left(|u^{(m)}(x)|^{p-2} u^{(m)}(x) \right)^{(m)} = r(x)|u(x)|^{p-2}u(x) \quad (a \leqslant x \leqslant b) \tag{18}$$

exists, and from (17), r satisfies

$$\int_a^b r(x)dx = \frac{1}{C(m,p)^p} + \varepsilon_1 \int_a^b \tilde{r}dx = \frac{1}{C(m,p)^p} + \varepsilon_2 \tag{19}$$

Hence (8) is sharp. □

Proof of Theorem 2. Using Lemma 2, (10) is proved quite the same as (8). To see that (10) is sharp, in this case, we define $\tilde{r}$ as

$$\tilde{r}(x) := \begin{cases} -\frac{x-a}{\delta}+1 & (a \leqslant x \leqslant a+\delta) \\ 0 & else. \end{cases}$$

Noting u^* takes its maximum at $x = a$ at this case, and following the similar argument as Theorem 1, we obtain the result. □

3. Appendix

LEMMA 3. *Let $\tilde{r} \in C([a,b],[0,\infty))$, then the minimizer of J exists.*

Proof. Let R be sufficiently large, and let W' and W'' be as

$$W' := \{u \in W_0^{m,p}(a,b) | \quad \int_a^b \tilde{r}|\phi|^p dx = 1\}$$
$$W'' := \{u \in W_0^{m,p}(a,b) | \quad \|u^{(m)}\|_{L^p(a,b)} \leqslant R\}$$

Since R is sufficiently large, if the minimizer of J exists, it is the element of $W := W' \cap W''$. Let us see that W' is weakly closed. Assume the sequence $\{\phi_n\} \subset W'$ such that $\phi_n \rightharpoonup \phi_0$ Since $W_0^{m,p}(a,b)$ is compactly embedded into $L^\infty(a,b)$, $\phi_n \to \phi_0$ in $L^\infty(a,b)$. Using this and Lebesgue convergence theorem, we see

$$\int_a^b \tilde{r}|\phi_0|^p dx = 1.$$

So, W' is weakly closed. In addition, W'' is weakly compact, thus W is also weakly compact. Moreover, $\|\phi^{(m)}\|_{L^p(a,b)}$ is weakly lower-semicontinuous in $W_0^{m,p}(a,b)$, hence J attains its minimum on W. □

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