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ON PRODUCTS OF GENERALIZED ORLICZ SPACES

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Abstract. In the context of generalized Orlicz spaces, the products $X_{\Phi_1} \odot X_{\Phi_2}$ and $X_{\Phi_1} \otimes X_{\Phi_2}$ are studied and conditions are obtained under which these spaces are contained in a suitable space X_Φ. These imbedding results (inequalities) are in a sense sharp and for the case $X = L_1$, the conditions are even necessary and sufficient. Moreover, a new Hölder type inequality is proved.

1. Introduction

In the sequel, X denotes a Banach function space (shortly written as BFS). In [7], Jain, Persson and Upreti studied the generalized Orlicz space X_Φ as a unification and extension of two spaces, namely, the X^p-space and the usual Orlicz space L_Φ. In fact, these spaces are special cases of the so called Calderon-Lozanovskii spaces $\rho(X,Y)$, where $Y = L_\infty$ in this case. In particular, one can find these spaces in Example 2 on pages 178-179 in the book [18] by L. Maligranda. Moreover, the spaces X^p are known as the p-convexification of X (see, e.g., [11] p. 143). There is a lot of basic information about Calderon-Lozanovskii spaces, e.g., in the books [18] by Maligranda and [14] by Lindestrauss-Tzafriri. Moreover, many papers are devoted to this interesting subject, e.g., [3], [4], [5], [10] and [11]. We also mention the early papers [15] and [16] by Lozanovskii. Let us just mention that the space X_Φ can be given with two norms, the Orlicz type norm and the Luxemburg type norm which are equivalent. Further, X_Φ is a Banach function space if X is so. Moreover, in [7], it was proved that a number of basic inequalities such as Hölder's, Minkowski's and Young's hold in the framework of these spaces. We also refer to [8] for some complementary results.

The present paper focuses on products of X_Φ-spaces. Two type of products are discussed, to be denoted by $X_{\Phi_1} \odot X_{\Phi_2}$ and $X_{\Phi_1} \otimes X_{\Phi_2}$. We find various conditions under which these product spaces are contained in some other appropriate generalized Orlicz space. Among these, some conditions for $X_{\Phi_1} \odot X_{\Phi_2}$ and $X_{\Phi_1} \otimes X_{\Phi_2}$ are both necessary and sufficient. In particular, some results generalize the results of Ando [1] and Krasnosel'skii and Rutickii [12] who considered the products $L_{\Phi_1} \odot L_{\Phi_2}$ and $L_{\Phi_1} \otimes L_{\Phi_2}$. Also a new Hölder type inequality is proved.

The paper is organized as follows : In Section 2, we give certain preliminaries required for subsequent sections. The main results concerning the product $X_{\Phi_1} \odot X_{\Phi_2}$ are presented and proved in Section 3 and the corresponding results concerning the product $X_{\Phi_1} \otimes X_{\Phi_2}$ can be found in Section 4.

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2. Preliminaries

The concept of BFS was introduced by Luxemburg [17]. For the definition and other properties of such spaces, one can refer to [2].

A BFS is said to satisfy the lower p-estimate with $p = 1$ if there exists a positive constant a such that the inequality

$$\left\| \sum_{n=1}^{\infty} u_n \right\|_X \geqslant a \sum_{n=1}^{\infty} \|u_n\|_X$$

holds for every sequence $\{u_n\}$, $u_n = u_n(t) > 0$ with

$$\operatorname{supp} u_n \cap \operatorname{supp} u_m = \emptyset, \qquad n \neq m,$$

see, e.g., ([11], p. 142) and note that the L-property in [7] is equivalent to this notion.

Examples of Banach function spaces are the classical Lebesgue spaces L^p, $1 \leqslant p \leqslant \infty$, the Orlicz spaces L_Φ, the classical Lorentz spaces $L_{p,q}$, $1 \leqslant p,q \leqslant \infty$, the generalized Lorentz spaces Λ_ϕ and the Marcinkiewicz spaces M_ϕ.

Let X be a BFS and $-\infty < p < \infty$, $p \neq 0$. We define the space X^p, usually called the p-convexification of X (see, e.g., ([11], p. 163), to be the space of all measurable functions f for which

$$\|f\|_{X^p} := \||f|^p\|_X^{\frac{1}{p}} < \infty.$$

For $1 < p < \infty$, X^p is a BFS. Note that for $X = L^1$, the space X^p coincides with the L^p-space. These spaces have been studied and used in several papers, e.g., [19], [20], [21]. Recently in [6], [9], Hardy type inequalities (and also geometric mean inequalities in some cases) have been studied in the context of X^p spaces.

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if

$$\Phi(s) = \int_0^s \phi(t)dt,$$

where $\phi : [0,\infty) \to [0,\infty]$, $\phi(0) = 0$ is an increasing, left continuous function which is neither identically zero nor identically infinite on $(0,\infty)$. A Young function Φ is continuous, convex, increasing and satisfies

$$\Phi(0) = 0, \quad \lim_{s\to\infty} \Phi(s) = \infty.$$

Moreover, a Young function Φ satisfies the following useful inequalities: for $s \geqslant 0$, we have

$$\begin{cases} \Phi(\alpha s) < \alpha\Phi(s), & \text{if } 0 \leqslant \alpha < 1 \\ \Phi(\alpha s) \geqslant \alpha\Phi(s), & \text{if } \alpha \geqslant 1. \end{cases} \tag{1}$$

Let Φ be a Young function generated by the function ϕ, i.e.,

$$\Phi(s) = \int_0^s \phi(t)dt.$$

Then the function Ψ generated by the function ψ, i.e.,

$$\Psi(s) = \int_0^s \psi(t)dt\,,$$

where

$$\psi(s) = \sup_{\phi(t)\leqslant s} t$$

is called the complementary function to Φ. It is known that Ψ is a Young function and that Φ is complementary to Ψ. The pair of complementary Young functions Φ, Ψ satisfies Young's inequality

$$u\cdot v \leqslant \Phi(u)+\Psi(v), \quad u,v\in[0,\infty). \tag{2}$$

Equality in (2.2) holds if and only if

$$v = \Phi(u) \quad \text{or} \quad u = \Psi(v)\,.$$

A Young function Φ is said to satisfy the Δ_2-condition, written $\Phi\in\Delta_2$, if there exist $k>0$ and $T\geqslant 0$ such that

$$\Phi(2t)\leqslant k\Phi(t) \quad \text{for all } \ t\geqslant T\,.$$

We shall also use the so called Δ'-condition: A Young function Φ is said to satisfy the Δ'-condition, written $\Phi\in\Delta'$, if there exist $c>0$ and $t_0\geqslant 0$ such that

$$\Phi(u\cdot v)\leqslant c\Phi(u)\Phi(v) \quad \text{for all } \ |u|,|v|\geqslant t_0.$$

Note that Δ'-condition implies Δ_2-condition.

Let Φ_1 and Φ_2 be two Young functions. We write $\Phi_2\prec\Phi_1$ if there exist constants $c>0$, $T\geqslant 0$ such that

$$\Phi_2(t)\leqslant\Phi_1(ct), \quad t\geqslant T\,.$$

The above mentioned concepts on Orlicz spaces are quite standard which can be found in any standard book on Orlicz space. Here we mention the celebrity monographs [12], [13] and [22].

The rest of the concepts we need are some of those introduced and studied in [7], [8]. We mention them here briefly.

Let X be a BFS and Φ denote a non-negative function on $[0,\infty)$. The generalized Orlicz class $\widetilde{X}_\Phi$ consists of all functions $u\in L^0(\Omega)$ such that

$$\rho_X(u,\Phi) = \|\Phi(|u|)\|_X < \infty\,.$$

For the case $\Phi(t)=t^p$, $0<p<\infty$, $\widetilde{X}_\Phi$ coincides algebraically with the space X^p endowed with the quasi-norm

$$\|u\|_{X^p} = \||u|^p\|_X^{\frac{1}{p}}\,.$$

Let X be a BFS and Φ, Ψ be a pair of complementary Young functions. The generalized Orlicz space, denoted by X_Φ, is the set of all $u \in L^0(\Omega)$ such that

$$\|u\|_\Phi := \sup_v \| |u \cdot v| \|_X ,$$

where the supremum is taken over all $v \in \widetilde{X}_\Psi$ for which $\rho_X(v;\Psi) \leqslant 1$.

It is known that for a Young function Φ, $\widetilde{X}_\Phi \subset X_\Phi$ and that X_Φ is a BFS, with the norm (2.4). Further, on the generalized Orlicz space X_Φ, a Luxemburg type norm can be defined in the following way:

$$\|u\|'_\Phi = \inf\left\{ k > 0 : \rho_X\left(\frac{|u|}{k}, \Phi\right) \leqslant 1 \right\}.$$

It is known that with this norm too, the space X_Φ is a BFS and that the two norms above are equivalent, i.e., there exists constants $c_1, c_2 > 0$ such that

$$c_1 \|u\|'_\Phi \leqslant \|u\|_\Phi \leqslant c_2 \|u\|'_\Phi .$$

In fact, it was proved in [7] that $c_2 = 2$.

Throughout, $\Omega \subset \mathbb{R}^n$ will be a set of finite measure.

3. The product $X_{\Phi_1} \odot X_{\Phi_2}$

We begin with the following definition:

DEFINITION 1. Let X_{Φ_1}, X_{Φ_2} be two generalized Orlicz spaces defined on $\Omega \subset \mathbb{R}^n, \Phi_1, \Phi_2$ being the Young functions. The product $X_{\Phi_1} \odot X_{\Phi_2}$ is defined to be space of all measurable functions defined on Ω which can be expressed as a product of the type $u \cdot v$, where $u \in X_{\Phi_1}$ and $v \in X_{\Phi_2}$.

Clearly, when $X = L^1$, the space $X_{\Phi_1} \odot X_{\Phi_2}$ becomes the standard product $L_{\Phi_1} \odot L_{\Phi_2}$.

In this section, we find conditions under which the space $X_{\Phi_1} \odot X_{\Phi_2}$ is contained in X_Φ, i.e., $X_{\Phi_1} \odot X_{\Phi_2} \subset X_\Phi$ for some Young function Φ.

THEOREM 1. *Let X be a BFS.*

(a) *If Φ, Φ_1, Φ_2 are Young functions such that there exists $\xi, \gamma > 0$ satisfying*

$$\Phi(\xi\alpha\beta) < \Phi_1(\alpha) + \Phi_2(\beta), \qquad \alpha, \beta \geqslant \gamma, \tag{1}$$

then the inclusion

$$X_{\Phi_1} \odot X_{\Phi_2} \subset X_\Phi \tag{2}$$

holds for any BFS X.

(b) *The statement in (a) is sharp in the sense that for the case $X = L_1$, in fact,* (3.1) *is a necessary and sufficient condition for the inclusion* (3.2).

REMARK 1. From the proof below we see that the equivalence estimate in (b) in fact holds for each X satisfying a lower p-statement with $p=1$. However, this is not an essentially stronger statement since it is easy to see that this assumption in fact implies that $X=L_1$.

Proof of Theorem 1. (a) Assume that (3.1) holds. Let $w \in X_{\Phi_1} \odot X_{\Phi_2}$. Then w can be expressed as

$$w = u.v$$

with $u \in X_{\Phi_1}$ and $v \in X_{\Phi_2}$. Consequently, there exists $k>0$ such that

$$\|\Phi_1(|ku|)\|_X + \|\Phi_2(|kv|)\|_X < \infty.$$

By using monotonicity of Young functions and (3.1) we have

$$\Phi(\xi k^2|u.v|) \leqslant \Phi_1(\gamma)+\Phi_1(|ku|)+\Phi_2(\gamma)+\Phi_2(|kv|),$$

which gives

$$\begin{aligned}\|\Phi(\xi k^2|u.v|)\|_X &\leqslant \Phi_1(\gamma)\|\chi_\Omega\|_X + \|\Phi_1(|ku|)\|_X + \Phi_2(\gamma)\|\chi_\Omega\|_X + \Phi_2(|kv|)\|_X \\ &< \infty\end{aligned}$$

and, consequently, $u \cdot v \in X_\Phi$, i.e., $w \in X_\Phi$, which means that (3.2) holds.

(b) Assume that (3.2) holds with $X=L_1$. We only need to prove the necessity of condition (3.1) for which it is sufficient to prove that there exist integer $k>0$ such that

$$\Phi\Big(\frac{\alpha\beta}{k}\Big) \leqslant 2^k\big(\Phi_1(\alpha)+\Phi_2(\beta)\big), \qquad \alpha,\beta \geqslant k. \tag{3}$$

Assume that (3.3) does not hold for any k. Then there exist sequences $\{\alpha_k\},\{\beta_k\}$ such that $\alpha_k,\beta_k \geqslant k$ and

$$\Phi\Big(\frac{\alpha_k\beta_k}{k}\Big) > 2^k[\Phi_1(\alpha_k)+\Phi_2(\beta_k)], \quad k=1,2,3,\ldots$$

Without any loss of generality, we can assume that $\Phi_1(1)+\Phi_2(1)>0$. Choose a sequence $\{\Omega_k\}$ of mutually disjoint measurable subsets of Ω such that

$$\|\,\chi_{\Omega_k}\|_X = \frac{\|\chi_\Omega\|_X[\Phi_1(1)+\Phi_2(1)]}{2^k[\Phi_1(\alpha_k)+\Phi_2(\beta_k)]}, \quad k=1,2,3,\ldots \tag{4}$$

Take

$$u(x)=\begin{cases}\alpha_k, & \text{if}\, x\in\Omega_k\\ 0, & \text{if}\, x\notin\bigcup_{k=1}^\infty\Omega_k\end{cases} \qquad \text{and} \qquad v(x)=\begin{cases}\beta_k, & \text{if}\, x\in\Omega_k\\ 0, & \text{if}\, x\notin\bigcup_{k=1}^\infty\Omega_k.\end{cases}$$

Then we have, by using (3.4) and the triangle inequality, that

$$\begin{aligned}\|\Phi_1(|u|)\|_X+\|\Phi_2(|v|)\|_X &= \left\|\sum_{k=1}^{\infty}\Phi_1(\alpha_k)\chi_{\Omega_k}\right\|_X+\left\|\sum_{k=1}^{\infty}\Phi_2(\beta_k)\chi_{\Omega_k}\right\|_X\\ &\leqslant \sum_{k=1}^{\infty}\Phi_1(\alpha_k)\|\chi_{\Omega_k}\|_X+\sum_{k=1}^{\infty}\Phi_2(\beta_k)\|\chi_{\Omega_k}\|_X\\ &= \sum_{k=1}^{\infty}[\Phi_1(\alpha_k)+\Phi_2(\beta_k)]\|\chi_{\Omega_k}\|_X\\ &= \|\chi_\Omega\|_X[\Phi_1(1)+\Phi_2(1)]<\infty,\end{aligned}$$

which yields that $u\in X_{\Phi_1}$ and $v\in X_{\Phi_2}$ and so $uv\in X_{\Phi_1}\odot X_{\Phi_2}$. But on the other hand we find using well known properties of the space $X=L_1$ and (3.4) that

$$\begin{aligned}\left\|\Phi(|\frac{uv}{k}|)\right\|_X &\geqslant \left\|\sum_{i=k}^{\infty}\Phi(\frac{\alpha_i\beta_i}{k})\chi_{\Omega_i}\right\|_X\geqslant\sum_{i=k}^{\infty}\left\|\Phi(\frac{\alpha_i\beta_i}{k})\chi_{\Omega_i}\right\|_X\\ &> \sum_{i=k}^{\infty}2^k[\Phi_1(\alpha_i)+\Phi_2(\beta_i)]\|\chi_{\Omega_i}\|_X\\ &= \sum_{i=k}^{\infty}\|\chi_\Omega\|_X[\Phi_1(1)+\Phi_2(1)]=\infty,\end{aligned}$$

i.e., $u\cdot v\notin X_\Phi$ which is a contradiction. Thus we conclude that (3.1) holds and the assertion follows. □

REMARK 2. The statement of Theorem 1(b) coincides with Theorem 1 in [1].

Our next result gives a Hölder type inequality. For this, we need the following lemma:

LEMMA 1. *Let u be an arbitrary measurable function defined on Ω. Suppose $u.v\in X_\Phi$ for all functions $v\in X_{\Phi_1}$. Then there exists a constant $k>0$ such that*

$$\|u.v\|_\Phi\leqslant K\|v\|_{\Phi_1}.$$

Proof. It is completely similar to the proof of Lemma 13.4 in [12], so we omit the details. □

THEOREM 2. *Let $u.v\in X_\Phi$ for all functions $u\in X_{\Phi_1}, v\in X_{\Phi_2}$. Then there exists a constant $k>0$ such that*

$$\|u.v\|_\Phi\leqslant k\|u\|_{\Phi_1}\|v\|_{\Phi_2}.$$

Proof. Define the linear operators

$$A_u(v) = u.v, \quad v \in X_{\Phi_2} \tag{5}$$

where $u \in X_{\Phi_1}, \|u\|_{\Phi_1} \leqslant 1$.

For each $u \in X_{\Phi_1}$, the operator A_u acts from X_{Φ_2} to X_{Φ_1} which, in view of Lemma 1, is bounded. Thus $\{A_u\}$ is a sequence of pointwise bounded operators and as such it is uniformly bounded since X_{Φ_2} is a complete space. Consequently $\|A_u\| \leqslant k$ for same constant k and for all $u \in X_{\Phi_1}$. Denote $u_0 = \frac{u}{\|u\|_{\Phi_1}}$. Clearly $u_0 \in X_{\Phi_1}$. Then, in view of (3.5), we have for $v \in X_{\Phi_2}$,

$$\|u.v\|_\Phi = \left\|\frac{u}{\|u\|_{\Phi_1}} v\right\|_\Phi \|u\|_{\Phi_1} = \left\|A_{u_0}(v)\right\|_\Phi \|u\|_{\Phi_1} \leqslant k\|v\|_{\Phi_2}\|u\|_{\Phi_1},$$

and hence the proof is complete. □

4. The product $X_{\Phi_1} \otimes X_{\Phi_2}$

In Section 2, the BFX X consisted of functions which are defined on $\Omega \subset \mathbb{R}^n$. In this section, we consider $2n$ dimensional space where the functions are defined on $\Omega \times \Omega \subset \mathbb{R}^n \times \mathbb{R}^n$. In order not to confuse our notation, we denote such a space by $X \times X$. Naturally, we can define generalized Orlicz space which has functions defined on $\Omega \times \Omega$. We denote such a space by $\hat{X}_\Phi$. Further we define the product $X_{\Phi_1} \otimes X_{\Phi_2}$ below.

DEFINITION 2. Let X_{Φ_1}, X_{Φ_2} be two generalized Orlicz spaces defined on $\Omega \subset \mathbb{R}^n, \Phi_1, \Phi_2$ being the Young functions. The product $X_{\Phi_1} \otimes X_{\Phi_2}$ is defined to be the space of all measurable functions w defined on $\Omega \times \Omega$ which can be expressed as a product of the type $u.v$ where $u \in X_{\Phi_1}, v \in X_{\Phi_2}$.

In this section, we find conditions under which the space $X_{\Phi_1} \otimes X_{\Phi_2}$ is contained in $\hat{X}_\Phi$ i.e $X_{\Phi_1} \otimes X_{\Phi_2} \subset \hat{X}_\Phi$.

THEOREM 3. *Consider the space $X \times X$. Assume that for all $w \in X \times X$ such that $w(x,y) = u(x)v(y)$ with $u, v \in X$, the following condition holds*

$$\|w\|_{X \times X} = \|u\|_X \|v\|_X. \tag{1}$$

(a) *If Φ, Φ_1, Φ_2 are Young functions such that there exists $\xi, \gamma > 0$ satisfying*

$$\Phi(\xi\alpha\beta) < \Phi_1(\alpha)\Phi_2(\beta), \qquad \alpha, \beta \geqslant \gamma, \tag{2}$$

then the inclusion

$$X_{\Phi_1} \otimes X_{\Phi_2} \subset \hat{X}_\Phi \tag{3}$$

holds

(b) *The statement in (a) is sharp in the sense that for the case $X = L_1$ in fact,* (4.2) *is a necessary and sufficient condition for the inclusion* (4.3).

REMARK 3. Also in this case it is easy to see that the equivalence statement in (b) indeed holds for each X satisfying a lower p-estimate with $p = 1$ but that this statement is not more general.

Proof of Theorem 3. (a) Let $w \in X_{\Phi_1} \otimes X_{\Phi_2}$. Then

$$w = u.v$$

with $u \in X_{\Phi_1}$, $v \in X_{\Phi_2}$ and both are defined on Ω. We have that there exists $k > 0$ such that

$$\|\Phi_1(|ku|)\|_X + \|\Phi_2(|kv|)\|_X < \infty.$$

By using monotonicity of Young functions and (4.2) we find that

$$\Phi(\xi k^2|u.v|) \leqslant \left[\Phi_1(\gamma) + \Phi_1(|ku|)\right]\left[\Phi_2(\gamma) + \Phi_2(|kv|)\right]$$

and, consequently, by (4.1), we obtain

$$\begin{aligned}\|\Phi(\xi k^2|u.v|)\|_{X\times X} &\leqslant \left\{\Phi_1(\gamma)\|\chi_\Omega\|_X + \|\Phi_1(|ku|)\|_X\right\} \times \left\{\Phi_2(\gamma)\|\chi_\Omega\|_X + \Phi_2(|kv|)\|_X\right\} \\ &< \infty.\end{aligned}$$

Thus $u \cdot v = w \in \hat{X}_\Phi$ and (4.3) holds.

(b) Let $X = L_1$. We only need to prove the necessity of condition (4.2) for which it is sufficient to prove that there exists an integer $k > 0$ such that

$$\Phi\Big(\frac{\alpha\beta}{k}\Big) \leqslant 2^{2k}\Phi_1(\alpha)\Phi_2(\beta), \qquad \alpha,\beta > k. \tag{4}$$

Assume, on the contrary, that (4.4) does not hold. Then there exist sequences $\{\alpha_k\}, \{\beta_k\}$ such that $\alpha_k, \beta_k \geqslant k$ and

$$\Phi\Big(\frac{\alpha_k\beta_k}{k}\Big) > 2^{2k}\Phi_1(\alpha_k)\Phi_2(\beta_k), \quad k = 1,2,3,... \tag{5}$$

Without any loss of generality, we can assume that $\Phi_1(1)\Phi_2(1) > 0$. Choose sequences $\{\Omega_k\}$ and $\{\Omega'_k\}$ of mutually disjoint measurable subsets of Ω such that

$$\|\,\chi_{\Omega_k}\|_X = \frac{\|\chi_\Omega\|_X\Phi_1(1)}{2^k\Phi_1(\alpha_k)}, \quad k = 1,2,3,... \tag{6}$$

and

$$\|\,\chi_{\Omega'_k}\|_X = \frac{\|\chi_\Omega\|_X\Phi_2(1)}{2^k\Phi_2(\beta_k)}, \quad k = 1,2,3,... \tag{7}$$

Take

$$u(x) = \begin{cases}\alpha_k, \text{ if } x \in \Omega_k \\ 0, \text{ if } x \notin \bigcup_{k=1}^\infty \Omega_k\end{cases} \qquad \text{and} \qquad v(x) = \begin{cases}\beta_k, \text{ if } x \in \Omega'_k \\ 0, \text{ if } x \notin \bigcup_{k=1}^\infty \Omega'_k.\end{cases}$$

Then we have using (4.6) and (4.7) that

$$\begin{aligned}\|\Phi_1(|u|)\|_X+\|\Phi_2(|v|)\|_X &= \left\|\sum_{k=1}^{\infty}\Phi_1(\alpha_k)\chi_{\Omega_k}\right\|_X+\left\|\sum_{k=1}^{\infty}\Phi_2(\beta_k)\chi_{\Omega'_k}\right\|_X\\ &\leqslant \sum_{k=1}^{\infty}\Phi_1(\alpha_k)\|\chi_{\Omega_k}\|_X+\sum_{k=1}^{\infty}\Phi_2(\beta_k)\|\chi_{\Omega'_k}\|_X\\ &= \sum_{k=1}^{\infty}\frac{\|\chi_\Omega\|_X\Phi_1(1)}{2^k}+\sum_{k=1}^{\infty}\frac{\|\chi_\Omega\|_X\Phi_2(1)}{2^k}\\ &= \|\chi_{\Omega_k}\|_X[\Phi_1(1)+\Phi_2(1)]<\infty,\end{aligned}$$

which yields that $u\in X_{\Phi_1}$ and $v\in X_{\Phi_2}$. But on the other hand by using integrability properties on $X\times X$ and (4.5), we obtain

$$\begin{aligned}\left\|\Phi(|\frac{u(x)v(y)}{k}|)\right\|_{X\times X} &\geqslant \left\|\sum_{i=k}^{\infty}\Phi(\frac{\alpha_i\beta_i}{k})\chi_{\Omega_i}\chi_{\Omega'_i}\right\|_{X\times X}\\ &\geqslant a\sum_{i=k}^{\infty}\left\|\Phi(\frac{\alpha_i\beta_i}{k})\chi_{\Omega_i}\chi_{\Omega'_i}\right\|_{X\times X}\\ &> a\sum_{i=k}^{\infty}2^{2k}\Phi_1(\alpha_i)\Phi_2(\beta_i)\|\chi_{\Omega_i}\chi_{\Omega'_i}\|_{X\times X}\\ &= a\sum_{i=k}^{\infty}2^{2k}\Phi_1(\alpha_i)\Phi_2(\beta_i)\|\chi_{\Omega_i}\|_X\|\chi_{\Omega'_i}\|_X\\ &= a\sum_{i=k}^{\infty}\|\chi_\Omega\|_X^2\Phi_1(1)\Phi_2(1)=\infty,\end{aligned}$$

i.e., $u\cdot v\notin\hat{X}_\Phi$, which is a contradiction and the result follows. □

REMARK 4. Theorem 3 is a generalization of Theorem 6 in [1].

THEOREM 4. *Let Φ be a Young function. Assume that for all $w\in X\times X$ such that $w(x,y)=u(x)v(y)$ with $u,v\in X$,* (4.1) *holds.*

(a) *If Φ satisfies the Δ'-condition, then the inclusion $X_\Phi\otimes X_\Phi\subset\hat{X}_\Phi$ holds.*

(b) *The statement in* (a) *is sharp in the sense that for the case $X=L_1$, in fact, the Δ'-condition is both necessary and sufficient for the inclusion in* (a) *to hold.*

REMARK 5. As in previous cases, the equivalence statement in (b) holds for each X satisfying a lower p-estimate with $p=1$ but this is not a more general statement.

Proof of Theorem 4. (a) Let $w\in X_\Phi\otimes X_\Phi$ so that w can be expressed as $w=u.v$ with $u,v\in X_\Phi$. Since the Young function Φ satisfies the Δ'-condition, there exists positive constants t_0 and c such that

$$\Phi(t_1t_2)\leqslant c\Phi(t_1)\Phi(t_2),\qquad t_1,t_2\geqslant t_0$$

and Φ, in particular, satisfies the Δ_2-condition which gives $\tilde{X}_\Phi = X_\Phi$. Let $u,v \in X_\Phi$. Denote

$$\Omega_u = \{x \in \Omega; |u| \geqslant t_0\} \qquad \text{and} \qquad \Omega_v = \{x \in \Omega; |v| \geqslant t_0\}.$$

Then, for $x \in \Omega_u$ and and $y \in \Omega_v$, we have

$$\Phi(|u.v|) \leqslant c\Phi(|u|)\Phi(|v|)$$

and using (4.1), we obtain

$$\begin{aligned}
\|\Phi(|u.v|)\|_{X\times X} &= \|\Phi(|u.v|)\chi_{\Omega_u\times\Omega_v} + \Phi(|u.v|)\chi_{\Omega\setminus\Omega_u\times\Omega_v}\|_{X\times X} \\
&\leqslant c\|\Phi(|u|)\|_X\|\Phi(|v|)\|_X + \|\Phi(|t_0v|)\chi_{\Omega_v}\|_X + \|\Phi(|t_0u|)\chi_{\Omega_u}\|_X \\
&\quad + \|(\Phi(t_0))^2\chi_{(\Omega\setminus\Omega_u)\times(\Omega\setminus\Omega_v)}\|_X \\
&< \infty
\end{aligned}$$

which gives that $w = u.v \in \hat{\hat{X}}_\Phi \subset \hat{X}_\Phi$ and the assertion follows.

(b) Let $X = L_1$ and assume that the inclusion $X_\Phi \otimes X_\Phi \subset \hat{X}_\Phi$ holds. Suppose that Φ does not satisfy the Δ'-condition. Then there exist sequences $\{\alpha_k\},\{\beta_k\}$ such that $\alpha_k,\beta_k \geqslant 0$ and

$$\Phi(\alpha_k\beta_k) > 2^{2k}\Phi(\alpha_k)\Phi(\beta_k), \quad k = 1,2,3,... \tag{8}$$

Choose sequences $\{\Omega_k\}$ and $\{\Omega_k'\}$ of mutually disjoint measurable subsets of Ω such that

$$\|\chi_{\Omega_k}\|_X = \frac{\|\chi_\Omega\|_X\Phi(\alpha_1)}{2^k\Phi(\alpha_k)}, \quad k = 1,2,3,...$$

and

$$\|\chi_{\Omega_k'}\|_X = \frac{\|\chi_\Omega\|_X\Phi(\beta_1)}{2^k\Phi(\beta_k)}, \quad k = 1,2,3,...$$

Take

$$u(x) = \begin{cases} \alpha_k, \text{ if } x \in \Omega_k \\ 0, \text{ if } x \notin \bigcup_{k=1}^\infty \Omega_k \end{cases} \qquad \text{and} \qquad v(x) = \begin{cases} \beta_k, \text{ if } x \in \Omega_k' \\ 0, \text{ if } x \notin \bigcup_{k=1}^\infty \Omega_k'. \end{cases}$$

Then we have

$$\begin{aligned}
\|\Phi(|u|)\|_X + \|\Phi(|v|)\|_X &= \left\|\sum_{k=1}^\infty \Phi(\alpha_k)\chi_{\Omega_k}\right\|_X + \left\|\sum_{k=1}^\infty \Phi(\beta_k)\chi_{\Omega_k'}\right\|_X \\
&\leqslant \sum_{k=1}^\infty \Phi(\alpha_k)\|\chi_{\Omega_k}\|_X + \sum_{k=1}^\infty \Phi(\beta_k)\|\chi_{\Omega_k'}\|_X \\
&= \sum_{k=1}^\infty \frac{\|\chi_\Omega\|_X\Phi(\alpha_1)}{2^k} + \sum_{k=1}^\infty \frac{\|\chi_\Omega\|_X\Phi(\beta_1)}{2^k} \\
&= \|\chi_{\Omega_k}\|_X[\Phi(\alpha_1) + \Phi(\beta_1)] < \infty,
\end{aligned}$$

so that $u,v \in X_\Phi$. But on the other hand by usual integrability rules in $X \times X$ and (4.8), we obtain

$$\begin{aligned}
\|\Phi(|uv|)\|_{X\times X} &\geqslant \left\| \sum_{i=k}^{\infty} \Phi\big(\alpha_i\beta_i\big)\chi_{\Omega_i}\chi_{\Omega_i'} \right\|_{X\times X} \\
&\geqslant \sum_{i=k}^{\infty} \left\| \Phi\big(\alpha_i\beta_i\big)\chi_{\Omega_i}\chi_{\Omega_i'} \right\|_{X\times X} \\
&> \sum_{i=k}^{\infty} 2^{2i}\Phi(\alpha_i)\Phi(\beta_i)\|\chi_{\Omega_i}\chi_{\Omega_i'}\|_{X\times X} \\
&= \sum_{i=k}^{\infty} 2^{2i}\Phi(\alpha_i)\Phi(\beta_i)\|\chi_{\Omega_i}\|_X\|\chi_{\Omega_i'}\|_X \\
&= \sum_{i=k}^{\infty} \|\chi_\Omega\|_X^2\Phi(\alpha_1)\Phi(\beta_1) = \infty,
\end{aligned}$$

i.e., $u \cdot v \notin \hat{X}_\Phi$, which is a contradiction and the necessity follows. $\square$

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