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ON THE DEGREES OF APPROXIMATION OF FUNCTIONS BELONGING TO $L^p(\widetilde{\omega})_\beta$ CLASS BY MATRIX MEANS OF CONJUGATE FOURIER SERIES

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Abstract. The results corresponding to some theorems of S. Lal [Tamkang J. Math., 31 (4) (2000), 279-288] and the results of the second and third authors [Banach Center Publ., in press] are shown. The same degrees of pointwise approximation as in mentioned papers by significantly weaker assumptions on considered functions are obtained. From presented pointwise results the estimation on norm approximation with significantly better degrees are derived.

1. Introduction

Let L^p $(1 \leqslant p < \infty)$ be the class of all 2π-periodic real–valued functions integrable in the Lebesgue sense with p-th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \left(\int_Q | f(t) |^p \, dt \right)^{1/p}. \tag{1}$$

Consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

and the conjugate one

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x)$$

with the partial sums $\widetilde{S}_k f$. We know that if $f \in L$ then

$$\widetilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\varepsilon \to 0^+} \widetilde{f}(x, \varepsilon),$$

where

$$\widetilde{f}(x, \varepsilon) := -\frac{1}{\pi} \int_\varepsilon^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt$$

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with

$$\psi_x(t) := f(x+t) - f(x-t),$$

exists for almost all x [10, Th.(3.1)IV].

Let $A := (a_{n,k})$ be an infinite lower triangular matrix of real numbers such that

$$a_{n,k} \geqslant 0 \text{ when } k = 0,1,2,...n, \ a_{n,k} = 0 \text{ when } k > n,$$

$$\sum_{k=0}^{n} a_{n,k} = 1, \text{ where } n = 0,1,2,...,$$

and denote, for $m = 0,1,2,...,n$,

$$A_{n,m} = \sum_{k=0}^{m} a_{n,k} \quad \text{and} \quad \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k}.$$

Let the $A-$transformation of $\left(\widetilde{S}_k f\right)$ be given by

$$\widetilde{T}_{n,A} f(x) := \sum_{k=0}^{n} a_{n,k} \widetilde{S}_k f(x) \qquad (\ n = 0,1,2,...).$$

We define two classes of sequences (see [3]).

A sequence $c := (c_k)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c \in RBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leqslant K(c)\, c_m,$$

for all positive integer m, where $K(c)$ is a constant depending only on c.

A sequence $c := (c_k)$ of nonnegative numbers will be called the Head Bounded Variation Sequence, or briefly $c \in HBVS$, if it has the property

$$\sum_{k=0}^{m-1} |c_k - c_{k+1}| \leqslant K(c)\, c_m,$$

for all positive integer m, or only for all $m \leqslant n$ if the sequence c has only finite nonzero terms and the last nonzero term is c_n.

Now, we define the another class of sequences.

Following by L. Leindler [4] a sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the Mean Rest Bounded Variation Sequence, or briefly $c \in MRBVS$, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leqslant K(c) \frac{1}{m+1} \sum_{k \geqslant m/2}^{m} c_k \tag{2}$$

for all positive integer m.

Analogously, a sequence $c := (c_n)$ of nonnegative numbers will be called the Mean Head Bounded Variation Sequence, or briefly $c \in MHBVS$, if it has the property

$$\sum_{k=0}^{n-m-1} |c_k - c_{k+1}| \leqslant K(c) \frac{1}{m+1} \sum_{k=n-m}^{n} c_k, \tag{3}$$

for all positive integers $m < n$, where the sequence c has only finite nonzero terms and the last nonzero term is c_n.

It is clear that (see [9])

$$RBVS \subsetneq MRBVS \quad \text{and} \quad HBVS \subsetneq MHBVS.$$

Consequently, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leqslant K(\alpha_n) \leqslant K$$

holds for all n, where $K(\alpha_n)$ denote the constants appearing in the inequalities (2) or (3) for the sequences $\alpha_n = (a_{n,k})_{k=0}^{n}$, $n = 0, 1, 2, \ldots$.

Now we can give the conditions to be used later on. We assume that for all n and $0 \leqslant m < n$

$$\sum_{k=m}^{n-1} \left|a_{n,k} - a_{n,k+1}\right| \leqslant K \frac{1}{m+1} \sum_{k \geqslant m/2}^{m} a_{n,k} \tag{4}$$

and

$$\sum_{k=0}^{n-m-1} \left|a_{n,k} - a_{n,k+1}\right| \leqslant K \frac{1}{m+1} \sum_{k=n-m}^{n} a_{n,k} \tag{5}$$

hold if $(a_{n,k})_{k=0}^{n}$ belongs to $MRBVS$ and $MHBVS$, for $n = 1, 2, \ldots$, respectively.

As a measure of approximation of $\widetilde{f}$ by $\widetilde{T}_{n,A} f$ we use the generalized modulus of continuity of f in the space L^p defined for $\beta \geqslant 0$ by the formula

$$\widetilde{\omega}_\beta f(\delta)_{L^p} := \sup_{0 \leqslant |t| \leqslant \delta} \left\{ \left|\sin \frac{t}{2}\right|^{\beta p} \int_0^\pi |\psi_x(t)|^p \, dx \right\}^{\frac{1}{p}}.$$

It is clear that for $\beta \geqslant \alpha \geqslant 0$

$$\widetilde{\omega}_\beta f(\delta)_{L^p} \leqslant \widetilde{\omega}_\alpha f(\delta)_{L^p},$$

and it is easily seen that $\widetilde{\omega}_0 f(\cdot)_{L^p} = \widetilde{\omega} f(\cdot)_{L^p}$ is the classical modulus of continuity.

The deviation $\widetilde{T}_{n,A} f - \widetilde{f}$ was estimated at the point as well as in the norm of L^p by K. Qureshi [7] and S. Lal, H. Nigam [1]. These results were generalized by K. Qureshi [8]. The next generalization was obtained by S. Lal [2] as follows:

THEOREM. *Let $A=(a_{n,k})$ be an infinite regular triangular matrix with nonnegative entries such that $(a_{n,k})_{k=0}^{n}$ are nondecreasing sequences, then the degree of approximation of function $\widetilde{f}$, conjugate to a periodic function f belonging to class*

$$W(L^p,\omega_0)=\left\{f\in L^p:\left(\int_0^{2\pi}|f(x+t)-f(x)|^p\left|\sin^\beta x\right|^p dx\right)^{\frac{1}{p}}=O(\omega_0(t))\right\},$$

is given by

$$\left\|\widetilde{T}_{n,A}f-\widetilde{f}\right\|=O\left((n+1)^{\beta+1/p}\omega_0\left(\frac{\pi}{n+1}\right)\right),$$

provided that ω_0 increases and satisfies

$$\left\{\int_0^{\pi/(n+1)}\left(\frac{t\,|\psi_x(t)|}{\omega_0(t)}\right)^p\sin^{\beta p}t\,dt\right\}^{1/p}=O\left((n+1)^{-1}\right)$$

and

$$\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}|\psi_x(t)|}{\omega_0(t)}\right)^p dt\right\}^{1/p}=O\left((n+1)^{\gamma}\right)$$

uniformly in x, where γ is an arbitrary positive number with $q(1-\gamma)-1>0$, and $p^{-1}+q^{-1}=1,\ 1\leqslant p\leqslant\infty$.

In this paper we shall consider the same deviation and the deviation $\widetilde{T}_{n,A}f(\cdot)-\widetilde{f}\left(\cdot,\frac{2\pi}{n}\right)$, additionally. In the theorems we formulate the new generalized conditions for the functions and the modulus of continuity obtaining the same degrees of approximation as above and sometimes the essentially better one. Finally, we also give some results on norm approximation with significantly better degrees of approximation. The obtained results generalize the results from [1, 2, 5, 6].

We shall write $I_1\ll I_2$ if there exists a positive constant K, sometimes depending on some parameters, such that $I_1\leqslant KI_2$.

2. Statement of the results

Let us consider a function ω of modulus of continuity type on the interval $[0,2\pi]$, i.e. a nondecreasing continuous function having the following properties: $\omega(0)=0$, $\omega(\delta_1+\delta_2)\leqslant\omega(\delta_1)+\omega(\delta_2)$ for any $0\leqslant\delta_1\leqslant\delta_2\leqslant\delta_1+\delta_2\leqslant 2\pi$. It is easy to conclude that the function $\delta^{-1}\omega(\delta)$ is almost nonincreasing function of δ. Let

$$L^p(\widetilde{\omega})_\beta=\left\{f\in L^p:\widetilde{\omega}_\beta f(\delta)_{L^p}\leqslant\widetilde{\omega}(\delta)\right\},$$

where $\widetilde{\omega}$ is also the function of modulus of continuity type. It is clear that for $\beta>\alpha\geqslant 0$

$$L^p(\widetilde{\omega})_\alpha\subset L^p(\widetilde{\omega})_\beta.$$

We can now formulate our main results.

At the beginning, we formulate the results on the degrees of pointwise summability of conjugate series.

THEOREM 1. *Let* $f \in L^p (1 < p < \infty)$, *and let* $\widetilde{\omega}$ *satisfy*

$$\sum_{m=1}^{n} (m+1)^{\beta+1-\frac{2}{q}} \left\{ \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x \left((n+1)^{\beta+1/p} \right) \tag{6}$$

and

$$\left\{ \int_0^{\frac{2\pi}{n}} \left(\frac{t\,|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x \left((n+1)^{-1} \right) \tag{7}$$

with $0 \leqslant \beta < 1 - \frac{1}{p}$ *and* $q = \frac{p}{p-1}$. *If* $(a_{n,k})_{k=0}^n \in MRBVS$, *then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{2\pi}{n}\right) \right| = O_x \left((n+1)^{\beta+\frac{1}{p}} \sum_{k=0}^{n} a_{n,k} \widetilde{\omega}\left(\frac{\pi}{k+1} \right) \right), \tag{8}$$

for considered x.

THEOREM 2. *Let* $f \in L^p (1 < p < \infty)$, *and let* $\widetilde{\omega}$ *satisfy* (6) *and* (7) *with* $0 \leqslant \beta < 1 - \frac{1}{p}$ *and* $q = \frac{p}{p-1}$. *If* $(a_{n,k})_{k=0}^n \in MHBVS$, *then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{2\pi}{n}\right) \right| = O_x \left((n+1)^{\beta+\frac{1}{p}} \sum_{k=0}^{n} a_{n,n-k} \widetilde{\omega}\left(\frac{\pi}{k+1} \right) \right), \tag{9}$$

for considered x.

THEOREM 3. *Let* $f \in L^p (1 < p < \infty)$, *and let* $\widetilde{\omega}$ *satisfy*

$$\left\{ \int_0^{\frac{2\pi}{n}} \left(\frac{\widetilde{\omega}(t)}{t \sin^{\beta} \frac{t}{2}} \right)^q dt \right\}^{1/q} = O \left((n+1)^{\beta+1/p} \widetilde{\omega}\left(\frac{\pi}{n+1} \right) \right) \tag{10}$$

and

$$\left\{ \int_0^{\frac{2\pi}{n}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x \left((n+1)^{-\frac{1}{p}} \right) \tag{11}$$

and (6) *with* $0 \leqslant \beta < 1 - \frac{1}{p}$ *and* $q = \frac{p}{p-1}$. *If* $(a_{n,k})_{k=0}^n \in MRBVS$, *then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right| = O_x \left((n+1)^{\beta+\frac{1}{p}} \sum_{k=0}^{n} a_{n,k} \widetilde{\omega}\left(\frac{\pi}{k+1} \right) \right), \tag{12}$$

for almost all considered x *such that* $\widetilde{f}(x)$ *exists*.

THEOREM 4. *Let* $f \in L^p (1 < p < \infty)$, *and let* $\widetilde{\omega}$ *satisfy* (6), (10) *and* (11) *with* $0 \leqslant \beta < 1 - \frac{1}{p}$ *and* $q = \frac{p}{p-1}$. *If* $(a_{n,k})_{k=0}^n \in MHBVS$, *then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right| = O_x \left((n+1)^{\beta+\frac{1}{p}} \sum_{k=0}^{n} a_{n,n-k} \widetilde{\omega}\left(\frac{\pi}{k+1} \right) \right), \tag{13}$$

for almost all considered x *such that* $\widetilde{f}(x)$ *exists*.

REMARK 1. Analyzing the proofs of Theorem 1 - 4 we can deduce that
(i) under the following assumption

$$\sum_{m=1}^{n}(m+1)^{\beta+1-\frac{2}{q}}\left\{\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^{p}\sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}=O_x\left((n+1)^{\beta}\right) \tag{14}$$

instead of (6) we obtain the estimates (12) and (13) with the expressions $(n+1)^{\beta}$ instead of $(n+1)^{\beta+\frac{1}{p}}$;

(ii) we get the estimates (8) and (9) with the expressions $(n+1)^{\beta}$ instead of $(n+1)^{\beta+\frac{1}{p}}$ under the assumptions (11) and (14) instead of (7) and (6), respectively;

(iii) taking the assumption $A_{n,k}=O\left(\frac{k}{n+1}\right)$ or $\overline{A}_{n,k}=O\left(\frac{k}{n+1}\right)$ we obtain the estimates (8), (9), (12) and (13) with the expression $\widetilde{\omega}\left(\frac{\pi}{n+1}\right)$ instead of $\sum_{k=0}^{n}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)$ or $\sum_{k=0}^{n}a_{n,n-k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)$, respectively.

REMARK 2. It can be note that the condition (6) is weaker than the condition

$$\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{|\psi_x(t)|}{t^{\gamma}\widetilde{\omega}(t)}\right)^{p}\sin^{\beta p}\frac{t}{2}dt\right\}^{\frac{1}{p}}=O_x\left((n+1)^{\gamma}\right), \tag{15}$$

from [6]. Namely, let $f\in L^{p}$ $(1<p<\infty)$, $\beta\geqslant 0$ and $0\leqslant\gamma<\beta+\frac{1}{p}$. If (15) holds then (6) is valid.

REMARK 3. From Remark 1, applying Remark 2, we can obtain the results from [6].

REMARK 4. Analyzing the proofs of Theorem 1 - 4 we can deduce that taking the assumption $(a_{n,k})_{k=0}^{n}\in RBVS$ or $(a_{n,k})_{k=0}^{n}\in HBVS$ instead of $(a_{n,k})_{k=0}^{n}\in MRBVS$ or $(a_{n,k})_{k=0}^{n}\in MHBVS$, respectively, we obtain the results from the paper [5].

Finally, we formulate the results on estimates of L^{p} norm of the deviation considered above.

THEOREM 5. *Let $f\in L^{p}(\widetilde{\omega})_{\beta}$ $(1<p<\infty)$ with $0\leqslant\beta<1-\frac{1}{p}$. If $(a_{n,k})_{k=0}^{n}\in MRBVS$, then*

$$\left\|\widetilde{T}_{n,A}f(\cdot)-\widetilde{f}\left(\cdot,\frac{2\pi}{n}\right)\right\|_{L^{p}}=\begin{cases}O\left((n+1)^{\beta}\sum_{k=0}^{n}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\right) & \text{for } \beta>0,\\ O\left((n+1)^{\frac{1}{p}}\sum_{k=0}^{n}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\right) & \text{for } \beta=0.\end{cases}$$

THEOREM 6. *Let $f\in L^{p}(\widetilde{\omega})_{\beta}$ $(1<p<\infty)$ with $0\leqslant\beta<1-\frac{1}{p}$. If $(a_{n,k})_{k=0}^{n}\in MHBVS$, then*

$$\left\|\widetilde{T}_{n,A}f(\cdot)-\widetilde{f}\left(\cdot,\frac{2\pi}{n}\right)\right\|_{L^{p}}=\begin{cases}O\left((n+1)^{\beta}\sum_{k=0}^{n}a_{n,n-k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\right) & \text{for } \beta>0,\\ O\left((n+1)^{\frac{1}{p}}\sum_{k=0}^{n}a_{n,n-k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\right) & \text{for } \beta=0.\end{cases}$$

REMARK 5. Taking the additionally condition (10) we obtain the same estimations for deviation $\widetilde{T}_{n,A}f-\widetilde{f}$ as in Theorem 5 and 6.

REMARK 6. If we consider the modulus of continuity in the form

$$\widetilde{\omega}f(\delta)_{L^p_\beta}:=\sup_{0\leqslant|t|\leqslant\delta}\left\{\int_0^\pi|\psi_x(t)|^p\left|\sin\frac{x}{2}\right|^{\beta p}dx\right\}^{\frac{1}{p}}$$

then Theorems 5, 6 and Remark 5 will be true under the assumption

$$f\in L^p_\beta(\widetilde{\omega})=\left\{f\in L^p:\widetilde{\omega}f(\delta)_{L^p_\beta}\leqslant\widetilde{\omega}(\delta)\right\}$$

and with the following norm

$$\|f\|_{L^p_\beta}:=\|f(\cdot)\|_{L^p_\beta}=\left(\int_Q|f(t)|^p\left|\sin\frac{t}{2}\right|^{\beta p}dt\right)^{1/p}\quad\text{when}\quad 1\leqslant p<\infty.$$

If we consider the conditions (6) and (7) (in case of Remark 5, the conditions (10) and (11) instead of (7) one), then we obtain the same estimates, as in Theorem 5, 6 and Remark 5, of our deviation in the usual L^p norm.

REMARK 7. In the paper [2] is considered $\sin t$ instead of $\sin\frac{t}{2}$, for $t\in(0,\pi)$. It generates some inconvenience since the inequality $\sin t\geqslant\frac{2}{\pi}t$ does not hold for all $t\in(0,\pi)$ and, among others, therefore the proofs in [2] are incorrect. We also note that the assumption (7) instead of (11) without the condition (10) leads us to the divergent integral of the form $\int_0^{\frac{\pi}{n+1}}t^{\frac{-(1+\beta)}{1-1/p}}dt$ $(\beta\geqslant 0)$.

REMARK 8. Under the above remarks we can observe that in the special case $\beta=0$, when our sequences (a_{nk}) are monotonic with respect to k we also have the corrected form of the result of S. Lal and H. K. Nigam [1].

3. Auxiliary results

We begin this section by some notations following A. Zygmund [10, Section 5 of Chapter II].

It is clear that

$$\widetilde{S}_kf(x)=-\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\widetilde{D}_k(t)\,dt$$

and

$$\widetilde{T}_{n,A}f(x)=-\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k}(t)\,dt,$$

where

$$\widetilde{D_k}(t)=\sum_{\nu=0}^{k}\sin\nu t=\frac{\cos\frac{t}{2}-\cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}.$$

Hence

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{2\pi}{n}\right) = -\frac{1}{\pi}\int_0^{\frac{2\pi}{n}}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k}(t)\,dt + \frac{1}{\pi}\int_{\frac{2\pi}{n}}^{\pi}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^{\circ}}(t)\,dt$$

and

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)=\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^{\circ}}(t)\,dt,$$

where

$$\widetilde{D_k^{\circ}}(t)=\frac{\cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}.$$

Now, we formulate some estimates for the conjugate Dirichlet kernels.

LEMMA 1. (see [10]) *If* $0<|t|\leqslant\pi/2$ *then*

$$\left|\widetilde{D_k^{\circ}}(t)\right|\leqslant\frac{\pi}{2|t|}\quad\textit{and}\quad\left|\widetilde{D_k}(t)\right|\leqslant\frac{\pi}{|t|}$$

and for any real t *we have*

$$\left|\widetilde{D_k}(t)\right|\leqslant\frac{1}{2}k(k+1)|t|\quad\textit{and}\quad\left|\widetilde{D_k}(t)\right|\leqslant k+1.$$

LEMMA 2. (see [6]) *If* $\left(a_{n,k}\right)_{k=0}^{n}\in MHBVS$, *then*

$$\left|\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^{\circ}}(t)\right|=O\left(t^{-1}\overline{A}_{n,n-2\tau}\right)$$

and if $\left(a_{n,k}\right)_{k=0}^{n}\in MRBVS$, *then*

$$\left|\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^{\circ}}(t)\right|=O\left(t^{-1}A_{n,\tau}\right),$$

for $\frac{2\pi}{n}\leqslant t\leqslant\pi$ $(n=2,3,\ldots)$, *where* $\tau=[\pi/t]$.

4. Proofs of the results

4.1. Proof of Theorem 1

We start with the obvious relations

$$\begin{aligned}\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{2\pi}{n}\right) &= -\frac{1}{\pi}\int_0^{\frac{2\pi}{n}}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k}(t)\,dt\\ &\quad+\frac{1}{\pi}\int_{\frac{2\pi}{n}}^{\pi}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^{\circ}}(t)\\ &=\widetilde{I_1}+\widetilde{I_2^{\circ}}\end{aligned}$$

and

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{2\pi}{n}\right)\right| \leqslant \left|\widetilde{I}_1\right|+\left|\widetilde{I}_2^{\circ}\right|.$$

By the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 1 and (7), for $\beta<1-\frac{1}{p}$, we have

$$\begin{aligned}
\left|\widetilde{I}_1\right| &\ll (n+1)^2\int_0^{\frac{2\pi}{n}} t\left|\psi_x(t)\right|dt \\
&\leqslant (n+1)^2\left\{\int_0^{\frac{2\pi}{n}}\left[\frac{t\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{2\pi}{n}}\left[\frac{\widetilde{\omega}(t)}{\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}} \\
&\ll (n+1)\left\{\int_0^{\frac{2\pi}{n}}\left[\frac{\widetilde{\omega}(t)}{t^\beta}\right]^q dt\right\}^{\frac{1}{q}} \ll (n+1)^{\beta+\frac{1}{p}}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right).
\end{aligned}$$

Using the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 2 and (6) for an even n

$$\begin{aligned}
\left|\widetilde{I}_2^{\circ}\right| &\ll \int_{\frac{2\pi}{n}}^{\pi}\frac{\left|\psi_x(t)\right|}{t}\sum_{k=0}^{\tau}a_{n,k}dt=\sum_{m=1}^{\frac{n}{2}-1}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{\left|\psi_x(t)\right|}{t}\sum_{k=0}^{\tau}a_{n,k}dt \\
&\leqslant \sum_{m=1}^{\frac{n}{2}-1}\sum_{k=0}^{m+1}a_{n,k}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{\left|\psi_x(t)\right|}{t}dt \leqslant \sum_{m=2}^{\frac{n}{2}}\sum_{k=0}^{m}a_{n,k}\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\frac{\left|\psi_x(t)\right|}{t}dt \\
&= \sum_{m=2}^{\frac{n}{2}}\sum_{k=2}^{m}a_{n,k}\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\frac{\left|\psi_x(t)\right|}{t}dt+\sum_{m=2}^{\frac{n}{2}}\left(a_{n,0}+a_{n,1}\right)\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\frac{\left|\psi_x(t)\right|}{t}dt \\
&= \sum_{k=2}^{\frac{n}{2}}a_{n,k}\sum_{m=k}^{\frac{n}{2}}\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\frac{\left|\psi_x(t)\right|}{t}dt+\sum_{k=0}^{1}a_{n,k}\sum_{m=2}^{\frac{n}{2}}\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\frac{\left|\psi_x(t)\right|}{t}dt \\
&\leqslant \sum_{k=2}^{\frac{n}{2}}a_{n,k}\sum_{m=k}^{\frac{n}{2}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{\widetilde{\omega}(t)}{t\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}} \\
&\quad+\sum_{k=0}^{1}a_{n,k}\sum_{m=2}^{\frac{n}{2}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{\widetilde{\omega}(t)}{t\sin^\beta\frac{t}{2}}\right]^q dt\right\}^{\frac{1}{q}} \\
&\leqslant \sum_{k=2}^{\frac{n}{2}}a_{n,k}\sum_{m=k}^{\frac{n}{2}}\widetilde{\omega}\left(\frac{\pi}{m-1}\right)\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}t^{(-1-\beta)q}dt\right\}^{\frac{1}{q}} \\
&\quad+\sum_{k=0}^{1}a_{n,k}\sum_{m=2}^{\frac{n}{2}}\widetilde{\omega}\left(\frac{\pi}{m-1}\right)\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\frac{1}{t^{(1+\beta)q}}dt\right\}^{\frac{1}{q}} \\
&\ll \sum_{k=0}^{\frac{n}{2}}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\sum_{m=2}^{\frac{n}{2}}m^{\beta+1-\frac{2}{q}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{\left|\psi_x(t)\right|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}} \\
&\ll (n+1)^{\beta+\frac{1}{p}}\sum_{k=0}^{n}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right),
\end{aligned}$$

and for an odd n $(n>2)$

$$\begin{aligned}\left|\widetilde{I_2^{\circ}}\right| &\ll \int_{\frac{2\pi}{n}}^{\pi}\frac{|\psi_x(t)|}{t}\sum_{k=0}^{\tau}a_{n,k}dt\\ &=\sum_{m=1}^{\left[\frac{n}{2}\right]-1}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{|\psi_x(t)|}{t}\sum_{k=0}^{\tau}a_{n,k}dt+\int_{\frac{2\pi}{n}}^{\frac{\pi}{\left[\frac{n}{2}\right]}}\frac{|\psi_x(t)|}{t}\sum_{k=0}^{\tau}a_{n,k}dt=\sum\nolimits_1+\sum\nolimits_2.\end{aligned}$$

The first sum we can estimate similarly like for an even n and therefore

$$\begin{aligned}\sum\nolimits_1 &\ll \sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\sum_{m=2}^{\left[\frac{n}{2}\right]}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^{\beta}\frac{t}{2}\right]^{p}dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}t^{(-1-\beta)q}dt\right\}^{\frac{1}{q}}\\ &\ll (n+1)^{\beta+\frac{1}{p}}\sum_{k=0}^{n}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right).\end{aligned}$$

For the second sum we have

$$\begin{aligned}\sum\nolimits_2 &\leqslant \sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,k}\int_{\frac{2\pi}{n}}^{\frac{\pi}{\left[\frac{n}{2}\right]}}\frac{|\psi_x(t)|}{t}dt\leqslant\sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,k}\int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}}\frac{|\psi_x(t)|}{t}dt\\ &\leqslant \sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,k}\left\{\int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}}\left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^{\beta}\frac{t}{2}\right]^{p}dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}}\left[\frac{\widetilde{\omega}(t)}{t\sin^{\beta}\frac{t}{2}}\right]^{q}dt\right\}^{\frac{1}{q}}\\ &\ll \sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\left(\left[\frac{n}{2}\right]+1\right)^{1+\beta-\frac{2}{q}}\left\{\int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}}\left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^{\beta}\frac{t}{2}\right]^{p}dt\right\}^{\frac{1}{p}}\\ &\ll \sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,k}\left(\left[\frac{n}{2}\right]+1\right)^{\beta+\frac{1}{p}}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\ll(n+1)^{\beta+\frac{1}{p}}\sum_{k=0}^{n}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right).\end{aligned}$$

Collecting these estimates we obtain the desired result. □

4.2. Proof of Theorem 2

Let as usual

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{2\pi}{n}\right)=\widetilde{I_1}+\widetilde{I_2^{\circ}}$$

and

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\right|\leqslant\left|\widetilde{I_1}\right|+\left|\widetilde{I_2^{\circ}}\right|.$$

The term $\widetilde{|I_1|}$ we can estimate by the same way as in the proof of Theorem 1. Therefore

$$\left|\widetilde{I_1}\right|\ll(n+1)^{\beta+\frac{1}{p}}\widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

Analogously to the above, by the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 2 and (6) for an even n

$$\begin{aligned}
\left|\widetilde{I_2^\circ}\right| &\ll \int_{\frac{2\pi}{n}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=n-2\tau}^{n} a_{n,k} dt = \sum_{m=1}^{\frac{n}{2}-1} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau} a_{n,n-2k} dt \\
&\leqslant \sum_{k=2}^{\frac{n}{2}} a_{n,n-2k} \sum_{m=k}^{\frac{n}{2}} \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} \frac{|\psi_x(t)|}{t} dt + \sum_{k=0}^{1} a_{n,n-2k} \sum_{m=2}^{\frac{n}{2}} \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} \frac{|\psi_x(t)|}{t} dt \\
&\leqslant \sum_{k=2}^{\frac{n}{2}} a_{n,n-2k} \sum_{m=k}^{\frac{n}{2}} \left\{ \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} \left[\frac{\widetilde{\omega}(t)}{t \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}} \\
&\quad + \sum_{k=0}^{1} a_{n,n-2k} \sum_{m=2}^{\frac{n}{2}} \left\{ \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} \left[\frac{\widetilde{\omega}(t)}{t \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}} \\
&\ll \sum_{k=0}^{\frac{n}{2}} a_{n,n-2k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right) \sum_{m=2}^{\frac{n}{2}} m^{\beta+1-\frac{2}{q}} \left\{ \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \\
&\ll n^{\beta+\frac{1}{p}} \sum_{k=0}^{\frac{n}{2}} a_{n,n-2k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right) \leqslant (n+1)^{\beta+\frac{1}{p}} \sum_{k=0}^{n} a_{n,n-k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right),
\end{aligned}$$

and for an odd n $(n>2)$

$$\begin{aligned}
\left|\widetilde{I_2^\circ}\right| &\ll \int_{\frac{2\pi}{n}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=n-2\tau}^{n} a_{n,k} dt \\
&= \sum_{m=1}^{\left[\frac{n}{2}\right]-1} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau} a_{n,n-2k} dt + \int_{\frac{2\pi}{n}}^{\frac{\pi}{\left[\frac{n}{2}\right]}} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau} a_{n,n-2k} dt = \sum\nolimits_1 + \sum\nolimits_2 .
\end{aligned}$$

The first sum we can estimate similarly like above

$$\begin{aligned}
\sum\nolimits_1 &\ll \sum_{k=0}^{\left[\frac{n}{2}\right]} a_{n,n-2k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right) \sum_{m=2}^{\left[\frac{n}{2}\right]} \left\{ \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}} t^{(-1-\beta)q} dt \right\}^{\frac{1}{q}} \\
&\ll (n+1)^{\beta+\frac{1}{p}} \sum_{k=0}^{n} a_{n,n-k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right).
\end{aligned}$$

For the second sum we have

$$\begin{aligned}
\Sigma_2 &\leqslant \sum_{k=0}^{\left[\frac{n}{2}\right]} a_{n,n-2k} \int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}} \frac{|\psi_x(t)|}{t} dt \\
&\leqslant \sum_{k=0}^{\left[\frac{n}{2}\right]} a_{n,n-2k} \left\{ \int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^{\beta} \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}} \left[\frac{\widetilde{\omega}(t)}{t \sin^{\beta} \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}} \\
&\ll \sum_{k=0}^{\left[\frac{n}{2}\right]} a_{n,n-2k} \left(\left[\frac{n}{2}\right] + 1 \right)^{\beta+\frac{1}{p}} \widetilde{\omega}\left(\frac{\pi}{n+1}\right) \ll (n+1)^{\beta+\frac{1}{p}} \sum_{k=0}^{n} a_{n,n-k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right).
\end{aligned}$$

Collecting these estimates we obtain the desired result. □

4.3. Proof of Theorem 3

We start with the obvious relations

$$\begin{aligned}
\widetilde{T}_{n,A} f(x) - \widetilde{f}(x) &= \frac{1}{\pi} \int_0^{\frac{2\pi}{n}} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D_k^{\circ}}(t)\, dt + \frac{1}{\pi} \int_{\frac{2\pi}{n}}^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D_k^{\circ}}(t)\, dt \\
&= \widetilde{I_1^{\circ}} + \widetilde{I_2^{\circ}}
\end{aligned}$$

and

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}(x) \right| \leqslant \left| \widetilde{I_1^{\circ}} \right| + \left| \widetilde{I_2^{\circ}} \right|.$$

By the Hölder inequality $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, Lemma 1, (11) and (10), for $\beta < 1 - \frac{1}{p}$

$$\begin{aligned}
\left| \widetilde{I_1^{\circ}} \right| &\ll \int_0^{\frac{2\pi}{n}} \frac{|\psi_x(t)|}{t} dt \\
&\leqslant \left\{ \int_0^{\frac{2\pi}{n}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^{\beta} \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{2\pi}{n}} \left[\frac{\widetilde{\omega}(t)}{t \sin^{\beta} \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}} \\
&\ll (n+1)^{\beta} \widetilde{\omega}\left(\frac{\pi}{n+1}\right).
\end{aligned}$$

The term $\left| \widetilde{I_2^{\circ}} \right|$ we can estimate by the same way like in the proof of Theorem 1.

So

$$\left| \widetilde{I_2^{\circ}} \right| \ll (n+1)^{\beta+\frac{1}{p}} \sum_{k=0}^{n} a_{n,k} \widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

Collecting these estimates we obtain the desired result. □

4.4. Proof of Theorem 4

Let as usual

$$\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)=\widetilde{I}_1^\circ+\widetilde{I}_2^\circ$$

and

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\right|\leqslant\left|\widetilde{I}_1^\circ\right|+\left|\widetilde{I}_2^\circ\right|.$$

For the first term, by the proof of Theorem 3, we have

$$\left|\widetilde{I}_1^\circ\right|\ll(n+1)^\beta\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right),$$

where $\beta<1-\frac{1}{p}$, and for the second one, by the proof of Theorem 2, we can write

$$\left|\widetilde{I}_2^\circ\right|\ll(n+1)^{\beta+\frac{1}{p}}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

Collecting these estimates we obtain the desired result. □

4.5. Proof of Remark 1

(i) We present the proof in the case of Theorem 3, only.

The estimate of the quantity $\left|\widetilde{I}_1^\circ\right|$ will be the same

$$\left|\widetilde{I}_1^\circ\right|\ll(n+1)^\beta\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

Similarly as in the proof of Theorem 3 using (14) we can show that for an even n

$$\begin{aligned}\left|\widetilde{I}_2^\circ\right|&\ll\sum_{k=0}^{\frac{n}{2}}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\sum_{m=2}^{\frac{n}{2}}m^{\beta+1-\frac{2}{q}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\\&\ll(n+1)^\beta\sum_{k=0}^{n}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\end{aligned}$$

and for an odd n $(n>2)$

$$\left|\widetilde{I}_2^\circ\right|\ll\sum\nolimits_1+\sum\nolimits_2.$$

The first sum we can estimate similarly like for an even n and therefore

$$\begin{aligned}\sum\nolimits_1&\ll\sum_{k=0}^{\left[\frac{n}{2}\right]}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right)\sum_{m=2}^{\left[\frac{n}{2}\right]}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}\left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\sin^\beta\frac{t}{2}\right]^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{\pi}{m}}^{\frac{\pi}{m-1}}t^{(-1-\beta)q}dt\right\}^{\frac{1}{q}}\\&\ll(n+1)^\beta\sum_{k=0}^{n}a_{n,k}\widetilde{\omega}\left(\frac{\pi}{k+1}\right).\end{aligned}$$

For the second sum we have

$$\sum\nolimits_2 \leqslant \sum_{k=0}^{\left[\frac{n}{2}\right]} a_{n,k} \left\{ \int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{\pi}{\left[\frac{n}{2}\right]+1}}^{\frac{\pi}{\left[\frac{n}{2}\right]}} \left[\frac{\widetilde{\omega}(t)}{t \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}}$$

$$\ll (n+1)^\beta \sum_{k=0}^{n} a_{n,k} \widetilde{\omega}\left(\frac{\pi}{k+1}\right).$$

(ii) We present the proof in the case of Theorem 1, only.

By the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, Lemma 1 and (11), for $\beta < 1-\frac{1}{p}$, we have

$$\left|\widetilde{I}_1\right| \ll (n+1)^2 \int_0^{\frac{2\pi}{n}} t\,|\psi_x(t)|\,dt$$

$$\ll (n+1) \left\{ \int_0^{\frac{2\pi}{n}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \left\{ \int_0^{\frac{2\pi}{n}} \left[\frac{\widetilde{\omega}(t)}{\sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}}$$

$$\ll (n+1)^\beta\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

The estimate of the quantity $\left|\widetilde{I}_2^\circ\right|$ will be the same as above.

(iii) We present the proof in the case of Theorem 1, only.

The estimate of the term $\left|\widetilde{I}_1\right|$ will be the same

$$\left|\widetilde{I}_1\right| \ll (n+1)^{\beta+\frac{1}{p}}\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

Using our additionally assumption we obtain

$$\left|\widetilde{I}_2^\circ\right| \ll \int_{\frac{2\pi}{n}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau} a_{n,k} dt \ll \sum_{m=1}^{\frac{n}{2}-1} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \frac{|\psi_x(t)|}{t} \frac{\tau}{n+1} dt$$

$$\ll \frac{1}{n+1} \sum_{m=1}^{\frac{n}{2}-1} \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \frac{|\psi_x(t)|}{t^2} dt$$

$$\leqslant \frac{1}{n+1} \sum_{m=1}^{\frac{n}{2}-1} \left\{ \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}} \left\{ \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \left[\frac{\widetilde{\omega}(t)}{t^2 \sin^\beta \frac{t}{2}} \right]^q dt \right\}^{\frac{1}{q}}$$

$$\ll \frac{1}{n+1} \sum_{m=1}^{\frac{n}{2}} \frac{\widetilde{\omega}\left(\frac{\pi}{m}\right)}{\frac{\pi}{m}} (m+1)^{1+\beta-\frac{2}{q}} \left\{ \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}}$$

$$\ll \widetilde{\omega}\left(\frac{\pi}{n+1}\right) \sum_{m=1}^{\frac{n}{2}} (m+1)^{1+\beta-\frac{2}{q}} \left\{ \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \left[\frac{|\psi_x(t)|}{\widetilde{\omega}(t)} \sin^\beta \frac{t}{2} \right]^p dt \right\}^{\frac{1}{p}}$$

$$\ll (n+1)^{\beta+\frac{1}{p}}\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right).$$

Thus the desired result follows. □

4.6. Proof of Remark 2

Let (15) holds. Then, using the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, we have

$$\sum_{m=1}^{n}(m+1)^{\beta+1-\frac{2}{q}}\left\{\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}$$
$$\ll \sum_{m=1}^{n}(m+1)^{\beta-\gamma+1-\frac{2}{q}}\left\{\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left(\frac{|\psi_x(t)|}{t^\gamma\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}$$
$$\leqslant \left\{\sum_{m=1}^{n}\left((m+1)^{\beta-\gamma+1-\frac{2}{q}}\right)^q\right\}^{\frac{1}{q}}\left\{\sum_{m=1}^{n}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left(\frac{|\psi_x(t)|}{t^\gamma\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p}$$
$$\leqslant \left\{\sum_{m=1}^{n}(m+1)^{\beta q-\gamma q+q-2}\right\}^{\frac{1}{q}}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{|\psi_x(t)|}{t^\gamma\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{\frac{1}{p}}$$
$$\ll (n+1)^{\beta+1/p}$$

when $\gamma<\beta-\frac{1}{p}$. □

4.7. Proofs of Theorems 5, 6 and Remark 5

The proofs are similar to these above with the applications of Remark 1. The expressions in the estimates under the L^p norm with respect to x will be like these on the left hand side of our conditions (7) (in case of Remark 5, the condition (11) instead of (7)) and (14) or (6) when $\beta>0$ or $\beta=0$, respectively. Since $f\in L^p(\widetilde{\omega})_\beta$, the such norm quantities will always have the same orders like these on the right hand side of the mentioned conditions. Therefore the proofs follow without any additionally assumptions. □

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