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QUASI–MONOTONE WEIGHT FUNCTIONS AND THEIR CHARACTERISTICS AND APPLICATIONS

LARS-ERIK PERSSON, NATASHA SAMKO AND PETER WALL

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Abstract. A weight function $w(x)$ on $(0,l)$ or (l,∞), is said to be quasi-monotone if $w(x)x^{-a_0} \leqslant C_0 w(y)y^{-a_0}$ either for all $x \leqslant y$ or for all $y \leqslant x$, for some $a_0 \in \mathbb{R}$, $C_0 \geqslant 1$. In this paper we discuss, complement and unify several results concerning quasi-monotone functions. In particular, some new results concerning the close connection to index numbers and generalized Bary-Stechkin classes are proved and applied. Moreover, some new regularization results are proved and several applications are pointed out, e.g. in interpolation theory, Fourier analysis, Hardy-type inequalities, singular operators and homogenization theory.

1. Introduction

Weight functions are very important for various applications. In many cases it is impossible to handle them with general weight functions and then it turns out that the class of quasi-monotone functions is very useful and applicable in many different areas of mathematics and its applications. Let us just mention the following:

– Fourier series, see e.g. [36], [37] and the recent thesis [20].

– Function spaces and classical operators of Harmonic Analysis, see e.g. [16], [18], [19], [17], [39], [42], [45], [52] and [53].

– Interpolation theory, see e.g. [38], [40] and [41].

– Operator theory and singular equations, see e.g. [44], [46], [48], [49], [50] and [51].

– Inequalities, see e.g. [2], [3], [4], [34] and [54].

– Homogenization theory, see e.g. [9], [11], [28], [56] and [57].

As far as we know there is no textbook with a complete theory concerning quasi-monotone functions and it can only be found in pieces in various books, papers and theses, see e.g. the references in this paper and the references there.

In Section 2 of this paper we present and complement some of this knowledge about quasi-monotone functions in a unified form.

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In Section 3 we present and complement some useful results concerning characteristics for quasi-monotone functions (e.g. their numerical characteristics, index numbers) of special interest for various applications. By using these characteristics we get a suitable control of the growth of the weights.

In Section 4 we prove some new regularization results for quasi-monotone (see Theorems 25 and 26), which seem to be extremely important for the applications mentioned above and also some new ones.

In Section 5 we present some applications. For example, in our Theorem 28 we prove a discrete Hardy-type inequality with rapidly decreasing or increasing weights and such a result can not be found e.g. in the books [23], [24] and [33] on this subject. We use our results to show how some results in the theory of Fourier series (see e.g. Theorem 33) and interpolation theory (see Theorems 34 and 35) can be defined in terms of index numbers. We also include results, see Theorems 36 and 37, which show how the notions related to quasi-monotone functions are used in operator theory, in particular in the theory of singular integral operators.

2. Preliminaries concerning quasi-monotone functions

We consider a weight function $\omega(x)$ on $(0,\ell), 0<\ell<\infty$, i.e. a positive and measurable function. However, all definitions and results can be formulated and proved also for weight functions defined on (ℓ,∞).

The notion of almost monotonicity goes back to S. Bernstein, see [7], who called a non-negative function $\omega(x)$ almost increasing, if $\omega(x) \leqslant C_0\omega(y), x<y$, and almost decreasing, if $\omega(y) \leqslant C_1\omega(x), x<y$ for some $C_0, C_1 \geqslant 1$. We think that his really first work with such notion is [6]. Note that every almost increasing (decreasing) function $\omega(x)$ is equivalent to an increasing (decreasing, respectively) function:

$$\omega(x) \leqslant \omega^*(x) \leqslant C\omega(x), \quad \frac{1}{C}\omega(x) \leqslant \omega_*(x) \leqslant \omega(x),$$

where $\omega^*(x)$ and $\omega_*(x)$ are the monotone majorant or the minorant of $\omega(x)$:

$$\omega^*(x) = \sup_{0<t\leqslant x} \omega(t), \quad \omega_*(x) = \inf_{x\leqslant t<\ell} \omega(t) \quad \text{and} \quad C = C_0$$

in the case $\omega(x)$ is almost increasing and

$$\omega^*(x) = \sup_{x\leqslant t<\ell} \omega(t), \quad \omega_*(x) = \inf_{0<t\leqslant x} \omega(t) \quad \text{and} \quad C = C_1$$

in the case it is almost decreasing.

We use a notion of such a generalized monotonicity in a more general context by saying that $\omega(x)$ is quasi-increasing, if there exists $a_0 \in \mathbb{R}$ such that

$$\omega(x)x^{-a_0} \leqslant C_0\omega(y)y^{-a_0}, \quad x<y, \tag{1}$$

for some $C_0 = C_0(\omega) \geqslant 1$, and quasi-decreasing, if there exists $a_1 \in \mathbb{R}$ such that

$$\omega(y)y^{-a_1} \leqslant C_1\omega(x)x^{-a_1}, \quad x<y, \tag{2}$$

for some $C_1 = C_1(\omega) \geqslant 1$. Quasi-increasing and quasi-decreasing functions will be referred to as quasi-monotone functions. The class of such functions will be denoted by Q.

In this paper we shall also introduce and discuss some important subclasses of the class Q.

DEFINITION 1. If $-\infty < a_0 < a_1 < \infty$, then the class $Q[a_0,a_1]$ consists of all $\omega(x) \in Q$ such that (1) and (2) hold. Moreover, we say that $\omega(x) \in Q(a_0,a_1)$, if $\omega(x) \in Q[a_0+\varepsilon, a_1-\varepsilon]$ for some $\varepsilon \in \left(0, \frac{a_1-a_0}{2}\right)$. The notation $\omega(x) \in Q[a_0,-)$ means that only (1) holds. We shall also permit hybrid cases, for example $Q[a_0,b_0)$, $Q(a_0,b_0]$, $Q(-,b_0]$ etc.

In the example below and in the sequel the notion of equivalent functions: $\omega(x) \approx \varphi(x)$ means that, for all x and some positive constants d_0 and d_1,

$$d_0\varphi(x) \leqslant \omega(x) \leqslant d_1\varphi(x).$$

EXAMPLE 2. The class of functions $\omega \in Q[0,1]$ with $C_0(\omega) = C_1(\omega) = 1$ (consisting of increasing functions $\omega(x)$ such that $\frac{\omega(x)}{x}$ is decreasing) is the class of quasi-concave functions $\omega(x)$, defined as the class of all the functions equivalent to a concave function: $\omega(x) \approx \varphi(x)$, where $\varphi(x)$ is concave (see e.g. [5], p. 117).

REMARK 3. The function $\varphi(x)$ in Example 2 is constructed as the least concave majorant and it always exists, if $\varphi(0) = 0$. In fact, it seems to be Stechkin who already in 1951 (see [10], Lemma 4) introduced this function via the formula

$$\varphi(u) = \sup\left\{\frac{u_2-u}{u_2-u_1}\omega(u_1) + \frac{u-u_1}{u_2-u_1}\omega(u_2), 0 \leqslant u_1 \leqslant u \leqslant u_2\right\} =$$

$$= \sup\{\alpha\omega(u_1) + (1-\alpha)\omega(u_2) : 0 \leqslant \alpha \leqslant 1, u_1, u_2 \geqslant 0, \alpha u_1 + (1-\alpha)u_2 = u\}.$$

Moreover, in 1970 Peetre, [35] introduced this function via the formula

$$\varphi(u) = \sup\left\{\sum_{i=1}^{n}\lambda_i\omega(u_i) : u = \sum_{i=1}^{n}\lambda_i u_i,\ u_i \geqslant 0,\ \sum_{i=1}^{n}\lambda_i = 1, \lambda_i \geqslant 0, n \in \mathbb{N}\right\}$$

and in 1978 Lozanovskii, [27] introduced it as

$$\varphi(u) = \inf_{s>0}\sup_{v>0}\frac{s+u}{s+v}u(v).$$

In the next section we introduce another regularized concave majorant function, which need not necessary to be the least one (see Remark 21).

REMARK 4. Many crucial objects in the analysis area and its applications are just quasi-concave. Let us just mention the Peetre K-functional

$$K(f,x) = K(f,x;A_0,A_1),$$

which is the most central notion in real interpolation theory (here A_0 and A_1 are two quasi-Banach spaces), the integral modulus of continuity $\omega_p(f,x)$ in approximation theory, the Hardy averaging operator

$$\frac{1}{x}\int_0^x f^*(t)dt,$$

where $f^*(t)$ is the rearrangement in decreasing order of $f(t)$, in the theory of Lorentz spaces and (weak) interpolation theory and the fundamental function $u(t) = u_E(t) = \|\chi_{(0,t)}\|_E$ (here E is a symmetric space and $\chi_{(0,t)}$ is the characteristic function of the interval $(0,t)$), see e.g. [22], Theorem 4.7.

We state some useful information about quasi-monotone functions in the following Proposition.

PROPOSITION 5. *Let* $\omega(x) \in Q[a_0,a_1], -\infty < a_0 < a_1 < \infty$. *Then*
a)

$$\omega(x^\alpha) \in \begin{cases} Q[a_0\alpha, a_1\alpha], & \text{if } \alpha > 0, \\ Q[a_1\alpha, a_0\alpha], & \text{if } \alpha < 0, \end{cases}$$

$$x^\alpha(\omega(x))^\beta \in \begin{cases} Q[\alpha + a_0\beta, \alpha + a_1\beta], & \text{if } \alpha \in \mathbb{R}, \beta > 0, \\ Q[\alpha + a_1\beta, \alpha + a_0\beta], & \text{if } \alpha \in \mathbb{R}, \beta < 0. \end{cases}$$

and $\omega^{-1}(x) \in Q[a_1^{-1}, a_0^{-1}]$ *whenever the inverse* $\omega^{-1}(x)$ *exists.*
b) *there exist* $\rho(x) \in Q[0,1]$ *and a concave function* $c(x)$ *such that*

$$\omega(x) = x^{a_0}\rho(x^{a_1-a_0}) \quad \text{and} \quad \omega(x) \approx x^{a_0}c(x^{a_1-a_0}).$$

Let $\psi(x)$ *be equivalent to some* $\omega(x) \in Q[a_0,a_1]$. *Then*
c) *also* $\psi(x) \in Q[a_0,a_1]$,
d) *the following representation formula holds:*

$$\omega(x) \approx \psi(x) \approx \alpha x^{a_0} + \beta x^{a_1} + \int_0^\ell \min(sx^{a_0}, x^{a_1})d\mu(s), \tag{3}$$

where $\alpha, \beta \geqslant 0$ *and* $\mu(s)$ *is a nondecreasing function on* $(0,\infty)$ *satisfying* $\lim\limits_{s\to\ell}\mu(s) < \infty$ *and* $\lim\limits_{s\to 0_+} s\mu(s) = 0$.

Proof. The statements **a)** and **c)** follow directly from the definitions, **b)** follows by using Example 1 and **a)** above. To get **d)**, it suffices to note that the representation formula for quasi-monotone functions ρ:

$$\rho(x) \approx \alpha + \beta x + \int_0^\ell \min(s,x)d\mu(s)$$

is known, being due to Peetre [35], p. 117; hence by using **c)** we obtain (3) and the proof is complete. □

REMARK 6. Let $\omega^\star(x) := x\omega(1/x)$ be the involution function. According to Proposition 5 we find that $\omega \in Q(0,1)$ if and only if $\omega^\star \in Q(0,1)$. This is very useful information in interpolation theory (see e.g. [41] and also [40]).

3. Relations to some useful characteristics

In this Section we shall discuss some relations between quasi-monotone functions and some indices used in various ways e.g. in interpolation theory, approximation theory, the theory of Orlicz spaces, singular integral operators, potentials, hypersingular operators etc.

DEFINITION 7. The Gustavsson-Peetre class P^{+-} consists of all functions $\omega(x)$ in $Q[0,1]$ with constants $C_0(\omega) = C_1(\omega) = 1$, such that

$$\overline{\omega}(x) := \sup_{s>0} \frac{\omega(sx)}{\omega(s)} = o(\max(1,x)) \quad as \quad x \to 0+ \quad and \quad x \to \infty.$$

DEFINITION 8. The class B_ω consists of all continuously differentiable functions $\omega(x)$ such that

$$0 < p(\omega) := \inf_{x>0} \frac{x\omega'(x)}{\omega(x)} \leqslant \sup_{x>0} \frac{x\omega'(x)}{\omega(x)} =: q(\omega) < 1.$$

The numbers $p(\omega)$ and $q(\omega)$ are also known in the literature as the Simonenko indices, see [55].

The following was proved in [41]:

PROPOSITION 9. *The following holds*

(a) $B_\omega \subset Q(0,1) \subset P^{+-}$,

(b) *for* $\omega \in P^{+-}$ *there exists a function* $\omega_0 \in B_\omega$ *such that* $\omega_0(x) \approx \omega(x)$.

REMARK 10. This proposition clearly connects the basic functions used in the theory of the Gustavsson-Peetre $\pm$ interpolation method (see [13]), the method of interpolation with a parameter function (see Persson [41]) and the Simonenko method [55] for interpolation and extrapolation of linear operators in Orlicz spaces.

DEFINITION 11. Let $\omega \in Q$. Then the numbers $m(\omega)$ and $M(\omega)$ are defined as follows:

$$m(\omega) := \lim_{x\to 0} \frac{\ln\left(\limsup_{h\to 0} \frac{\omega(hx)}{\omega(h)}\right)}{\ln x}, \qquad M(\omega) := \lim_{x\to\infty} \frac{\ln\left(\limsup_{h\to 0} \frac{\omega(hx)}{\omega(h)}\right)}{\ln x}.$$

REMARK 12. The index numbers $m(\omega)$ and $M(\omega)$ were introduced in this form and in the context of quasi-monotone functions in [43], [44] as numerical characteristics of the Bary-Stechkin class (see below) and will be referred to as the lower and the upper index numbers of $\omega(x)$; in the case of Young functions $\omega(x)$ for Orlicz spaces, such type indices go back to [31].

DEFINITION 13. Let $0<\ell\leqslant\infty$. The Bary-Stechkin class Φ (see [1]) consists of all $\omega(x)\in Q$ such that the Zygmund conditions

$$\int_0^h \frac{\omega(x)}{x}dx \leqslant C\omega(h), \qquad \int_h^\ell \frac{\omega(x)}{x^2}dx \leqslant C\frac{\omega(h)}{h}, \tag{$Z_{0,\ell}$}$$

hold, where $C=C(\omega)>0$ does not depend on $h>0$.

The class of functions satisfying both the conditions in $(Z_{0,\ell})$ was studied in detail in the paper [1] of N. Bary-S. Stechkin.

The following result was proved by N. Samko in [43], [45]:

PROPOSITION 14. *Let $0<\ell<\infty$. A function $\omega\in Q$ belongs to the Bary-Stechkin class Φ if and only if $0<m(\omega)\leqslant M(\omega)<1$ and for $\omega\in\Phi$ and any $\varepsilon>0$ there exist constants $c_1=c_1(\omega,\varepsilon)$ and $c_2=c_2(\omega,\varepsilon)$ such that*

$$c_1 x^{M(\omega)+\varepsilon} \leqslant \omega(x) \leqslant c_2 x^{m(\omega)-\varepsilon}.$$

The Matuszewska-Orlicz type indices can be defined in different (equivalent) ways and here we will choose one which shows the close connection to quasi-monotone functions in some class $Q(a,b)$. Moreover, in order to be able to compare with the result in Proposition 14, we choose a variant yielding for functions on $(0,\ell), 0<\ell\leqslant\infty$.

DEFINITION 15. Let ω be a quasi-monotone function on $(0,\ell), 0<\ell\leqslant\infty$. The lower and upper Matuszewska-Orlicz indices are defined by

$$\alpha(\omega)=\sup\{p\in\mathbb{R}:\omega(\lambda u)\leqslant C\lambda^p\omega(u) \text{ for some } C>0 \text{ and all } u\in(0,\ell),\ 0<\lambda<1\},$$

$$\beta(\omega)=\inf\{p\in\mathbb{R}:\omega(\lambda u)\leqslant C\lambda^p\omega(u) \text{ for some } C>0 \text{ and all } \lambda u\in(0,\ell),\ \lambda\geqslant 1\},$$

respectively (see [14], [15], [25], [26], [29] and also the original paper [30] from 1960). Note also that the constructions used in the definition of these indices are close to those used in the so called Lozinskii conditions from 1956, see conditions (L) and (L_k) in [1].

REMARK 16. It is well-known and easy to prove that if $\omega\approx\omega_0$ then $\alpha(\omega)=\alpha(\omega_0)$ and $\beta(\omega)=\beta(\omega_0)$.

Partly guided by the result in Proposition 14 and the results in [14], see also [15], we will now state some new relations between the indices $\alpha(\omega)$ and $\beta(\omega)$ and our quasi-monotone classes. We will formulate our results in a more general situation, namely when we are dealing with so called generalized Bary-Stechkin class Φ_γ^β defined as follows:

DEFINITION 17. Let $0<\ell\leqslant\infty$. The class $\Phi_{\gamma_0}^{\gamma}, -\infty<\gamma_0<\gamma<\infty$ consists of all $\omega\in Q$ such that the conditions

$$\int_0^h \frac{\omega(x)}{x^{1+\gamma_0}}dx \leqslant C\frac{\omega(h)}{h^{\gamma_0}} \quad \text{and} \quad \int_h^\ell \frac{\omega(x)}{x^{1+\gamma}}dx \leqslant C\frac{\omega(h)}{h^{\gamma}} \tag{4}$$

hold, where $C = C(\omega)$ does not depend on $h \in (0,\ell)$ (see e.g. [47] and the references given there).

Note that a statement of the type of Proposition 14 for the generalized Bary-Stechkin class $\Phi^{\gamma}_{\gamma_0}$ may be found in Theorem 3.5 of [16]. In Theorem 3.6 of [16] there was also shown that $\omega \in \Phi^{\gamma}_{\gamma_0}$ implies the following formulas for the indices $m(\omega)$ and $M(\omega)$:

$$m(\omega) = \sup\left\{\delta > \gamma_0 : \ \frac{\omega(x)}{x^\delta} \ \text{ is almost increasing}\right\}, \tag{5}$$

$$M(\omega) = \inf\left\{\delta < \gamma) : \ \frac{\omega(x)}{x^\delta} \ \text{ is almost decreasing}\right\}. \tag{6}$$

In this connection we define the following functions:

$$\Phi_{\gamma_0}(h) := h^{\gamma_0} \int_0^h \frac{\omega(x)}{x^{1+\gamma_0}} dx \quad \text{and} \quad \Psi_\gamma(h) := h^\gamma \int_h^\ell \frac{\omega(x)}{x^{1+\gamma}} dx, \tag{7}$$

which play a crucial role in our next results.

THEOREM 18. *Let* $\gamma_0 \in \mathbb{R}$, $\omega \in Q(-,\gamma_0+1]$ *be defined on* $(0,\ell), 0 < \ell \leqslant \infty$. *Then the following conditions are equivalent:*

(a) *the left inequality in* (4) *holds for all* $h \in (0,\ell)$,

(b) $\alpha(\omega) > \gamma_0$,

(c) $\alpha(\Phi_{\gamma_0}) > \gamma_0$,

(d) $\alpha(\omega) = \alpha(\Phi_{\gamma_0}) > \gamma_0$,

(e) $\Phi_{\gamma_0} \approx \omega$,

(f) $\omega \in Q(\gamma_0, 1+\gamma_0]$.

Proof. According to the assumption $\omega \in Q(-,\gamma_0+1]$ and the definitions of $\alpha(\omega)$ and the class $Q(\gamma_0,-)$, it follows that (b) and (f) are equivalent.

Next we note that the assumption $\omega \in Q(-,1+\gamma_0]$ means that $\omega(x)x^{-\gamma_0-1} \geqslant C_0\omega(h)h^{-\gamma_0-1}$ for $0 \leqslant x \leqslant h$, and some $C_0 > 0$ so that

$$\Phi_{\gamma_0}(h) \geqslant C_0\omega(h). \tag{8}$$

Let (b) hold. Then there exists $\varepsilon > 0$ such that $\omega(x)x^{-\gamma_0-\varepsilon} \leqslant C\omega(h)h^{-\gamma_0-\varepsilon}$ for $0 < x \leqslant h$ and some $C > 0$ and, thus,

$$\Phi_{\gamma_0}(h) \leqslant h^{\gamma_0} C\omega(h)h^{-\gamma_0-\varepsilon} \int_0^h x^{\varepsilon-1} dx = \frac{C}{\varepsilon}\omega(h).$$

We conclude that $\Phi_{\gamma_0} \approx \omega$ which means that (e) holds. Moreover, when (e), or equivalently (b) holds, we have that $\alpha(\omega) = \alpha(\Phi_0)$, which means that the conditions $(b),(c),(d)$ and (e) are equivalent. Next we note that the left inequality in (4) means that $\Phi_{\gamma_0}(h) \leqslant C\omega(h)$ for $0 < h \leqslant \ell$. Therefore, according to (8), the conditions (a) and (e) are equivalent. The proof is complete. $\square$

REMARK 19. For the case $\gamma_0 = 0$ and $\ell = \infty$ Theorem 18 was also proved in [14] (see Proposition 1), but our proof here is different and simpler.

Next we state and prove a similar result for the β index corresponding to the right end point.

THEOREM 20. *Let $\gamma \in \mathbb{R}, \omega \in Q[\gamma,-)$ be defined on $(0,\ell), 0 < \ell \leqslant \infty$. Then the following conditions are equivalent:*

(a) the right inequality in (4) holds for all $h \in (0,\ell)$,

(b) $\beta(\omega) < \gamma$,

(c) $\beta(\Psi_\gamma) < \gamma$,

(d) $\beta(\omega) = \beta(\Psi_\gamma) < \gamma$,

(e) $\Psi_\gamma \approx \omega$,

(f) $\omega \in Q[\gamma, 1+\gamma)$.

Proof. In view of the assumption $\omega \in Q[\gamma,-)$ and the definition of $\beta(\omega)$, we find that (b) and (f) are equivalent. Assume that (b) holds. Then there exists $\varepsilon > 0$ so that $\omega(x)x^{-\gamma+\varepsilon} \leqslant C\omega(h)h^{-\gamma+\varepsilon}$. We first assume that $\ell = \infty$. The assumption $\omega \in Q[\gamma,-]$ implies that $\omega(x)x^{-\gamma} \leqslant C\omega(h)h^{-\gamma}$ for $t \geqslant h$. This gives that

$$\Psi_\gamma(h) = h^\gamma \int_h^\infty \frac{\omega(x)}{x^{\alpha+1}} \geqslant h^\gamma C\omega(h)h^{-\gamma} \int_h^{2h} \frac{1}{t} dt = C \ln 2\, \omega(h) = C_0\omega(h). \tag{9}$$

Let (b) hold. Then there exists $\varepsilon > 0$ so that $\omega(x)x^{-\gamma+\varepsilon} \leqslant C\omega(h)h^{-\gamma+\varepsilon}$ for $x \geqslant h$ and some $C > 0$. It follows that

$$\Psi_\gamma(h) \leqslant h^\gamma C\omega(h)h^{-\gamma+\varepsilon} \int_h^\infty x^{-1-\varepsilon} dx = \frac{C}{\varepsilon}\omega(h) = C_0\omega(h).$$

and this combined with (9) implies (e). Moreover, (e) implies that $\beta(\omega) = \beta(\Psi_\gamma)$ so we conclude that the conditions (b), (c), (d) and (e) are equivalent.

Finally, the condition (4) means that $\Psi_\gamma(h) \leqslant C\omega(h)$ for $h \geqslant \ell$ and this fact combined with (9) (which holds according to our assumption) shows that (a) and (e) are equivalent so the proof is complete for the case $\ell = \infty$.

The proof of the case $\ell < \infty$ is essentially the same. The only difference is that here we must also use the fact that the condition $Q(-,-)$ in fact implies that $\varphi(x) \approx \varphi(xt)$ for $1 \leqslant t \leqslant 2$ and also $\Psi_\gamma(x) \approx \Psi_\gamma(xt)$ for $1 \leqslant t \leqslant 2$. The proof is complete. □

REMARK 21. The functions Φ_{γ_0} and Ψ_γ defined in (7) and equipped with the generalized Bary-Stechkin conditions can obviously be regarded as some regularized majorants of the weight function ω. In the next Section we will prove some complementary or even stronger regularization results.

By now using our results above for the special case $\gamma_0 = 0$ and $\gamma = 1$ we can get a result which is similar to Proposition 14 and also rediscover and extend some results from [14].

COROLLARY 22. *Let* ω *be a quasi-concave function on* $(0,\ell), 0 < \ell \leqslant \infty$ *and let* Φ_0 *and* Ψ_1 *be defined by* (7) *with* $\gamma_0 = 0$ *and* $\gamma = 1$*, respectively. Then the following conditions are equivalent:*

(a) The Bary-Stechkin conditions (Z_ℓ) *hold,*

(b) $0 < \alpha(\omega) \leqslant \beta(\omega) < 1$,

(c) $0 < \alpha(\Phi_0) \leqslant \beta(\Psi_1) < 1$,

(d) $\Phi_0 \approx \Psi_1 \approx \omega$,

(e) $\omega \in Q(0,1)$.

REMARK 23. For the case $\ell = \infty$ this result can also be derived from the result in [14] but for the case $\ell < \infty$ the result seems to be new.

4. Some regularization results

We first state and prove the following crucial regularization result for sequences of independent interest:

LEMMA 24. *Let* $\sum_{k=0}^{\infty} a_k$ *be a convergent series with positive terms and let* $c > 1$ *be arbitrary. Then there exist a majorant sequence* $b_k, k = 0,1,2,\ldots$ *such that* $a_k \leqslant b_k, c^{-1} \leqslant \frac{b_{k+1}}{b_k} \leqslant c, k = 0,1,2,\ldots$ *and*

$$\sum_{k=0}^{\infty} b_k \leqslant \frac{c+1}{c-1} \sum_{k=0}^{\infty} a_k.$$

Proof. We choose $b_k = \sum_{n=0}^{\infty} a_n c^{-|k-n|}, k = 0,1,2,\ldots.$ Then

$$\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} \sum_{n=0}^{k} a_n c^{n-k} + \sum_{k=0}^{\infty} \sum_{n=k+1}^{\infty} a_n c^{k-n}$$

$$= \sum_{n=0}^{\infty} a_n c^n \sum_{k=n}^{\infty} c^{-k} + \sum_{n=1}^{\infty} a_n c^{-n} \sum_{k=0}^{n-1} c^k \leqslant \frac{c+1}{c-1} \sum_{n=0}^{\infty} a_n.$$

Trivially we have that $a_k \leqslant b_k$. Moreover,

$$b_{k+1} = \sum_{n=0}^{\infty} a_n c^{-|k+1-n|} = \sum_{n=0}^{k} a_n c^{n-k-1} + \sum_{n=k+1}^{\infty} a_n c^{k+1-n}$$
$$= c^{-1} \sum_{n=0}^{k} a_n c^{-(k-n)} + c \sum_{n=k+1}^{\infty} a_n c^{k-n}.$$

We conclude that $b_{k+1} \leqslant c b_k$ and $b_{k+1} \geqslant c^{-1} b_k$ and the proof is complete. □

Now we state our regularization results in two different (but equivalent) forms for the intervals $(0,\ell)$ and $(\ell,\infty), 0<\ell<\infty$, which both are useful for applications.

THEOREM 25. *Let $\delta > 0$ be arbitrary and let $\omega(x)$ be a positive, integrable and quasi-monotone function on $(0,\ell), 0<\ell<\infty$. Then there exists a majorant function $\omega_1(x)$ with the following properties:*

1) $\omega_1(x) \geqslant \omega(x), \forall x \in (0,\ell)$,

2) $\omega_1(x) \in Q[-1-\delta, -1+\delta]$ *(even with constants $C_0(\omega_1) = C_1(\omega_1) = 1$),*

3) $\int_0^\ell \omega_1(x)dx \leqslant K \int_0^\ell \omega(x)dx$, *where the constant K only depends on δ and the constant in the definition that $\omega(x)$ is quasi-monotone.*

THEOREM 26. *Let $\delta > 0$ be arbitrary and let $\omega(x)$ be a positive, integrable and quasi-monotone function on $(\ell,\infty), 0<\ell<\infty$. Then there exists a majorant function $\omega_1(x)$ with the following properties:*

1′) $\omega_1(x) \geqslant \omega(x)$ *for all* $x \in (\ell,\infty)$,

2′) $\omega_1(x) \in Q[-1-\delta, -1+\delta]$ *(even with constants $C_0(\omega_1) = C_1(\omega_1) = 1$),*

3′) $\int_\ell^\infty \omega_1(x)dx \leqslant K \int_\ell^\infty \omega(x)dx$, *where the constant K only depends on δ and the constant equipped with the assumption that $\omega(x)$ is quasi-monotone.*

Proof. We first assume that $\ell = 1$. Let $\omega(x)$ be quasi-increasing i.e. $\omega(x)x^b \leqslant C\omega(y)y^b, x \leqslant y$, and we may without loss of generality assume that $b > 0$. Then, for $x \in [2^k, 2^{k+1}], k = 0,1,2,\ldots$, we have that

$$\frac{1}{C} 2^{-b} \omega(2^k) \leqslant \omega(x) \leqslant C 2^b \omega(2^{k+1}). \tag{10}$$

Therefore, in particular,

$$\sum_{k=0}^{\infty} \omega(2^k) 2^k \leqslant C 2^b \sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} \omega(x)dx = C 2^b \int_1^\infty \omega(x)dx < \infty. \tag{11}$$

Next we apply Lemma 24 with $C = 2^\delta$ to find real numbers $d_k, k = 1,2,\ldots$ such that $d_k \geqslant \omega(2^k)$, $\sum_{k=0}^{\infty} d_k 2^k < \infty$,

$$2^{-(1+\delta)} \leqslant d_{k+1}/d_k \leqslant 2^{-1+\delta}, \tag{12}$$

and

$$\sum_{k=0}^{\infty} d_k 2^k \leqslant \frac{2^\delta + 1}{2^\delta - 1} \sum_{k=0}^{\infty} \omega(2^k) 2^k. \tag{13}$$

The majorant function $\omega_1(x)$ we are seeking for, is now constructed as follows: For

$$x = 2^{k+u}, k = 1, 2, ..., 0 \leqslant u \leqslant 1, \quad \omega_1(x) = C2^b (d_k)^{1-u} (d_{k+1})^u.$$

We observe that for $0 \leqslant u_1 \leqslant u_2 \leqslant 1$ and $k = 0, 1, 2, ...,$

$$2^{-(\delta+1)(u_2-u_1)} \leqslant \frac{\omega_1(2^{k+u_2})}{\omega_1(2^{k+u_1})} = \left(\frac{d_{k+1}}{d_k}\right)^{u_2-u_1} \leqslant 2^{(\delta-1)(u_2-u_1)}, \tag{14}$$

and, for $k_2 > k_1$,

$$2^{-(\delta+1)(k_2-k_1)} \leqslant \frac{\omega_1(2^{k_2})}{\omega_1(2^{k_1})} = \left(\frac{d_{k_2}}{d_{k_1}}\right) \leqslant 2^{(\delta-1)(k_2-k_1)}. \tag{15}$$

According to (14) and (15) we conclude that $\omega_1(x) x^{1+\delta}$ is increasing and $\omega_1(x) x^{1-\delta}$ is decreasing i.e. $\omega_1(x) \in Q[-1, 1+\delta]$ (with constants $C_0 = C_1 = 1$), i.e., $2')$ holds.

We may without loss of generality assume that $\delta < 1$. Then, by (10), (12) and the fact that $d_{k+1} \geqslant \omega(2^{k+1})$, we get that, for $2^k \leqslant x \leqslant 2^{k+1}$, $k = 0, 1, 2, ...,$

$$\begin{aligned} \omega_1(x) = \omega_1(2^{k+u}) &= C2^b (d_k)^{1-u} (d_{k+1})^u \geqslant C2^b 2^{(1-\delta)(1-u)} d_{k+1}) \\ &\geqslant C2^b d_{k+1} \geqslant C2^b \omega(2^{k+1}) \geqslant \omega(x), \end{aligned}$$

which means that also $1')$ holds.

Finally, according to (11), (13) and (14), we obtain that

$$\begin{aligned} \int_1^{\infty} \omega_1(x) dx = \sum_{k=0}^{\infty} \int_{2^k}^{2^{k+1}} \omega_1(x) dx &\leqslant 2^{\delta-1} \sum_{k=0}^{\infty} \omega_1(2^k) 2^k \leqslant C2^{b+\delta-1} \sum_{k=0}^{\infty} d_k 2^k \\ &\leqslant C2^{b+\delta-1} \frac{2^\delta+1}{2^\delta-1} \sum_{k=0}^{\infty} \omega(2^k) 2^k \leqslant C^2 2^{2b+\delta-1} \frac{2^\delta+1}{2^\delta-1} \int_1^{\infty} \omega(x) dx, \end{aligned}$$

i.e. $3')$ holds with constant $K = C^2 2^{2b+\delta-1} \frac{2^\delta+1}{2^\delta-1}$. The case when $\omega(x)$ is quasi-decreasing can be proved completely analogously so the proof is complete for the case $\ell = 1$.

For the case $\ell \neq 1$ we define $\omega_1^*(x) = \omega_1(\ell x)$ and $\omega^*(x) = \omega(\ell x)$ where ω_1 and ω are functions on $(0, 1)$ as considered above. We just note that $1')$, $2')$ and $3')$ hold with ω_1 and ω replaced by ω_1^* and ω^*, respectively. Hence, by just clearing the notation the proof is complete also in this case. □

Proof of Theorem 25. Again we first assume that $\ell = 1$ and introduce now the auxiliary weights $\omega_1^*(x) = \omega_1(1/x) \cdot x^{-2}$ and $\omega^*(x) = \omega(1/x) \cdot x^{-2}$, where ω_1 and ω

are the functions in Theorem 26 with the properties $1') - 3')$ (with $\ell = 1$.) By making some straightforward calculations we find that ω^* and ω_1^* satisfy $1) - 3)$. Hence, according to Theorem 26 we have proved that Theorem 25 holds with $\ell = 1$. Finally, by making a similar dilation of the weights as in the end of the proof of Theorem 26 we obtain the proof also for the case $\ell \neq 1$. The proof is complete. □

REMARK 27. The proof above shows that indeed Theorems 25 and 26 are in a sense equivalent. Moreover, by using both Theorems simultaneously with $\ell = 1$ we can formulate the similar Theorem also for the case when the weight $\omega(x)$ is quasi-monotone on the whole interval $(0,\infty)$.

5. Applications

The first application is a new result in the theory of Hardy-type inequalities.

THEOREM 28. *Let $\{c_k\}_{k=1}^{\infty}$ be a non-negative sequence, $p \in \mathbb{R}$ and let $\lambda(t)$ be a quasi-monotone function defined on $[0,\infty)$.*

(a) If $\lambda \in Q(-,0)$ with $C = 1$, then

$$\sum_{n=1}^{\infty} \lambda(2^n) \left(\sum_{k=1}^{n} c_k \right)^p \leqslant K_0 \sum_{n=1}^{\infty} \lambda(2^n) c_n^p \tag{16}$$

where K_0 does not depend on the sequence $\{c_k\}_{k=1}^{\infty}$.

(b) If $\lambda \in Q(0,-)$ with $C = 1$, then

$$\sum_{n=1}^{\infty} \lambda(2^n) \left(\sum_{k=n}^{\infty} \right)^p \leqslant K_1 \sum_{n=1}^{\infty} \lambda(2^n) c_n^p \tag{17}$$

where K_1 does not depend on the sequence $\{c_k\}_{k=1}^{\infty}$.

(for the case $p < 0$ we assume that $c_k > 0, k = 1,2,\ldots$)

REMARK 29. Note that Hardy type inequalities usually hold only for $p > 1$ and in the reversed direction for $0 < p < 1$ but with these weights both inequalities (a) and (b) indeed hold for all $p \in \mathbb{R}$ and in the same direction. This fact can not be seen in the standard literature, see e.g. the books [23] and [24] and the references there. By using these discrete inequalities we can derive some corresponding Hardy type inequalities also in the continuous case yielding e.g. for all $p > 0$ in classes of quasi-monotone functions and weights (for example Lemma 2.5 in [38], p. 296, is a special case of what can be proved in this way).

Proof. (a) Let $p > 0$. The condition $\lambda \in Q(-,0)$ implies that, for $n = 1,2,\ldots$,

$$\lambda(2^{n+1}) 2^{(n+1)\varepsilon} \leqslant \lambda(2^n) 2^{n\varepsilon} \tag{18}$$

for small $\varepsilon > 0$. Now we use the regularization Lemma 24 with $a_n = \lambda(2^n)c_n^p$.

We choose $c = 2^\delta, 0 < \delta < \varepsilon < \infty$ and note that, by Lemma 24, there exists a majorant sequence $\{c_n^*\}_1^\infty$ such that

$$\sum_{n=1}^{\infty}(c_n^*)^p\lambda(2^n) \leqslant \frac{2^\delta+1}{2^\delta-1}\sum_{n=1}^{\infty}(c_n)^p\lambda(2^n) \tag{19}$$

and

$$\left(\frac{\lambda(2^n)}{\lambda(2^{n+1})}\right)^{1/p}2^{-\delta/p} \leqslant \frac{c_{n+1}^*}{c_n^*} \leqslant \left(\frac{\lambda(2^n)}{\lambda(2^{n+1})}\right)^{1/p}2^{\delta/p}. \tag{20}$$

By using (18) and (20) we find that

$$\frac{c_{n+1}^*}{c_n^*} \geqslant 2^{\frac{\varepsilon-\delta}{p}} > 1.$$

Hence, in particular,

$$\sum_{k=1}^{n} c_k^* \leqslant Cc_n^*, n = 1,2,..., \tag{21}$$

where C is independent of n.

Finally, by using (19) and (21), we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda(2^n)\left(\sum_{k=1}^{n}c_k\right)^p &\leqslant \sum_{n=1}^{\infty}\lambda(2^n)\left(\sum_{k=1}^{n}c_k^*\right)^p \\ &\leqslant C^p\sum_{n=1}^{\infty}\lambda(2^n)(c_n^*)^p \leqslant \frac{2^\delta+1}{2^\delta-1}C^p\sum_{n=1}^{\infty}\lambda(2^n)c_n^p\end{aligned}$$

so (16) holds with

$$K_0 = \frac{2^\delta+1}{2^\delta-1}C^p.$$

For the case $p < 0$ it trivially holds with constant 1, so the proof is complete. The proof of (b) is completely similar, so we omit the details. □

REMARK 30. For $p > 0$ obviously (16) holds in the reversed direction with $K_0 = 1$. As noted before for $p < 0$ indeed (16) holds even with $K_0 = 1$. Moreover, from the proof above it is obvious that in this case (16) holds in the reversed direction for some $K_0 > 0$. Hence, in fact, if $\lambda \in Q(-,0)$ it even yields that

$$\sum_{n=1}^{\infty}\lambda(2^n)\left(\sum_{k=1}^{n}c_k\right)^p \approx \sum_{n=1}^{\infty}\lambda(2^n)c_n^p$$

for all $p \in \mathbb{R}$. Similarly, in the case (b) we have the following more precise statement that for every $\lambda \in Q(0,-)$:

$$\sum_{n=1}^{\infty}\lambda(2^n)\left(\sum_{k=n}^{\infty}c_k\right)^p \approx \sum_{n=1}^{\infty}\lambda(2^n)c_n^p$$

for all $p \in \mathbb{R}$. (In these two estimates we assume that $c_k > 0$ when $p < 0$)

For the special case when $\lambda(x)$ is a power function, the inequalities (16) and (17), the information above implies the following two sided Hardy-type inequalities:

COROLLARY 31. *Let $p \in \mathbb{R}$ and let $\varepsilon > 0$ be arbitrary and let $\{c_n\}_1^\infty$ be a positive sequence. Then*

$$\text{(a)} \quad \sum_{n=1}^{\infty} 2^{-n\varepsilon} \left(\sum_{k=1}^{n} c_k \right)^p \approx \sum_{n=1}^{\infty} 2^{-n\varepsilon} c_n^p$$

and

$$\text{(b)} \quad \sum_{n=1}^{\infty} 2^{n\varepsilon} \left(\sum_{k=n}^{\infty} c_k \right)^p \approx \sum_{n=1}^{\infty} 2^{n\varepsilon} c_n^p,$$

where the equivalence constants only depend on ε and p and not on the sequence $\{c_n\}_1^\infty$ (for the case $p < 0$ we assume that $c_n > 0, n = 1, 2, \ldots$)

REMARK 32. The close connection between quasi-monotone functions and indices we have pointed out in this paper can be used to formulate many results in the literature in terms of their characteristic index numbers. We just give two such examples from the theory of interpolation and theory of Fourier series.

We say that the orthonormal system $\Phi = \{\varphi_k(x)\}_{k=1}^\infty$ is regular if there exists a constant B_0 such that

1) for every segment e from $[0,1]$ and $k \in \mathbb{N}$ it yields that

$$\left| \int_e \varphi_k(x) dx \right| \leqslant B_0 \min(|e|, 1/k),$$

2) for every segment ω from N and $t \in (0,1]$ we have that

$$\left(\sum_{k \in \omega} \varphi_k(\cdot) \right)^* (t) \leqslant B_0 \min(|\omega|, 1/t),$$

where $\left(\sum_{k\in\omega} \varphi_k(\cdot)\right)^*(t)$ as usual denotes the non-increasing rearrangement of function $\sum_{k\in\omega} \varphi_k(x)$.

The Fourier coefficients of the periodic function f with period 1 with respect to the system Φ are defined by

$$a_n = a_n(f) = \int_0^1 f(x) \bar{\varphi}_n(x) dx, n \in \mathbb{Z}_+.$$

Almost all results concerning Fourier series in the recent thesis [20] can, according to the results in this paper, equivalently, be formulated in terms of index numbers. For example, Theorem 2.1 in the paper [21] (see also [20]) can be formulated in the following new way:

THEOREM 33. *Let* $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ *be an orthonormal regular system and let* $1 \leqslant p \leqslant \infty$. *If* $\omega(t)$ *is quasi-monotone and* $0 < m(\omega) \leqslant M(\omega) < 1$, *then*

$$\left(\sum_{n=1}^{\infty}(\bar{a}_n)\omega(n))^p\frac{1}{n}\right)^{1/p} \leqslant c\left(\int_0^1\left(f^*(t)t\omega\left(\frac{1}{t}\right)\right)^p\frac{dt}{t}\right)^{1/p},$$

where $\bar{a}_n = sup_{r\geqslant n}\frac{1}{r}|\sum_{m=1}^{r}a_m(f)|$, *and* $a_n(f)$ *are the Fourier coefficients with respect to the system* Φ. *Here, as usual* $f^*(t)$ *is the non-increasing rearrangement of* $f(t), 0 \leqslant t \leqslant 1$.

Concerning classical real interpolation theory in general we refer to the book of Bergh and Löfström [5] and specially concerning real interpolation with a parameter function to the paper [41] by Persson.

The definition of a parameter function is more or less equivalent with our definition of quasi-monotone function in the class $Q(0,1)$ in this paper. Hence, by using the results in this paper also all results in the paper [41] can be formulated in terms of indices. For example, we have:

THEOREM 34. (The equivalence theorem) *If* ω *is quasi-monotone and* $0 < m(\omega) \leqslant M(\omega) < 1$, *then*

$$(A_0,A_1)_{\omega,q;K} = (A_0,A_1)_{\omega,q;J}.$$

Here and in the sequel (A_0,A_1) is a compatible couple of Banach spaces and $1 \leqslant q < \infty$. Moreover, K and J denote the Peetre K and J methods.

Hence, in the sequel we need not to distinguish between those two situations and just write $(A_0,A_1)_{\omega,q}$. According to our results in this paper we can now formulate a main result (see [41] and the references given there) in terms of index numbers.

THEOREM 35. *Let* (A_0,A_1) *and* (B_0,B_1) *be comparable Banach couples and let* ω *be a quasi-monotone function such that* $0 < m(\omega) \leqslant M(\omega) < 1$.
a) (The interpolation property) If T *is a bounded sublinear operator from* A_i *to* $B_i, i = 0,1$, *with norms* M_0 *and* M_1, *respectively, then* T *is a bounded operator from* $(A_0,A_1)_{\omega,q}$ *to* $(B_0,B_1)_{\omega,q}$ *with a bound* $M \leqslant M_0\bar{\rho}(M_1/M_0)$, *where* $\bar{\rho}(s) = \sup_{t>0}(\rho(st)/\rho(t))$.
b) (The power property) If $0 < p < \infty$, *then* $(A_0^p,A_1^p)_{\omega,q}^{1/p} = (A_0,A_1)_{\omega_1,qp}$, *where* $\omega_1(t) = (\omega(t^p))^{1/p}$.
c) (The duality property) If $A_0 \bigcap A_1$ *is dense in both* A_0 *and* A_1, *then* $(A_0,A_1)'_{\omega,q} = (A'_0,A'_1)_{\omega,q}$, *where* $\omega_1(t) = 1/\omega(1/t)$ *and* $\frac{1}{q}+\frac{1}{q'} = 1$.

In the next theorem we will consider the Riemann-Liouville fractional integration operator

$$I_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \quad x > a, \tag{22}$$

in weighted generalized Hölder spaces $H_0^\omega(\Omega,\rho)$ (see (23) below) of functions with a given dominant of their modulus of continuity $\omega(t)$, where $\rho(x)$ is a quasi-monotone weight.

Let $\Omega=[a,b], -\infty<a<b<\infty$. The generalized Hölder space $H^\omega(\Omega)$ is introduced as

$$H^\omega(\Omega)=\{f(x):\ \omega(f,h)\leqslant c\omega(h),\ 0<h<\ell=b-a\},$$

where $\omega(f,h)=\max_{\substack{x\in\Omega,y\in\Omega\\ |x-y|\leqslant h}}|f(x)-f(y)|$. The function $\omega(h)$, referred to in the sequel as *the characteristic function of the space, or characteristics*, will be supposed to belong to the Bary-Stechkin class Φ.

Let $\rho(x)=\psi(x-a)$ be a weight function on $[a,b]$, which is finite and positive for $x\in(a,b]$ and may vanish or be infinite at $x=a$. We define the weighted space $H_0^\omega(\Omega,\rho)$ as

$$H_0^\omega(\Omega,\rho)\ =\ \left\{f(x):\rho(x)f(x)\in H^\omega(\Omega)\ ,\ \lim_{x\to a}[\rho(x)f(x)]=0\right\}. \tag{23}$$

Equipped with the norm

$$\|f\|_{H_0^\omega(\Omega,\rho)}=\|\rho f\|_{H_0^\omega(\Omega)}=\|\rho f\|_{C(\Omega)}+\sup_{h>0}\frac{\omega(\rho f,h)}{\omega(h)},$$

this is a Banach space.

A weight function ψ is said to belong to the class W_μ, $\mu\in\mathbb{R}_+$, if $\frac{\psi(x)}{x^\mu}$ is almost decreasing and $\psi(x)$ satisfies the condition

$$\left|\frac{\psi(x)-\psi(y)}{x-y}\right|\leqslant c\frac{\psi(x^*)}{x^*},\qquad x^*=\max(x,y),\quad c>0. \tag{24}$$

Observe that condition (24) is satisfied automatically, if $\frac{\psi(x)}{x^\mu}$ is decreasing (instead of being almost decreasing). Note also that $W_\mu\subset Q(-,\mu]$, see the notation of Definition 1. The following theorem was proved in [16].

Let also

$$W:=\{\varphi\in C([0,\ell]):\varphi(0)=0,\ \varphi(x)>0\ \text{ for }\ x>0,\ \ \varphi(x)\ \text{ is almost increasing}\}.$$

THEOREM 36. *Let $\rho=\psi(x-a)$, where $\psi\in W_\mu,\mu>0$. The Riemann-Liouville fractional integration operator I_{a+}^α with $0<\alpha<1$, maps boundedly the space $H_0^\omega(\Omega,\rho)$ with $\omega\in W$ onto the space $H_0^{\omega_\alpha}(\Omega,\rho)$ with $\omega_\alpha(h)=h^\alpha\omega(h)$:*

$$I_{a+}^\alpha[H_0^\omega(\Omega,\rho)]=H_0^{\omega_\alpha}(\Omega,\rho), \tag{25}$$

if

$$0<m(\omega)\leqslant M(\omega)<1-\alpha\quad \textit{and}\quad 0<\mu<1+m(\omega). \tag{26}$$

Another application we mention here is related to the solvability theory of singular integral equations

$$a(t)u(t)+b(t)(Su)(t)=f(t), \qquad (Su)(t)=\frac{1}{\pi i}\int_{\Gamma}\frac{u(\tau)}{\tau-t}d\tau, \quad t\in\Gamma, \tag{27}$$

in weighted spaces. This theory, which has a wide area of applications, depends much on the assumptions on the coefficients $a(t)$ and $b(t)$ and the curve Γ. It has a long history, we refer to the books [8], [12] and [32]. One of the main points in that theory is to reveal the interrelation between the behavior of the weight at the points of discontinuity of the coefficients and the parameters of the space which ensure the normal solvability of the equation (27). We choose a recent result on the normal solvability of the equation (27) in weighted Morrey spaces.

The Morrey spaces $\mathscr{L}^{p,\lambda}(\Gamma)$ on Γ are defined via the norm

$$\|f\|_{p,\lambda}:=\|f\|_{\mathscr{L}^{p,\lambda}(\Gamma)}=\sup_{t\in\Gamma,r>0}\left\{\frac{1}{r^{\lambda}}\int\limits_{\Gamma(t,r)}|f(\tau)|^{p}\,d\mu(\tau)\right\}^{\frac{1}{p}}, \tag{28}$$

where $\Gamma(t,r)=\{\tau\in\Gamma:|\tau-t|<r\}$, $1\leqslant p<\infty$ and $0\leqslant\lambda\leqslant 1$. For a non-negative weight function $\rho(t)$, the weighted space Morrey space is defined as

$$\mathscr{L}^{p,\lambda}(\Gamma,\rho)=\{f:\rho f\in\mathscr{L}^{p,\lambda}(\Gamma)\} \tag{29}$$

with $\|f\|_{\mathscr{L}^{p,\lambda}(\Gamma,\rho)}:=\|\rho f\|_{\mathscr{L}^{p,\lambda}(\Gamma)}$.

We take $\Gamma=[0,1]$ for simplicity and also because of various applications of such equations along an interval, and

$$\rho(x)=\varphi_0(x)\varphi_1(1-x), \tag{30}$$

where φ_0 and φ_1 are quasi-monotone weights. We assume that $a(x)$ and $b(x)$ are real-valued functions and consider the equation in the form

$$a(x)u(x)+\frac{b(x)}{\pi}\int\limits_{0}^{1}\frac{u(t)\,dt}{t-x}=f(x), \quad x\in(0,1). \tag{31}$$

Let

$$\theta(x):=\arg\frac{a(x)-ib(x)}{a(x)+ib(x)} \tag{32}$$

with the choice $\theta(0)\in[0,2\pi)$. The following theorem was proved in [51].

THEOREM 37. *Let $a,b\in C([0,1])$ and $\rho(t)$ be weight (30). The singular integral equation (31) is Fredholm in the weighted Morrey space $\mathscr{L}^{p,\lambda}([0,1],\rho)$, $1<p<\infty, 0\leqslant\lambda<1$, if*

$$\inf_{0<x<1}(|a(x)|+|b(x)|)\neq 0,$$

and

$$\frac{\theta(0)}{2\pi}-\frac{1-\lambda}{p}\notin[m(\varphi_0),M(\varphi_0)]+\mathbb{Z},\quad \frac{\theta(1)}{2\pi}+\frac{1-\lambda}{p}\notin[-M(\varphi_1),-m(\varphi_1)]+\mathbb{Z}. \tag{33}$$

Under these conditions, the Fredholm index of the equation (31) in the space $\mathscr{L}^{p,\lambda}([0,1],\rho)$ is given by the formula

$$\varkappa=\frac{1}{2\pi}\Delta\theta(x)\Big|_{[0,1]}+\frac{\theta(1)}{2\pi}-\frac{\theta(0)}{2\pi}, \tag{34}$$

where $\Delta\theta(x)|_{[0,1]}$ is the increment of $\theta(x)$ along $[0,1]$ and the values of $\theta(0)$ and $\theta(1)$ are chosen by the rules

$$M(\varphi_0)-1<\frac{\theta(0)}{2\pi}-\frac{1-\lambda}{p}<m(\varphi_0),\quad m(\varphi_1)<\frac{\theta(1)}{2\pi}+\frac{1-\lambda}{p}<1-M(\varphi_1). \tag{35}$$

If $\varkappa\geqslant 0$, the number of linearly independent solutions in $\mathscr{L}^{p,\lambda}([0,1],\rho)$ of the homogeneous singular integral equation is equal to $\varkappa$, and the non-homogeneous equation is unconditionally solvable. If $\varkappa<0$, then for the non-homogeneous equation there exist $|\varkappa|$ linearly independent solvability conditions.

REMARK 38. Formula for the Fredholm index $\varkappa$ may be recalculated in terms more convenient for applications:

$$\varkappa=\left[\xi_0-\frac{\theta(0)}{2\pi}\right]+\left[\xi_1+\frac{\theta(1)}{2\pi}\right] \tag{36}$$

where the brackets denote the integer part of a number, $\theta(0)$ and $\theta(1)$ are the end-point values of an arbitrary, but the same branch of $\theta(x)=\arg\frac{a(x)-ib(x)}{a(x)+ib(x)}$, while ξ_0 and ξ_1 are arbitrarily chosen points of the intervals

$$\left[\frac{1-\lambda}{p}+m(\varphi_0),\frac{1-\lambda}{p}+M(\varphi_0)\right],\quad \left[\frac{1-\lambda}{p}+m(\varphi_1),\frac{1-\lambda}{p}+m(\varphi_1)\right], \tag{37}$$

respectively. The right-hand side of formula (36) does not depend on the choice of auxiliary parameters ξ_0 and ξ_1, because by conditions (33) the numbers $\frac{\theta(0)}{2\pi}$ and $-\frac{\theta(1)}{2\pi}$ are not allowed to take values from the intervals where ξ_0 and ξ_1 are chosen. In the case where the weight functions have coinciding index numbers: $m(\varphi_0)=M(\varphi_0)$ and $m(\varphi_1)=M(\varphi_1)$, the "prohibited" intervals (37) degenerate to the points $\frac{1-\lambda}{p}+m(\varphi_0)$ and $\frac{1-\lambda}{p}+m(\varphi_1)$, and the formula for the Fredholm index $\varkappa$ takes the form

$$\varkappa=\left[\frac{1-\lambda}{p}+m(\varphi_0)-\frac{\theta(0)}{2\pi}\right]+\left[\frac{1-\lambda}{p}+m(\varphi_1)+\frac{\theta(1)}{2\pi}\right]. \tag{38}$$

REMARK 39. Many results in classical homogenization theory are studied with weights involved (e.g. to be able to handle singularities in the underlying differential equations), see [9], [11], [28], [56] and [57]. Quasi-monotone functions can be a very suitable class of weight functions to get control of the growth properties. We aim to investigate this question in a forthcoming paper.

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Lars-Erik Persson
Luleå University of Technology
Sweden
and
Narvik University College
Norway
e-mail: Lars-Erik.Persson@ltu.se

Natasha Samko
Research center CEAF
Instituto Superior Técnico
Portugal
e-mail: nsamko@gmail.com

Peter Wall
Luleå University of Technology
Sweden
e-mail: Peter.Wall@ltu.se