Mathematical
Inequalities
& Applications
Volume 15, Number 3 (2012), 707–716

doi:10.7153/mia-15-62

NEW CONVOLUTIONS AND NORM INEQUALITIES

L. P. CASTRO AND S. SAITOH

(*Communicated by S. Varošanec*)

Abstract. We introduce new three types of convolutions for which – together with the classical convolution – we obtain new convolution inequalities. This is done within a framework from the theory of reproducing kernels which helps us to perform the mentioned inequalities in a very global way.

1. Introduction

In Fourier analysis and operator theory, the convolution and convolution type operators have been studied for a long time due to their fundamental role in modelling and solving a wide range of mathematical physics problems. A huge list of examples could be indicated in this line. Additionally, generalized convolutions for integral transforms were firstly considered R.V. Churchill [11] (cf. also [4]), and methods for generalized convolutions of arbitrary integral transforms appeared already in [18]. After this a huge variety of extensions has appeared in different contexts. In a general perspective, we should refer to the systematic study of so-called smooth Fourier integral operators initiated in the classical paper of L. Hörmander [17]. In particular, G. I. Èskin [15] and L. Hörmander [17] showed the local L_2 boundedness of Fourier integral operators with non-degenerate phase functions. After this several extensions followed for Fourier integral operators, and in connection with the Hörmander's local L_2 result the works of R. Beals [1] and A. Greenleaf and G. Uhlmann [16] are relevant. Specific classes of those have been extensively studied during the last decades. This was the case of the so-called Wiener-Hopf (or Toeplitz) and Hankel integral operators which – in an isolated way or as algebraic combinations of both types (cf., e.g., [2, 3, 6, 8, 9, 10]) – can also be recognized in several applications. All this contains formulations which depend on convolutions. Products of shift operators and convolution type operators [5, 7] are

Mathematics subject classification (2010): Primary 44A35; Secondary 30C40, 42A85, 45E10, 46E22, 47A30, 47B32, 47B38, 47G10.

Keywords and phrases: Integral transform, Hilbert space, linear transform, reproducing kernel, linear mapping, convolution, Toeplitz kernel, Hankel kernel, norm inequality.

This work was supported in part by *FEDER* funds through *COMPETE*–Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. S. Saitoh is supported in part by the Grant-in-Aid for Scientific Research (C)(2)(No. 215401111) from the Japan Society for the Promotion Science.

Paper MIA-15-62

also object of past and recent intensive studies – once again in view of their need in the applications.

Inspired by the Wiener-Hopf plus Hankel integral type operators and shift actions over them, in this paper we consider convolution type integral equations which combine the classical convolution with three additional new types of convolution operations. Taking profit of the reproducing kernels theory, our main goal in here is to derive the consequent convolution inequalities for the new types of convolutions.

It is worthwhile mention that even for the elementary L_2 functions case, we did not know their convolution properties for a long time and their precise meanings were given by the theory of reproducing kernels; see the series of papers [22, 23, 24, 26]. However, when we consider the situation carefully, we realize that we have further three natural types of convolutions in view of Fourier analysis purposes. Namely, in the present paper, we will present the related natural function spaces for the new convolution types holding the related convolution norm inequalities.

In order to state our main theorem, we shall first introduce the relevant function spaces $\mathscr{F}(\rho)$ which are dependent on non-negative and integrable functions ρ on $\mathbb{R}$. We will say that $F \in \mathscr{F}(\rho)$ if and only if

$$\int \frac{|F(t)|^2}{\rho(t)} dt < \infty \qquad \text{on the support of } \rho \,,$$

and $F = 0$ on the outside of the support of ρ.

We will consider the usual convolution in the just presented spaces,

$$((F_1) *_1 (F_2))(t) = \int_{\mathbb{R}} F_1(\xi) F_2(t-\xi) d\xi \,,$$

and will additionally introduce the following three types:

$$((F_1) *_2 (F_2))(t) = \int_{\mathbb{R}} F_1(\xi) \overline{F_2(\xi - t)} d\xi,$$
$$((F_1) *_3 (F_2))(t) = \int_{\mathbb{R}} \overline{F_1(\xi)} F_2(\xi + t) d\xi,$$
$$((F_1) *_4 (F_2))(t) = \int_{\mathbb{R}} \overline{F_1(\xi) F_2(-\xi - t)} d\xi.$$

We have already all the sufficient notation to state the main result of the present paper.

THEOREM 1.1. *Let ρ_1 and ρ_2 be non-negative and integrable functions on $\mathbb{R}$ which allow us to consider the spaces $\mathscr{F}(\rho_1)$ and $\mathscr{F}(\rho_2)$, respectively. The generalized convolution inequality*

$$\int_{\mathbb{R}} \frac{|((F_1) *_1 (F_2) + (F_1) *_2 (F_2) + (F_1) *_3 (F_2) + (F_1) *_4 (F_2))(t)|^2}{(\rho_1 *_1 \rho_2)(t) + (\rho_1 *_2 \rho_2)(t) + (\rho_1 *_3 \rho_2)(t) + (\rho_1 *_4 \rho_2)(t)} dt$$
$$\leqslant 4 \int_{\mathbb{R}} \frac{|F_1(t)|^2}{\rho_1(t)} dt \cdot \int_{\mathbb{R}} \frac{|F_2(t)|^2}{\rho_2(t)} dt$$

holds true, for functions $F_j \in \mathscr{F}(\rho_j)$, $j = 1, 2$.

In some special cases, for these convolution inequalities, we found some important and fundamental applications to the related integral equations containing corresponding types of convolutions as integral kernels. Moreover, in some cases, notice that the above convolution type operators are non-linear. However, even for such cases, we can sometimes "modify" them to linear operators, and with some additional reasoning, we are therefore able to derive their boundedness from the above convolution type inequalities.

2. Reproducing kernel Hilbert spaces machinery

Following [25, 29], we shall introduce a general theory for linear mappings in the framework of Hilbert spaces.

Let $\mathscr{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathscr{H}$-valued function on E. Then we shall consider the linear transform

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathscr{H}}, \quad \mathbf{f} \in \mathscr{H}, \tag{2.1}$$

from $\mathscr{H}$ into the linear space $\mathscr{F}(E)$ comprising all the complex valued functions on E. In order to investigate the linear mapping (2.1), we form a positive definite quadratic form function $K(p,q)$ on $E \times E$ defined by

$$K(p,q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathscr{H}} \quad \text{on} \quad E \times E. \tag{2.2}$$

Then, we obtain the following:

(I) The range of the linear mapping (2.1) by $\mathscr{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(p,q)$ whose characterization is given by the two properties: $K(\cdot,q) \in H_K(E)$ for any $q \in E$ and, for any $f \in H_K(E)$ and for any $p \in E$, $(f(\cdot), K(\cdot,p))_{H_K(E)} = f(p)$.

(II) In general, we have the inequality

$$\|f\|_{H_K(E)} \leqslant \|\mathbf{f}\|_{\mathscr{H}}.$$

Here, for any member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathscr{H}$ satisfying

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathscr{H}} \quad \text{on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathscr{H}}. \tag{2.3}$$

(III) In general, we have the inversion formula in (2.1) in the form

$$f \mapsto \mathbf{f}^* \tag{2.4}$$

in (II) by using the reproducing kernel Hilbert space $H_K(E)$.

However, this formula (2.4) is, in general, involved and delicate.

Next, note that in general: For any two positive definite quadratic form functions $K_1(p,q)$ and $K_2(p,q)$ on $E\times E$, the usual product $K(p,q)=K_1(p,q)K_2(p,q)$ is again a positive definite quadratic form function on E by the Schur's theorem. Then, the reproducing kernel Hilbert space H_K admitting the kernel $K(p,q)$ is the restriction of the tensor product $H_{K_1}(E)\otimes H_{K_2}(E)$ to the diagonal set; that is given by

PROPOSITION 2.1. *Let $\{f_j^{(1)}\}_j$ and $\{f_j^{(2)}\}_j$ be some complete orthonormal systems in $H_{K_1}(E)$ and $H_{K_2}(E)$, respectively, then the reproducing kernel Hilbert space H_K is comprised of all functions on E which are represented as, in the sense of absolutely convergence on E,*

$$f(p)=\sum_{i,j}\alpha_{i,j}f_i^{(1)}(p)f_j^{(2)}(p) \quad on \quad E, \qquad \sum_{i,j}|\alpha_{i,j}|^2<\infty \tag{2.5}$$

and its norm in H_K is given by

$$\|f\|_{H_K}^2=\min\sum_{i,j}|\alpha_{i,j}|^2$$

where $\{\alpha_{i,j}\}$ are considered satisfying (2.5).

In particular, we obtain the inequality:

$$\|f_1f_2\|_{H_{K_1K_2}(E)}\leqslant\|f_1\|_{H_{K_1}(E)}\|f_2\|_{H_{K_2}(E)}.$$

We note the following sum version.

PROPOSITION 2.2. *For two positive definite quadratic form functions $K_1(p,q)$ and $K_2(p,q)$ on E, the sum $K_S(p,q)=K_1(p,q)+K_2(p,q)$ is a positive definite quadratic form function on E. The reproducing kernel Hilbert space H_{K_S} admitting the reproducing kernel $K_S(p,q)$ on E is composed of all functions*

$$f=f_1+f_2, \quad f_j\in H_{K_j}(E), \tag{2.6}$$

and the norm in H_{K_S} is given by

$$\|f\|_{H_{K_S}}^2=\min\left\{\|f_1\|_{H_{K_1(E)}}^2+\|f_2\|_{H_{K_2(E)}}^2\right\},$$

where the minimum is taken over all the expressions (2.6) for f. In particular, we obtain the triangle inequality, for $f_j\in K_{K_j}(E)$, $j=1,2$,

$$\|f_1+f_2\|_{H_{K_S}}^2\leqslant\|f_1\|_{H_{K_1}(E)}^2+\|f_2\|_{H_{K_2}(E)}^2. \tag{2.7}$$

3. New convolutions as integral kernels and norm inequalities

As very general reproducing kernels represented by the Fourier integral containing the Paley-Wiener spaces, finite and infinite orders Sobolev Hilbert spaces, for any given non-negative integrable functions ρ_1, ρ_2 on $\mathbb{R}$ that are measurable functions and are not zero identically, we define the positive definite quadratic form functions K_j by

$$K_j(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(i(x-y)\cdot t)\rho_j(t)\,dt,$$

for $j = 1,2$. Then, we consider the induced integral transforms $L_j : L_2(\mathbb{R};\rho_j) \to H_{K_j}$ by

$$f_j(x) = (L_jF_j)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} F_j(t)\exp(it\cdot x)\,\rho_j(t)dt \tag{3.1}$$

for the functions F_j satisfying

$$\frac{1}{2\pi}\int_{\mathbb{R}} |F_j(t)|^2\rho_j(t)dt < \infty, \tag{3.2}$$

respectively. Then, for the reproducing kernel Hilbert spaces H_{K_j} admitting the kernels K_j, we have the isometric identities:

$$\|f_j\|^2_{H_{K_j}} = \frac{1}{2\pi}\int_{\mathbb{R}} |F_j(t)|^2\rho_j(t)dt, \tag{3.3}$$

respectively. Now, we shall consider the non-linear operator φ_{f_1,f_2}, for $f_j \in H_{K_j}$ $(j = 1,2)$

$$\begin{aligned}\varphi(f_1,f_2)(x) &= (f_1(x)+\overline{f_1(x)})(f_2(x)+\overline{f_2(x)})\\ &= f_1(x)f_2(x)+f_1(x)\overline{f_2(x)}+\overline{f_1(x)}f_2(x)+\overline{f_1(x)f_2(x)}.\end{aligned} \tag{3.4}$$

Then, we obtain the identity

$$\begin{aligned}\varphi(f_1,f_2)(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} &\exp(ix\cdot t)((F_1\rho_1)*_1(F_2\rho_2)+(F_1\rho_1)*_2(F_2\rho_2)\\ &+(F_1\rho_1)*_3(F_2\rho_2)+(F_1\rho_1)*_4(F_2\rho_2))(t)\,dt.\end{aligned} \tag{3.5}$$

Following the operator $\varphi(f_1,f_2)$, we shall consider the identity

$$\begin{aligned}\mathbf{K}(x,y) &:= K_1(x,y)K_2(x,y)+K_1(x,y)\overline{K_2(x,y)}+\overline{K_1(x,y)}K_2(x,y)+\overline{K_1(x,y)K_2(x,y)}\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}} \exp(i(x-y)\cdot t)\cdot\Omega(t;\rho_1,\rho_2)\,dt\end{aligned}$$

for

$$\Omega(t;\rho_1,\rho_2) = (\rho_1 *_1 \rho_2)(t)+(\rho_1 *_2 \rho_2)(t)+(\rho_1 *_3 \rho_2)(t)+(\rho_1 *_4 \rho_2)(t).$$

Then, by the structure of the reproducing kernel Hilbert spaces of sum and product, we see that the image of the nonlinear operator $\varphi(f_1,f_2)$ belongs to the reproducing kernel Hilbert space $H_{\mathbf{K}}$ with the kernel $\mathbf{K}(x,y)$ and, furthermore, we obtain the inequality

$$\|\varphi(f_1,f_2)\|^2_{H_{\mathbf{K}}} \leqslant 4\|f_1\|^2_{H_{K_1}}\|f_2\|^2_{H_{K_2}}. \tag{3.6}$$

Meanwhile, note that the reproducing kernel Hilbert space $H_{\mathbf{K}}$ itself is realized explicitly as we see from the representation of $\mathbf{K}(x,y)$ in terms of the Fourier integral: Any function $g \in H_{\mathbf{K}}$ is represented by the integral

$$g(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} G(t)\exp(ix\cdot t)\Omega(t;\rho_1,\rho_2)\,dt \tag{3.7}$$

for a function G satisfying

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}} |G(t)|^2\Omega(t;\rho_1,\rho_2)\,dt < \infty, \tag{3.8}$$

and we obtain the isometric identity

$$\|g\|^2_{H_{\mathbf{K}}} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}} |G(t)|^2\Omega(t;\rho_1,\rho_2)\,dt. \tag{3.9}$$

Therefore we obtain in the t space the desired convolution inequality:

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{\Omega(t;\rho_1,\rho_2)}&|((F_1\rho_1)*_1(F_2\rho_2)+(F_1\rho_1)*_2(F_2\rho_2)\\ &+(F_1\rho_1)*_3(F_2\rho_2)+(F_1\rho_1)*_4(F_2\rho_2))(t)|^2\,dt\\ &\leqslant 4\int_{\mathbb{R}}|F_1(t)|^2\rho_1(t)\,dt\cdot\int_{\mathbb{R}}|F_2(t)|^2\rho_2(t)\,dt.\end{aligned} \tag{3.10}$$

This result for the usual convolution was expanded in various directions with applications to inverse problems and partial differential equations through L_p $(p>1)$ versions and converse inequalities. See, for example, [12, 13, 14, 19, 20, 21, 27, 28].

In particular, for each term we obtain the following norm inequalities:

COROLLARY 3.1. *For the 4 convolutions $*$ we have the norm inequalities:*

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{(\rho_1*\rho_2)(t)}&\left\{|((F_1\rho_1)*(F_2\rho_2))(t)|^2\right\}\,dt\\ &\leqslant \int_{\mathbb{R}}|F_1(t)|^2\rho_1(t)\,dt\cdot\int_{\mathbb{R}}|F_2(t)|^2\rho_2(t)\,dt.\end{aligned}$$

By a similar method, we can obtain modified versions. For example, by considering the reproducing kernel

$$\begin{aligned}|K_1(x,y)+K_2(x,y)|^2 = &K_1(x,y)\overline{K_1(x,y)}+K_1(x,y)\overline{K_2(x,y)}\\ &+\overline{K_1(x,y)}K_2(x,y)+K_2(x,y)\overline{K_2(x,y)}\end{aligned}$$

and the related operator

$$\begin{aligned}\psi(f_1,f_2)(x) &= |f_1(x)+f_2(x)|^2\\ &= f_1(x)\overline{f_1(x)}+f_1(x)\overline{f_2(x)}+\overline{f_1(x)}f_2(x)+f_2(x)\overline{f_2(x)}\end{aligned}$$

we obtain the following corresponding norm inequalities.

COROLLARY 3.2. *Within the just mentioned framework, we have the norm inequalities:*

$$\| |f_1+f_2|^2 \|^2_{H_{|K_1+K_2|^2}} \leqslant (\|f_1\|^2_{K_1} + \|f_2\|^2_{K_2})^2$$

and

$$\int_{\mathbb{R}} \frac{|((F_1\rho_1)*_2(F_1\rho_1)+(F_1\rho_1)*_2(F_2\rho_2)+(F_2\rho_2)*_2(F_1\rho_1)+(F_2\rho_2)*_2(F_2\rho_2))(t)|^2}{(\rho_1*_2\rho_1+\rho_1*_2\rho_2+\rho_2*_2\rho_1+\rho_2*_2\rho_2)(t)}\,dt$$
$$\leqslant \left(\int_{\mathbb{R}} |F_1(t)|^2\rho_1(t)\,dt + \int_{\mathbb{R}} |F_2(t)|^2\rho_2(t)\,dt\right)^2. \tag{3.11}$$

4. Inequalities and equality problems derived by the theory of reproducing kernels

We have just derived many and entirely new inequalities by applying the theory of reproducing kernels. In part, this exemplifies the power of reproducing kernels theory. Within this scope, for typical examples, we would like to refer to [29]. Furthermore, complete characterizations of the situations where the corresponding equalities are attained (in the above inequalities) are – in general – very difficult problems. See e.g. the deep theory of A. Yamada ([33]) in view of this. Despite such difficulties, significant steps in this way were given in the informal communication and the manuscript [32] where N.D.V. Nhan and D.T. Duc were able to derive generalizations and many concrete applications to the boundedness of various integral transforms and the estimates of the solutions of integral equations that solved the consequent equality problems within the L_p framework. However, our results gave basic contributions to their paper already by creating entirely new type inequalities.

It is also worth mentioning that all the inequalities in the present paper are the best possible within our framework. This is simply because we can recognize cases where the equality takes place. For example, in (3.6) and Corollary 3.2, for the related reproducing kernels, the equalities hold, as we see from the theory of reproducing kernels.

5. Basic application of Theorem 1.1

Theorem 1.1 gives the basic fundamental theory for the corresponding induced convolution integral equations. We would like to show this with the concrete example results whose proof is not simple.

In order to state the example, we shall first fix some notation. In coherence with above, for non-negative and integrable functions ρ_j on $\mathbb{R}$, $j=1,2,3$, we say that $F_j \in \mathscr{F}(\rho_j)$ if

$$\int \frac{|F_j(t)|^2}{\rho_j(t)}dt < \infty$$

on the support of ρ_j, and $F_j = 0$ in the outside of the support of ρ_j.

For the space $\mathscr{F}(\rho_1)$, we will impose additional assumptions for the natural requests of our method. We set

$$\Omega(t;\rho) = \rho_1 * (2\pi + \rho_2) + \int_{\mathbb{R}} \rho_1(\xi)\rho_3(\xi + t)d\xi,$$

for the usual convolution $*$.

We assume that $\mathscr{F}(\rho_1)$ is the real-valued function space and the support of ρ_1 is $[a,b)$ $(-\infty < a < b \leqslant +\infty)$ and on this interval, ρ_1 is a positive continuous function.

For any fixed $F_j \in \mathscr{F}(\rho_j)$, $j = 2,3$ (so that $F_2 \pm F_3$ are not zero identically), there exists a uniquely determined solution F_1 (satisfying $F_1 \in \mathscr{F}(\rho_1)$) of the equation

$$2\pi\alpha F_1(t) + \int_{\mathbb{R}} F_1(\xi)F_2(t-\xi)d\xi + \int_{\mathbb{R}} F_1(\xi)F_3(t+\xi)d\xi = \widetilde{G}(t), \tag{5.1}$$

for any function $\widetilde{G}$ satisfying

$$\int_{\mathbb{R}} |\widetilde{G}(\tau)|^2 \Omega(\tau;\rho)^{-1} d\tau < \infty,$$

in the sense of the Moore-Penrose generalized inverse (cf. [30]).

The first step to deduce this example is to establish some bounded linear operator from a certain reproducing kernel Hilbert space into some Hilbert space. This is derived from the related convolution inequality. Theorem 1.1 will give the basic theory in the related convolution integral equation with the four convolution types.

Meanwhile, when we identify what is possible to obtain with the classical methods by using Fredholm and Wiener-Hopf techniques, we realize the power of the just exemplified situation (see, for example, [31]).

Acknowledgement

The authors are thankful to the referee for the constructive comments and suggestions which gave rise to the two last sections.

REFERENCES

[1] R. BEALS, *Spatially inhomogeneous pseudodifferential operators II*, Comm. Pure Appl. Math. **27** (1974), 161–205.

[2] G. BOGVERADZE AND L. P. CASTRO, *Toeplitz plus Hankel operators with infinite index*, Integral Equations Operator Theory **62**, 1 (2008), 43–63.

[3] G. BOGVERADZE AND L. P. CASTRO, *Invertibility characterization of Wiener-Hopf plus Hankel operators via odd asymmetric factorizations*, Banach J. Math. Anal. **3**, 1 (2009), 1–18.

[4] J. W. BROWN AND R. V. CHURCHILL, *Fourier Series and Boundary Value Problems* (4th edition), McGraw-Hill Book Co., New York, 1987.

[5] L. P. CASTRO, R. DUDUCHAVA AND F.-O. SPECK, *Solvability of singular integro-differential equations with multiple complex shifts*, Complex Anal. Oper. Theory **2**, 2 (2008), 327–343.

[6] L. P. CASTRO AND A. P. NOLASCO, *A semi-Fredholm theory for Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols*, J. Operator Theory **62**, 1 (2009), 3–31.

[7] L. P. CASTRO AND E. M. ROJAS, *Reduction of singular integral operators with flip and their Fredholm property*, Lobachevskii J. Math. **29**, 3 (2008), 119–129.

[8] L. P. CASTRO AND E. M. ROJAS, *Explicit solutions of Cauchy singular integral equations with weighted Carleman shift*, J. Math. Anal. Appl. **371**, 1 (2010), 128–133.

[9] L. P. CASTRO AND E. M. ROJAS, *On the invertibility of singular integral equations with reflection on the unit circle*, Integral Equations and Operator Theory **70** (2011), 63–99.

[10] L. P. CASTRO AND A. S. SILVA, *Invertibility of matrix Wiener-Hopf plus Hankel operators with symbols producing a positive numerical range*, Z. Anal. Anwend. **28**, 1 (2009), 119–127.

[11] R. V. CHURCHILL, *Fourier Series and Boundary Value Problems*, McGraw-Hill, New York, 1941.

[12] D. T. DUC AND N. D. V. NHAN, *On some convolution norm inequalities in weighted $L_p(\mathbb{R}^n,\rho)$ spaces and their applications*, Math. Inequal. Appl. **11**, 3 (2008), 495–505.

[13] D. T. DUC AND N. D. V. NHAN, *On some reverse weighted $L_p(\mathbb{R}^n)$-norm inequalities in convolutions and their applications*, Math. Inequal. Appl. **12**, 1 (2009), 67–80.

[14] D. T. DUC AND N. D. V. NHAN, *Some applications of convolution inequalities in weighted L_p spaces*, Integral Transforms Spec. Funct. **19**, 7–8 (2008), 471–480.

[15] G. I. ÈSKIN, *Degenerate elliptic pseudodifferential equations of principal type* (Russian), Mat. Sb. (N.S.) **82** (124) (1970), 585–628.

[16] A. GREENLEAF AND G. UHLMANN, *Estimates for singular Radon transforms and pseudodifferential operators with singular symbols*, J. Funct. Anal. **89**, 1 (1990), 202–232.

[17] L. HÖRMANDER, *Fourier integral operators I*, Acta Math. **127**, 1–2 (1971), 79–183.

[18] V. A. KAKICHEV, *On the convolutions for integral transforms* (Russian), Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk **22** (1967), 48–57.

[19] N. D. V. NHAN, D. T. DUC AND V. K. TUAN, *Weighted L_p-norm inequalities for various convolution type transformations and their applications*, Armen. J. Math. **1**, 4 (2008), 1–18.

[20] N. D. V. NHAN AND D. T. DUC, *Fundamental inequalities for the iterated Laplace convolution in weighted L_p spaces and their applications*, Integral Transforms Spec. Funct. **19**, 9–10 (2008), 655–664.

[21] N. D. V. NHAN AND D. T. DUC, *Reverse weighted L_p-norm inequalities and their applications*, J. Math. Inequal. **2**, 1 (2008), 57–73.

[22] N. D. V. NHAN AND D. T. DUC, *Weighted L_p-norm inequalities in convolutions and their applications*, J. Math. Inequal. **2**, 1 (2008), 45–55.

[23] S. SAITOH, *A fundamental inequality in the convolution of L_2 functions on the half line*, Proc. Amer. Math. Soc. **91** (1984), 285–286.

[24] S. SAITOH, *On the convolution of L_2 functions*, Kodai-Math. J. **9** (1986), 50–57.

[25] S. SAITOH, *Integral Transforms, Reproducing Kernels and their Applications*, Pitman Research Notes in Mathematics Series **369**. Longman, Harlow, 1997.

[26] S. SAITOH, *Inequalities in the most simple Sobolev space and convolutions of L_2 functions with weights*, Proc. Amer. Math. Soc. **118** (1999), 515–520.

[27] S. SAITOH, *Weighted L_p-norm inequalities in convolutions. Survey on Classical Inequalities*, 225–234, Math. Appl. **517**, Kluwer Acad. Publ., Dordrecht, 2000.

[28] S. SAITOH, V. K. TUAN, AND M. YAMAMOTO, *Reverse convolution inequalities and applications to inverse heat source problems*, J. Inequal. Pure Appl. Math. **3**, 5 (2002), Article 80, 11 pp.

[29] S. SAITOH, *Theory of reproducing kernels; applications to approximate solutions of bounded linear operator equations on Hilbert spaces*, Amer. Math. Soc. Transl. **230** (2010), 107–134.

[30] L. P. CASTRO, A. SILVA AND S. SAITOH, *Integral equations with mixed Toeplitz and Hankel kernels*, manuscript.

[31] H. HOCHSTADT, *Integral Equations*, John Wiley & Sons, N. Y., 1973.

[32] N. D. V. NHAN AND D. T. DUC, *Norm inequalities for new convolutions and their applications*, manuscript.

[33] A. YAMADA, *Equality conditions for general norm inequalities in reproducing kernel Hilbert spaces,* Advances in Analysis, World Scientific, 2005, 447–455.

(Received May 10, 2011)

Luís Castro
Center for R&D in Mathematics and Applications
Department of Mathematics
University of Aveiro
3810-193 Aveiro
Portugal
e-mail: castro@ua.pt

Saburou Saitoh
Center for R&D in Mathematics and Applications
Department of Mathematics
University of Aveiro
3810-193 Aveiro
Portugal
e-mail: saburou.saitoh@gmail.com