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INEQUALITIES OF HLAWKA TYPE FOR FUZZY REAL NUMBERS

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Abstract. In this paper, we give and prove inequalities of Hlawka type and related results in the fuzzy real number space.

1. Introduction

For arbitrary real numbers x, y, z, the inequality

$$|x+y|+|y+z|+|z+x| \leqslant |x|+|y|+|z|+|x+y+z| \tag{1.1}$$

is well known as Hlawka's inequality. It can be extended to an n-dimensional Euclidean vector space and more generally, a Hilbert space (see [5, p. 521]). It also has other various extensions which are referred as inequalities of Hlawka type (see [1], [3], [5–10]).

In this paper, we give a version of Hlawka's inequality and some results related to this inequality in the fuzzy real number space. First we introduce some notations.

Denote by $\mathbb{R}_{\mathscr{F}}$ the class of fuzzy subsets u of the real axis $\mathbb{R}$ (i.e., $u:\mathbb{R}\to[0,1]$) satisfying the following properties:

(i) u is normal, i.e., $\exists x_u \in \mathbb{R}$ with $u(x_u)=1$;

(ii) u is a convex fuzzy set, i.e., $u(tx+(1-t)y) \geqslant \min\{u(x),u(y)\}$ for any $t\in[0,1]$, $x,y\in\mathbb{R}$;

(iii) u is upper semi-continuous on $\mathbb{R}$;

(iv) $\overline{\{x\in\mathbb{R} : u(x)>0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\mathbb{R}_{\mathscr{F}}$ is called the space of fuzzy real numbers and any $u\in\mathbb{R}_{\mathscr{F}}$ is called a fuzzy real number (see e.g. [4]).

For $0<\alpha\leqslant 1$ and $u\in\mathbb{R}_{\mathscr{F}}$, let

$$[u]^\alpha = \{x\in\mathbb{R} : u(x)\geqslant\alpha\} \quad \text{and} \quad [u]^0=\overline{\{x\in\mathbb{R} : u(x)>0\}}.$$

Then for each $\alpha\in[0,1]$, $[u]^\alpha=[u^\alpha_-,u^\alpha_+]$ is a bounded closed interval (u^α_-,u^α_+ denote the endpoints of the $\alpha-$level set). For $u,v\in\mathbb{R}_{\mathscr{F}}$ and $\lambda\in\mathbb{R}$, we have the sum $u\oplus\upsilon$ and the product $\lambda\cdot u$ defined by $[u\oplus v]^\alpha=[u]^\alpha+[\upsilon]^\alpha$, $[\lambda\cdot u]^\alpha=\lambda[u]^\alpha, \forall\alpha\in[0,1]$, where $[u]^\alpha+[\upsilon]^\alpha$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see e.g. [4], [11]).

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REMARK 1. Obviously, $\mathbb{R} \subset \mathbb{R}_{\mathscr{F}}$ because any real number $x_0 \in \mathbb{R}$ can be described as the fuzzy real number $\tilde{x_0} = \chi_{\{x_0\}}$ whose value is 1 for $x = x_0$ and 0 otherwise. Especially, we denote $\tilde{0} = \chi_{\{0\}}$. Then $\tilde{0} \in \mathbb{R}_{\mathscr{F}}$ is a neutral element with respect to $\oplus$, i.e., $u \oplus \tilde{0} = \tilde{0} \oplus u = u$ for all $u \in \mathbb{R}_{\mathscr{F}}$.

It is easy to see that $(\mathbb{R}_{\mathscr{F}}, \oplus, \cdot)$ is not a linear space since the following distribution law

$$(a+b) \cdot u = a \cdot u \oplus b \cdot u, \ \ a,b \in \mathbb{R}, \ \ u \in \mathbb{R}_{\mathscr{F}}$$

holds only for $ab \geqslant 0$. We can define a metric D on $\mathbb{R}_{\mathscr{F}}$ by

$$D(u,\upsilon) = \sup_{\alpha \in [0,1]} \max\{|u_-^\alpha - \upsilon_-^\alpha|, |u_+^\alpha - \upsilon_+^\alpha|\}, \ \ u,\upsilon \in \mathbb{R}_{\mathscr{F}}.$$

Then $(\mathbb{R}_{\mathscr{F}}, D)$ is a complete metric space (see [2], [4]).

For any $u \in \mathbb{R}_{\mathscr{F}}$, we denote $\|u\|_{\mathscr{F}} = D(u,\tilde{0})$. Then $\|\cdot\|_{\mathscr{F}}$ has the properties of the usual norm on $\mathbb{R}_{\mathscr{F}}$, i.e., for $u \in \mathbb{R}_{\mathscr{F}}$, $\|u\|_{\mathscr{F}} \geqslant 0$ and $\|u\|_{\mathscr{F}} = 0$ if and only if $u = \tilde{0}$; for any $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}_{\mathscr{F}}$, $\|\lambda \cdot u\|_{\mathscr{F}} = |\lambda| \cdot \|u\|_{\mathscr{F}}$; for any $u,\upsilon \in \mathbb{R}_{\mathscr{F}}$, $\|u \oplus \upsilon\|_{\mathscr{F}} \leqslant \|u\|_{\mathscr{F}} + \|\upsilon\|_{\mathscr{F}}$. However, $\|\cdot\|_{\mathscr{F}}$ is not a norm, since $\mathbb{R}_{\mathscr{F}}$ is not a linear space.

This paper is devoted to proving inequalities of Hlawka type for fuzzy real numbers. Our main result are formulated as follows:

THEOREM 1. *For any $u,v,w \in \mathbb{R}_{\mathscr{F}}$, we have*

$$\|u\|_{\mathscr{F}} + \|v\|_{\mathscr{F}} + \|w\|_{\mathscr{F}} + \|u \oplus v \oplus w\|_{\mathscr{F}} \geqslant \|u \oplus v\|_{\mathscr{F}} + \|u \oplus w\|_{\mathscr{F}} + \|v \oplus w\|_{\mathscr{F}}. \quad (1.2)$$

Based on Theorem 1, by the standard method (see e.g. [3], [5, pp. 522–528], [6]) we have the following generalization of Hlawka's inequality, which is usually called Djoković's inequality. We omit the proof.

THEOREM 2. *For any natural numbers n,k with $2 \leqslant k \leqslant n-1$, the inequality*

$$\sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \|u_{i_1} \oplus \cdots \oplus u_{i_k}\|_{\mathscr{F}} \leqslant \binom{n-1}{k-1} \sum_{i=1}^{n} \|u_i\|_{\mathscr{F}} + \binom{n-2}{k-2} \|u_1 \oplus \cdots \oplus u_n\|_{\mathscr{F}}$$

holds for all $u_1, \ldots, u_n \in \mathbb{R}_{\mathscr{F}}$.

2. Proof of Theorem 1

In order to prove Theorem 1, we need to prove the following lemma, which may have its own independent interest.

LEMMA 1. *For $a_1,a_2,b_1,b_2,c_1,c_2 \in \mathbb{R}$, we have*

$$\begin{aligned}\max\{|a_1|,|a_2|\} &+ \max\{|b_1|,|b_2|\} + \max\{|c_1|,|c_2|\} + \max\{|a_1+b_1+c_1|,|a_2+b_2+c_2|\} \\ &\geqslant \max\{|a_1+b_1|,|a_2+b_2|\} + \max\{|a_1+c_1|,|a_2+c_2|\} \\ &\quad + \max\{|b_1+c_1|,|b_2+c_2|\}. \qquad (2.1)\end{aligned}$$

REMARK 2. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, we denote $\|\mathbf{a}\|_\infty = \max_{1 \leqslant i \leqslant n} |a_i|$. Then for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, the inequality

$$\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty + \|\mathbf{c}\|_\infty + \|\mathbf{a}+\mathbf{b}+\mathbf{c}\|_\infty \geqslant \|\mathbf{a}+\mathbf{b}\|_\infty + \|\mathbf{a}+\mathbf{c}\|_\infty + \|\mathbf{b}+\mathbf{c}\|_\infty \tag{2.2}$$

holds only when $n = 1$ or 2. Indeed, (2.2) with $n = 2$ or $n = 1$ is (2.1) or (1.1), and if $n \geqslant 3$ and $\mathbf{a} = (-1, 1, 1, 0, \dots, 0)$, $\mathbf{b} = (1, -1, 1, 0, \dots, 0)$, $\mathbf{c} = (1, 1, -1, 0, \dots, 0) \in \mathbb{R}^n$, then (2.2) does not hold.

Proof of Lemma 1. Due to the symmetry and the property that $|x| = |-x|$, we may assume that $a_1, b_1 \geqslant 0$ and at most one number of a_2, b_2, c_2 is negative. So it suffices to consider the following four cases:

(A) $a_1, b_1, c_1, a_2, b_2, c_2 \geqslant 0$;
(B) $a_1, b_1, c_1, a_2, b_2 \geqslant 0,\ c_2 \leqslant 0$;
(C) $a_1, b_1, a_2, b_2 \geqslant 0,\ c_1, c_2 \leqslant 0$;
(D) $a_1, b_1 \geqslant 0,\ c_1 \leqslant 0,\ a_2 \geqslant 0,\ b_2 \leqslant 0,\ c_2 \geqslant 0$.

First, we consider Case (A). In this case, $a_1, b_1, c_1, a_2, b_2, c_2 \geqslant 0$. Using the equality $\max\{x, y\} = \frac{x+y+|x-y|}{2}$, we get that (2.1) is equivalent to the follow inequality

$$\begin{aligned} &|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |(a_1 - a_2) + (b_1 - b_2) + (c_1 - c_2)| \\ \geqslant\ &|(a_1 - a_2) + (b_1 - b_2)| + |(a_1 - a_2) + (c_1 - c_2)| + |(b_1 - b_2) + (c_1 - c_2)|. \end{aligned} \tag{2.3}$$

Clearly, (2.3) follows directly from (1.1) with $x = a_1 - a_2$, $y = b_1 - b_2$, $z = c_1 - c_2$.

Next, we consider Case (B). We replace c_2 by $-c_2$ in this case. Then $a_1, b_1, c_1, a_2, b_2, c_2 \geqslant 0$ and (2.1) is equivalent to the following inequality

$$\begin{aligned} &|a_2 - a_1| + |b_2 - b_1| + c_2 + |c_2 - c_1| + |a_2 + b_2 - c_2| + \big|a_1 + b_1 + c_1 - |a_2 + b_2 - c_2|\big| \\ \geqslant\ &|a_2 - a_1 + b_2 - b_1| + |a_2 - c_2| + |b_2 - c_2| + \big|b_1 + c_1 - |b_2 - c_2|\big| + \big|a_1 + c_1 - |a_2 - c_2|\big|. \end{aligned} \tag{2.4}$$

We notice that for $a, b \geqslant 0$ and $t \in \mathbb{R}$, (1.1) reduces to

$$|a + b + t| + |t| \geqslant |a + t| + |b + t|. \tag{2.5}$$

We divide Case (B) into the following four subcases (i)-(iv):
Subcase (i): $c_2 \geqslant a_2 + b_2$. In this subcase, (2.4) is equivalent to

$$\begin{aligned} &|a_2 - a_1| + |b_1 - b_2| + |c_2 - c_1| + |a_1 + b_1 + a_2 + b_2 + c_1 - c_2| \\ \geqslant\ &|a_1 - a_2 + b_1 - b_2| + |b_1 + b_2 + c_1 - c_2| + |a_1 + a_2 + c_1 - c_2|. \end{aligned} \tag{2.6}$$

Using (2.5) with $a = a_1 + a_2,\ b = b_1 + b_2,\ t = c_1 - c_2$ and the triangle inequality

$$|a_1 - a_2 + b_1 - b_2| \leqslant |a_1 - a_2| + |b_1 - b_2|,$$

we get (2.6).

The remaining three subcases satisfy the condition $c_2 \leqslant a_2+b_2$. If $c_2 \leqslant a_2+b_2$, by (1.1) we get

$$\begin{aligned}\text{Left hand of (2.4)} &= |a_1-a_2|+|b_1-b_2|+c_1+c_2+\big|a_1-a_2+b_1-b_2+c_1+c_2\big| \\ &\quad +|c_1-c_2|+a_2+b_2-c_2-c_1 \\ &\geqslant |a_1-a_2+b_1-b_2|+|a_1-a_2+c_1+c_2|+|b_1-b_2+c_1+c_2| \\ &\quad +|c_1-c_2|+a_2+b_2-c_2-c_1.\end{aligned}$$

This means that in order to prove (2.4), it suffices to prove that

$$\begin{aligned}&|a_1-a_2+c_1+c_2|+|b_1-b_2+c_1+c_2|+|c_1-c_2|+a_2+b_2 \\ \geqslant&|a_2-c_2|+|b_2-c_2|+\big|b_1+c_1-|b_2-c_2|\big|+\big|a_1+c_1-|a_2-c_2|\big|+c_1+c_2. \quad (2.7)\end{aligned}$$

Subcase (ii): $\min\{a_2,b_2\} \geqslant c_2$. In this subcase, (2.7) is equivalent to

$$|c_1-c_2| \geqslant c_1-c_2,$$

which is true.

Subcase (iii): $\min\{a_2,b_2\} \leqslant c_2 \leqslant \max\{a_2,b_2\}$. We may assume $a_2 \leqslant c_2 \leqslant b_2$. In this subcase, (2.7) is equivalent to

$$a_1+a_2+|c_1-c_2| \geqslant |a_1+c_1-c_2+a_2|,$$

which follows from the triangle inequality.

Subcase (iv): $\max\{a_2,b_2\} \leqslant c_2$ and $a_2+b_2 \geqslant c_2$. In this subcase, (2.7) is equivalent to

$$|c_1-c_2|+a_1+a_2+b_1+b_2+c_1-c_2 \geqslant |b_1+b_2+c_1-c_2|+|a_1+a_2+c_1-c_2|,$$

which can be deduced from (2.5) with $a=a_1+a_2,\ b=b_1+b_2,\ t=c_1-c_2$.

Now we consider Case (C). We replace c_1 and c_2 by $-c_1$ and $-c_2$ in this case. Then $a_1,b_1,c_1,a_2,b_2,c_2 \geqslant 0$ and (2.1) is equivalent to the following inequality

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+c_1+c_2+|a_1+b_1-c_1|+|a_2+b_2-c_2| \\ &+\big||a_1+b_1-c_1|-|a_2+b_2-c_2|\big| \ \geqslant\ |a_1-a_2+b_1-b_2|+|a_1-c_1|+|a_2-c_2| \\ &+\big||a_1-c_1|-|a_2-c_2|\big|+|b_1-c_1|+|b_2-c_2|+\big||b_2-c_2|-|b_1-c_1|\big|. \quad (2.8)\end{aligned}$$

We divide Case (C) into the following two subcases (I)-(II):

Subcase (I): $(a_1+b_1-c_1)(a_2+b_2-c_2) \geqslant 0$. In this subcase, by (1.1), the triangle inequality, and (2.5), we have

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_2-c_1|+\big||a_1+b_1-c_1|-|a_2+b_2-c_2|\big| \\ \geqslant\ &|a_1-a_2+b_1-b_2|+|a_1-a_2+c_2-c_1|\ +|b_1-b_2+c_2-c_1| \\ \geqslant\ &|a_1-a_2+b_1-b_2|+\big||a_1-c_1|-|a_2-c_2|\big|+\big||b_1-c_1|-|b_2-c_2|\big|; \quad (2.9)\end{aligned}$$

$$|a_1+b_1-c_1|+c_1 \geqslant |a_1-c_1|+|b_1-c_1|; \tag{2.10}$$

and

$$|a_2+b_2-c_2|+c_2 \geqslant |a_2-c_2|+|b_2-c_2|,$$

which, together with (2.9) and (2.10), implies (2.8).

Subcase (II): $(a_1+b_1-c_1)(a_2+b_2-c_2) \leqslant 0$. We may suppose $a_1+b_1-c_1 \leqslant 0$, $a_2+b_2-c_2 \geqslant 0$. In this subcase, (2.8) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+a_2+b_2+|c_1+c_2-a_1-a_2-b_1-b_2|\\ &\geqslant |a_1-a_2+b_1-b_2|+\big|c_1-a_1-|c_2-a_2|\big|+|c_2-a_2|+\big|c_1-b_1-|c_2-b_2|\big|+|c_2-b_2|.\end{aligned} \tag{2.11}$$

(1) If $|c_2-a_2| \leqslant c_1-a_1$ and $|c_2-b_2| \leqslant c_1-b_1$, then (2.11) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+a_1+a_2+b_1+b_2+|c_1+c_2-a_1-a_2-b_1-b_2|\\ &\geqslant 2c_1+|a_1-a_2+b_1-b_2|,\end{aligned}$$

which follows from the inequalities

$$|a_1-a_2|+|b_1-b_2| \geqslant |a_1-a_2+b_1-b_2|$$

and

$$|c_1-c_2|+a_1+a_2+b_1+b_2+|c_1+c_2-a_1-a_2-b_1-b_2| \geqslant 2c_1$$

immediately.

(2) If $|c_2-a_2| \geqslant c_1-a_1$ and $|c_2-b_2| \geqslant c_1-b_1$, then (2.11) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+2a_2+2b_2+2c_1+|c_1-c_2|+|c_1+c_2-a_1-a_2-b_1-b_2|\\ &\geqslant |a_1-a_2+b_1-b_2|+2|a_2-c_2|+2|b_2-c_2|+a_1+b_1+a_2+b_2.\end{aligned} \tag{2.12}$$

Using the inequalities $|a_1-a_2|+|b_1-b_2| \geqslant |a_1-a_2+b_1-b_2|$,

$$2(a_2+b_2)=2|a_2+b_2-c_2|+2c_2 \geqslant 2|a_2-c_2|+2|b_2-c_2|,$$

and

$$2c_1+|c_1-c_2|+|c_1+c_2-a_1-a_2-b_1-b_2| \geqslant a_1+a_2+b_1+b_2,$$

we get (2.12).

(3) Now we consider the subcase $c_1-a_1 \geqslant |a_2-c_2|$ and $c_1-b_1 \leqslant |c_2-b_2|$ or the subcase $c_1-a_1 \leqslant |a_2-c_2|$ and $c_1-b_1 \geqslant |c_2-b_2|$. We may assume that $c_1-a_1 \geqslant |a_2-c_2|$ and $c_1-b_1 \leqslant |c_2-b_2|$ without loss of generality. Then (2.11) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+a_1+a_2+b_1+b_2+|c_1+c_2-a_1-a_2-b_1-b_2|\\ &\geqslant 2|c_2-b_2|+2b_1+|a_1-a_2+b_1-b_2|.\end{aligned} \tag{2.13}$$

(3-1) If $c_2 \leqslant b_2$, then $|b_2-c_2|=b_2-c_2 \geqslant c_1-b_1$ and thence $b_1+b_2 \geqslant c_1+c_2$. Hence, (2.13) is equivalent to

$$|a_1-a_2|+|b_1-b_2|+2a_1+2a_2+|c_1-c_2| \geqslant |a_1-a_2+b_1-b_2|+c_1-c_2,$$

which follows immediately from the equalities $|a_1-a_2|+|b_1-b_2|\geqslant|a_1-a_2+b_1-b_2|$, $2a_1+2a_2\geqslant 0$, and $|c_1-c_2|\geqslant c_1-c_2$.

(3-2) If $c_2\geqslant b_2$, then $a_1\leqslant c_1-b_1\leqslant c_2-b_2\leqslant a_2$. In this case, we use in (2.13) the inequality

$$|c_1+c_2-a_1-a_2-b_1-b_2|+a_1+a_2+b_1+b_2\geqslant c_1+c_2.$$

Then in order to show (2.13), it suffices to prove that

$$|c_1-c_2|+c_1+c_2+a_2-a_1+|b_1-b_2|\geqslant 2c_2-2b_2+2b_1+|a_1-a_2+b_1-b_2|. \quad (2.14)$$

If $a_2-a_1+b_2-b_1\geqslant 0$, then (2.14) is equivalent to

$$|c_1-c_2|+|b_1-b_2|\geqslant c_2-c_1-b_2+b_1,$$

which is true.

On the other hand, if $a_2-a_1+b_2-b_1\leqslant 0$, then $c_2\leqslant a_2+b_2\leqslant a_1+b_1\leqslant c_1$. Hence, (2.14) is equivalent to

$$2c_1+2a_2+2b_2+|b_1-b_2|\geqslant 2c_2+2b_1+2a_1+b_1-b_2,$$

which can be deduced immediately from the inequalities $2c_1\geqslant 2a_1+2b_1$, $2a_2+2b_2\geqslant 2c_2$ and $|b_1-b_2|\geqslant b_1-b_2$.

This completes the proof of (2.1) in Case (C).

Finally, we consider Case (D). We replace c_1 and b_2 by $-c_1$ and $-b_2$ in this case. Then $a_1,b_1,c_1,a_2,b_2,c_2\geqslant 0$, and (2.1) is equivalent to the following inequality

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+c_1+b_2+|a_1+b_1-c_1|+|a_2-b_2+c_2|\\&+\big||a_1+b_1-c_1|-|a_2+c_2-b_2|\big| \ \geqslant\ |a_2-b_2|+|c_2-b_2|+|c_1-a_1|+|c_1-b_1|\\&+\big|a_1+b_1-|a_2-b_2|\big|+\big|a_2+c_2-|a_1-c_1|\big|+\big||b_1-c_1|-|b_2-c_2|\big|.\end{aligned} \quad (2.15)$$

We divide Case (D) into the following three subcases (a)-(c):

Subcase (a): $a_1+b_1\leqslant c_1$, $a_2+c_2\leqslant b_2$. In this subcase, we have

$$|a_1+b_1-c_1|+c_1=|c_1-a_1|+|c_1-b_1|,\ \ |a_2+c_2-b_2|+|b_2|=|a_2-b_2|+|c_2-b_2|.$$

Then (2.15) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+|c_1+c_2-b_1-b_2+a_2-a_1|\\&\geqslant|a_1+b_1+a_2-b_2|+|a_2+c_2+a_1-c_1|+|c_1+c_2-b_1-b_2|.\end{aligned} \quad (2.16)$$

Using (1.1) with $x=a_1+a_2$, $y=b_1-b_2$, $z=c_2-c_1$ and noting that

$$y+z\leqslant x+y+z=-b_2-c_1+a_1+a_2+b_1+c_2\leqslant 0,$$

we get

$$|b_1-b_2|+|c_2-c_1|\geqslant|a_1+a_2+b_1-b_2|+|a_1+a_2+c_2-c_1|,$$

which, combined with the triangle inequality

$$|c_1+c_2-b_1-b_2| \leqslant |a_1-a_2|+|c_1+c_2-b_1-b_2+a_2-a_1|,$$

gives (2.16).

Subcase (b): $a_1+b_1 \geqslant c_1$, $a_2+c_2 \geqslant b_2$. In this subcase, we consider the following subcases:

(1) If $|a_2-b_2| \leqslant a_1+b_1$ and $|a_1-c_1| \leqslant a_2+c_2$, then (2.15) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+|a_1-a_2-c_1-c_2+b_1+b_2|\\ &\geqslant |b_1-c_1|+|b_2-c_2|+\big||b_1-c_1|-|b_2-c_2|\big| = 2\max\{|b_1-c_1|, |b_2-c_2|\},\end{aligned}$$

which follows from the triangle inequality and the facts that

$$2(b_2-c_2) = (a_2-a_1)+(b_2-b_1)+(c_1-c_2)+(a_1-a_2-c_1-c_2+b_1+b_2)$$

and

$$2(b_1-c_1) = (a_2-a_1)+(b_1-b_2)+(c_2-c_1)+(a_1-a_2-c_1-c_2+b_1+b_2).$$

(2) If $|a_2-b_2| \geqslant a_1+b_1$ and $|a_1-c_1| \geqslant a_2+c_2$, then (2.15) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+2a_1+2b_1+2a_2+2c_2+|a_1-a_2+b_1+b_2-c_1-c_2|\\ &\geqslant\ 2|a_2-b_2|+2|a_1-c_1|+2\max\{|b_1-c_1|,|b_2-c_2|\}.\end{aligned} \tag{2.17}$$

It follows from (2.5) that

$$|a_1-c_1|+|b_1-c_1| \leqslant a_1+b_1, \quad |a_2-b_2|+|c_2-b_2| \leqslant a_2+c_2.$$

Hence, we have

$$\begin{aligned}&|a_2-b_2|+|a_1-c_1|+\max\{|b_1-c_1|,|b_2-c_2|\}\\ &\leqslant |a_2-b_2|+|a_1-c_1|+|b_1-c_1|+|b_2-c_2| \leqslant a_1+b_1+a_2+c_2,\end{aligned}$$

which gives (2.17).

(3) Now we consider the subcase $|a_2-b_2| \geqslant a_1+b_1, |a_1-c_1| \leqslant a_2+c_2$ or the subcase $|a_2-b_2| \leqslant a_1+b_1, |a_1-c_1| \geqslant a_2+c_2$. We may assume that $|a_2-b_2| \geqslant a_1+b_1$ and $|a_1-c_1| \leqslant a_2+c_2$ without loss of generality. Then (2.15) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+2a_1+2b_1+|a_1-a_2+b_1+b_2-c_1-c_2|\\ &\geqslant 2|a_2-b_2|+2\max\{|b_1-c_1|,|b_2-c_2|\}.\end{aligned} \tag{2.18}$$

(3-1) If $\max\{|b_1-c_1|,|b_2-c_2|\} = |b_2-c_2|$, then by (2.5) we have

$$|a_2-b_2|+\max\{|b_1-c_1|,|b_2-c_2|\} = |a_2-b_2|+|c_2-b_2| \leqslant |a_2+c_2-b_2|+b_2 = a_2+c_2.$$

So, to show (2.18) it suffices to prove that

$$2c_2+2a_2 \leqslant |a_1-a_2|+|b_1-b_2|+|c_1-c_2|+2a_1+2b_1+|a_1-a_2+b_1+b_2-c_1-c_2|,$$

which follows from the triangle inequality and

$$2c_2+2a_2=(a_2-a_1+c_2+c_1-b_1-b_2)+(a_2-a_1)+(c_2-c_1)+(b_2-b_1)+2a_1+2b_1.$$

(3-2) If $\max\{|b_1-c_1|,|b_2-c_2|\}=|b_1-c_1|$ and $a_2\geqslant b_2$, then $a_2-a_1-b_1-b_2\geqslant 0$ and $a_2-a_1+c_1+c_2\geqslant 0$. It follows from (1.1) that

$$\begin{aligned}&|a_2-a_1|+b_1+b_2+c_1+c_2+|a_2-a_1-b_1-b_2+c_1+c_2|\\ \geqslant\ &|a_2-a_1-b_1-b_2|+|a_2-a_1+c_1+c_2|+|-b_1-b_2+c_1+c_2|,\end{aligned}$$

which gives that

$$|a_2-a_1|+|a_2-a_1-b_1-b_2+c_1+c_2|\geqslant 2a_2-2a_1-2b_1-2b_2+|c_1+c_2-b_1-b_2|, \tag{2.19}$$

Hence, by (2.19) and the triangle inequality we have

$$\begin{aligned}\text{Left hand of (2.18)}\geqslant\ &2(a_2-b_2)+|c_1+c_2-b_1-b_2|+|b_1-b_2|+|c_1-c_2|\\ \geqslant\ &2|a_2-b_2|+2|b_1-c_1|,\end{aligned}$$

proving (2.18).

(3-3) If $\max\{|b_1-c_1|,|b_2-c_2|\}=|b_1-c_1|$ and $a_2\leqslant b_2$, then $b_2\geqslant a_1+a_2+b_1$, $|a_2-b_2|\leqslant c_2$, and $c_2\geqslant b_2-a_2\geqslant a_1+b_1\geqslant c_1$.

If $c_1\leqslant b_1$, then (2.18) is equivalent to

$$|a_2-a_1|+2a_1+2a_2+c_1+c_2+|a_2-a_1-b_1-b_2+c_1+c_2|\geqslant\ b_1+b_2,$$

which follows from the triangle inequality.

If $c_1\geqslant b_1$, then $|b_1-c_1|\leqslant a_1$. By the triangle inequality, we have

$$\begin{aligned}&\text{Right hand of (2.18)}\\ &\quad\leqslant 2c_2+2a_1\\ &\quad\leqslant |a_2-a_1+c_2+c_1-b_1-b_2|+|a_1-a_2|+|c_2-c_1|+|b_2-b_1|+2a_1+2b_1,\end{aligned}$$

proving (2.18).

Subcase (c): $a_1+b_1\leqslant c_1,\ a_2+c_2\geqslant b_2$ or $a_1+b_1\geqslant c_1,\ a_2+c_2\leqslant b_2$. We may assume that $a_1+b_1\leqslant c_1$ and $a_2+b_2\geqslant c_2$. In this subcase, (2.15) is equivalent to

$$\begin{aligned}&|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+a_2+c_2+|a_1+a_2+c_2-c_1+b_1-b_2|\\ \geqslant&|a_2-b_2|+|c_2-b_2|+\big|a_1+b_1-|a_2-b_2|\big|+|a_2+c_2+a_1-c_1|+\big|c_1-b_1-|b_2-c_2|\big|.\end{aligned} \tag{2.20}$$

In order to prove (2.20), we use the triangle inequality

$$|b_1-b_2|+|a_1+a_2+c_2-c_1+b_1-b_2|\geqslant |a_2+c_2+a_1-c_1|.$$

It suffices to prove that

$$\begin{aligned}&|a_1-a_2|+|c_1-c_2|+a_2+c_2\\ \geqslant&|a_2-b_2|+|c_2-b_2|+\big|a_1+b_1-|a_2-b_2|\big|+\big|c_1-b_1-|b_2-c_2|\big|.\end{aligned} \tag{2.21}$$

Now we consider the following subcases:

(1) If $|a_2-b_2| \leqslant a_1+b_1$ and $|b_2-c_2| \leqslant c_1-b_1$, then (2.21) is equivalent to

$$|a_1-a_2|+|c_1-c_2|+a_2+c_2 \geqslant a_1+c_1,$$

which is obvious.

(2) If $|a_2-b_2| \geqslant a_1+b_1$ and $|b_2-c_2| \geqslant c_1-b_1$, then (2.21) is equivalent to

$$|a_1-a_2|+|c_1-c_2|+a_2+c_2+a_1+c_1 \geqslant 2|a_2-b_2|+2|b_2-c_2|. \tag{2.22}$$

Using (2.5) we get that

$$2|a_2-b_2|+2|b_2-c_2| \leqslant 2|a_2+c_2-b_2|+2b_2 = 2a_2+2c_2.$$

Then (2.22) follows from $|a_1-a_2|+|c_1-c_2|+a_1+c_1 \geqslant a_2+c_2$ immediately.

(3-1) If $|a_2-b_2| \leqslant a_1+b_1$ and $|b_2-c_2| \geqslant c_1-b_1$, and $|b_2-c_2| \leqslant a_2$, then (2.21) is equivalent to

$$|a_1-a_2|+|c_1-c_2|+c_1+c_2+a_2+a_1 \geqslant 2a_1+2b_1+2|b_2-c_2|,$$

which follows from the inequalities

$$2c_1+2a_2 \geqslant 2a_1+2b_1+2|b_2-c_2|$$

and

$$|a_1-a_2|+|c_1-c_2|+c_1+c_2+a_2+a_1 \geqslant 2c_1+2a_2.$$

(3-2) If $|a_2-b_2| \leqslant a_1+b_1$ and $|b_2-c_2| \geqslant c_1-b_1$, and $|b_2-c_2| > a_2$, then $c_2 > a_2+b_2$ and $c_2-c_1-b_2+b_1 \geqslant 0$. Hence, (2.20) is equivalent to

$$|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+2a_2 \geqslant |a_1+a_2+c_2-c_1|+b_1-b_2,$$

which is true.

(4-1) If $|a_2-b_2| \geqslant a_1+b_1$ and $|b_2-c_2| \leqslant c_1-b_1$, and $a_2 \geqslant b_2$, then (2.21) is equivalent to

$$|a_1-a_2|+|c_1-c_2|+2b_1+2b_2 \geqslant c_1-c_2+a_2-a_1,$$

which holds.

(4-2) If $|a_2-b_2| \geqslant a_1+b_1$ and $|b_2-c_2| \leqslant c_1-b_1$, and $a_2 \leqslant b_2$, then $a_1+b_1+a_2-b_2 \leqslant 0$ and $c_1-b_1-c_2+b_2 \geqslant 0$. Hence, (2.20) is equivalent to

$$\begin{aligned}|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+a_1+3a_2+c_2+2b_1+|a_1+a_2+c_2-c_1+b_1-b_2|\\ \geqslant\ 2b_2+c_1+|a_2+c_2+a_1-c_1|.\end{aligned} \tag{2.23}$$

Using (1.1), we get

$$\begin{aligned}&|b_1-b_2|+|c_2-c_1|+a_1+a_2+|a_1+a_2+c_2-c_1+b_1-b_2|\\ &\geqslant |a_1+a_2+c_2-c_1|+(-b_1+b_2-c_2+c_1)+(b_2-a_1-a_2-b_1).\end{aligned} \tag{2.24}$$

Hence, by (2.24) we have

$$\begin{aligned}\text{Left hand of (2.23)} \geqslant\ & |a_1-a_2|+a_2-a_1+2b_2+c_1+|a_2+c_2+a_1-c_1| \\ \geqslant\ & 2b_2+c_1+|a_2+c_2+a_1-c_1|,\end{aligned}$$

proving (2.23).

The proof of Lemma 1 is complete. □

Proof of Theorem 1. It is easy to see that for $u \in \mathbb{R}_{\mathscr{F}}$,

$$\|u\|_{\mathscr{F}} = \max\{|u_-^0|,|u_+^0|\}.$$

Then for u, v, $w \in \mathbb{R}_{\mathscr{F}}$, (1.2) is equivalent to the following inequality

$$\begin{aligned}&\max\{|u_-^0|,|u_+^0|\}+\max\{|v_-^0|,|v_+^0|\}+\max\{|w_-^0|,|w_+^0|\}\\ &\quad+\max\{|u_-^0+v_-^0+w_-^0|,|u_+^0+v_+^0+w_+^0|\}\\ &\geqslant \max\{|u_-^0+v_-^0|,|u_+^0+v_+^0|\}+\max\{|u_-^0+w_-^0|,|u_+^0+w_+^0|\}\\ &\quad+\max\{|v_-^0+w_-^0|,|v_+^0+w_+^0|\},\end{aligned}$$

which follows from Lemma 1 immediately. □

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