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A NEW REVERSE ISOPERIMETRIC INEQUALITY AND ITS STABILITY

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Abstract. In this paper, we deal with the reverse isoperimetric inequality for a closed and strictly convex curve in the Euclidean plane $\mathbb{R}^2$ involving the following geometric functionals associated to the given convex curve: length, areas of the region respectively included by the curve and the locus of curvature centers, and the integral of the radius of curvature. In fact, a stronger and sharp version of the reverse isoperimetric inequality proved by Pan and Yang in [1] is established with a simple Fourier series proof. Furthermore, we investigate the stability property of such an inequality (almost equality implies that the curve is nearly circular).

1. Introduction and main results

We recall the classical isoperimetric inequality (see [2]) in the Euclidean plane $\mathbb{R}^2$, given by:

THEOREM 1.1. (Isoperimetric Inequality) *Let γ be a simple closed curve of length L, enclosing a region of area A. Then*

$$L^2 - 4\pi A \geqslant 0, \tag{1}$$

and equality holds if and only if γ is a circle.

This fact was known to the ancient Greeks, and the first mathematical proof was only given in the 19th century by Steiner in [2]. Since then authors obtained many new proofs, sharpened forms, generalizations, and applications of this famous inequality.

Recently, in [3] S. L. Pan and H. Zhang derived an interesting reverse isoperimetric inequality for closed convex curves.

THEOREM 1.2. *Let γ be a C^2 closed strictly convex plane curve with length L and enclosed area A. Then*

$$L^2 \leqslant 4\pi\left(A + |\tilde{A}|\right), \tag{2}$$

where $\tilde{A}$ denotes the oriented area of the locus of its curvature centers and equality holds if and only if γ is a circle.

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REMARK 1. It is obvious that if γ is a circle, then the locus of its curvature centers is only a point, and thus its area $\tilde{A} = 0$. Conversely, if $\tilde{A} = 0$, then from the classical isoperimetric inequality (1) and the reverse isoperimetric inequality (2), it follows that the area A and the length L of γ satisfy $L^2 = 4\pi A$, which implies that γ is a circle, and therefore the locus of curvature centers of γ is a point.

Furthermore, in [1], in order to estimate the isoperimetric deficit of the evolving curve

$$X_t = \left(\frac{L}{2\pi} - \frac{1}{k}\right)\overrightarrow{N},$$

where k is the signed curvature of the evolving curve and $\overrightarrow{N}$ is the unit inward pointing normal vector along the curve, S. L. Pan and J. N. Yang established a new reverse isoperimetric inequality.

THEOREM 1.3. *Let γ be a C^2 closed strictly convex curve in the plane $\mathbb{R}^2$ with length L and enclosed area A. Then*

$$\int_0^{2\pi} \rho(\theta)^2 d\theta \geqslant \frac{L^2 - 2\pi A}{\pi}, \tag{3}$$

where ρ is the curvature radius of γ and equality holds if and only if γ is a circle.

REMARK 2. Recall that in [3] S. L. Pan and H. Zhang also proved that

$$\int_0^{2\pi} \rho(\theta)^2 d\theta = 2\left(A + |\tilde{A}|\right), \tag{4}$$

thus the inequality (3) in Theorem 1.3 can be written as

$$L^2 \leqslant 4\pi A + 2\pi|\tilde{A}|,$$

which is actually a stronger version of the Pan and Zhang's result (2).

In this paper, we establish a stronger and sharp version of (3), and one of our main results is formulated as follows:

THEOREM 1.4. (Main Theorem) *Let γ be a C^2 closed strictly convex curve in the plane $\mathbb{R}^2$ with length L and enclosed area A. Then*

$$\int_0^{2\pi} \rho(\theta)^2 d\theta \geqslant \frac{L^2 - 2\pi\left(A - \varepsilon|\tilde{A}|\right)}{\pi}, \tag{5}$$

satisfied for any $\varepsilon \leqslant \frac{1}{2}$, where $\tilde{A}$ denotes the oriented area of the locus of its curvature centers. Moreover, if we select the parameter $\varepsilon = \frac{1}{2}$, the inequality

$$L^2 \leqslant 4\pi A + \pi|\tilde{A}| \tag{6}$$

is actually a sharp version of (3). *Equality in* (5) *holds if γ is a circle. Conversely, for $\varepsilon = \frac{1}{2}$, if equality in* (5) *holds, then the Minkowski support function of γ is of the form*

$$p(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta.$$

For arbitrary $\varepsilon < \frac{1}{2}$, *if equality holds, then the Minkowski support function is of the form*

$$p(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta.$$

REMARK 3. As in Remark 2, by using (4), the inequality (5) in Theorem 1.4 can be written as

$$L^2 \leqslant 4\pi A + 2\pi|\tilde{A}|(1-\varepsilon),$$

which shows that our inequality (5) is stronger than Pan and Yang's result (3) if $0 < \varepsilon \leqslant \frac{1}{2}$.

The stability problem associated with an isoperimetric inequality is also interesting and significant. A well-known and the most frequently used example is the Steiner disc [4].

DEFINITION 1.5. The Steiner disc of a convex body K, denoted by $S(K)$, is the circular disc with radius $\frac{L(K)}{2\pi}$ and center at the Steiner point $\overrightarrow{s}(K)$ which can be defined in terms of the Minkowski support function $p_K(\theta)$ of the convex body K by:

$$\overrightarrow{s}(K) = \frac{1}{\pi}\int_0^{2\pi} \overrightarrow{u}(\theta)\, p_K(\theta) d\theta,$$

where $\overrightarrow{u}(\theta)$ is a unit pointing tangent vector along the curve, and $L(K)$ denotes the perimeter of the convex body K.

Recently, in [5] S. L. Pan and H. P. Xu established the following stability estimates for the reverse isoperimetric inequality (2) by comparing a convex body K with its Steiner disk.

THEOREM 1.6. *Let* $\gamma(\theta)$ *be a* C^2 *closed and strictly convex plane curve enclosed a convex body K with length L and enclosed area A. Then*

$$\begin{aligned} h_1(K, S(K))^2 &= \left(\max_u \left|p_K(u) - p_{S(K)}(u)\right|\right)^2 \\ &\leqslant \frac{4\pi^2 - 33}{96\pi^2}\left(4\pi\left(A(K) + |\tilde{A}(K)|\right) - L^2(K)\right) \end{aligned}$$

and

$$\begin{aligned} h_2(K, S(K))^2 &= \int_0^{2\pi} \left|p_K(\theta) - p_{S(K)}(\theta)\right|^2 d\theta \\ &\leqslant \frac{1}{18\pi}\left(4\pi\left(A(K) + |\tilde{A}(K)|\right) - L^2(K)\right), \end{aligned}$$

where $p_K(\theta)$ *denotes the Minkowski support function of the convex body K such that* $p(\theta) + p''(\theta) \neq 0$ *and* $S(K)$ *denotes the Steiner disc associated with K such that*

$$4\pi\left(A(S(K)) + |\tilde{A}(S(K))|\right) - L^2(S(K)) = 0. \tag{7}$$

REMARK 4. For arbitrary $\varepsilon > 0$ such that

$$\varphi(K) = 4\pi\left(A(K) + |\tilde{A}(K)|\right) - L^2(K) < \varepsilon,$$

since

$$\frac{4\pi^2 - 33}{96\pi^2} < \frac{1}{18\pi},$$

by using Theorem 1.6 and (7) it follows that

$$\begin{aligned}\max\left\{h_1(K,S(K))^2, h_2(K,S(K))^2\right\} &\leqslant \frac{1}{18\pi}|\varphi(K) - \varphi(S(K))| \\ &< \frac{\varepsilon}{18\pi},\end{aligned}$$

which implies that the reverse isoperimetric inequality (2) does have good stability properties with respect to both the distances h_1 and h_2.

The paper is organized as follows. In section 2, we recall some basic facts about plane convex geometry. In section 3, we present a simpler proof of Theorem 1.4 by using Fourier series, which is different from the approach in [1]. In section 4, we investigate a stability property of the reverse isoperimetric inequality (5).

2. Geometric quantities and their Fourier series

In this section, we recall some basic facts about plane convex geometry, which will be used later. In this paper, we always assume that γ is a closed and strictly convex plane curve which is sufficiently regular. Actually it should be a C^2 closed and strictly convex curve in the plane $\mathbb{R}^2$, such that the radius of curvature can be defined and the Fourier series needed in the proof converges uniformly. The details can be found in the classical literature [5].

Let $p(\theta)$ denote the Minkowski support function of the curve $\gamma(\theta)$, where θ is the angle between the x-axis and the pointing outward normal vector along the curve, which gives us the parametrization of $\gamma(\theta)$ in terms of θ as follows:

$$\gamma(\theta) = (\gamma_1(\theta), \gamma_2(\theta)) = \left(p(\theta)\cos\theta - p'(\theta)\sin\theta, p(\theta)\sin\theta + p'(\theta)\cos\theta\right).$$

The curvature k and the curvature radius ρ of $\gamma(\theta)$ can be calculated respectively by

$$k(\theta) = \frac{d\theta}{ds} = \frac{1}{p(\theta) + p''(\theta)} > 0 \tag{8}$$

and

$$\rho(\theta) = \frac{ds}{d\theta} = p(\theta) + p''(\theta) > 0. \tag{9}$$

The length L of γ and the area A that it bounds can be also calculated respectively by

$$L = \int_\gamma ds = \int_0^{2\pi} p(\theta)d\theta \tag{10}$$

and

$$A = \frac{1}{2}\int_{\gamma} p(\theta)\,ds = \frac{1}{2}\int_{0}^{2\pi}\left(p(\theta)^2 - p'(\theta)^2\right)d\theta. \tag{11}$$

The locus of curvature centers of γ is as follow:

$$\begin{aligned}\beta(\theta) &= \gamma(\theta) + \rho(\theta)N(\theta)\\ &= \left(-p'(\theta)\sin\theta - p''(\theta)\cos\theta, p'(\theta)\cos\theta - p''(\theta)\sin\theta\right),\end{aligned}$$

where $N(\theta) = (-\cos\theta, -\sin\theta)$ is the unit inward pointing normal vector along the curve, and the oriented area of the domain enclosed by β can be given by [3]:

$$\tilde{A} = \frac{1}{2}\int_{0}^{2\pi}\left(p'(\theta)^2 - p''(\theta)^2\right)d\theta. \tag{12}$$

Since the curve γ is a C^2 closed and strictly convex curve, the Minkowski support function of the convex body K included by γ is always C^2 bounded and 2π-periodic, which yields the Fourier series in the following form:

$$p(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta). \tag{13}$$

Differentiation of (13) with respect to θ gives us

$$p'(\theta) = \sum_{n=1}^{\infty} n(-a_n\sin n\theta + b_n\cos n\theta) \tag{14}$$

and

$$p''(\theta) = -\sum_{n=1}^{\infty} n^2(a_n\cos n\theta + b_n\sin n\theta). \tag{15}$$

Thus, by (13), (14), (15) and the Parseval equality we can express these geometric quantities in terms of the Fourier coefficients of $p(\theta)$.

$$\rho(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta) - \sum_{n=1}^{\infty} n^2(a_n\cos n\theta + b_n\sin n\theta), \tag{16}$$

$$L(K) = 2\pi a_0, \tag{17}$$

$$A(K) = \pi a_0^2 - \frac{\pi}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)\left(a_n^2+b_n^2\right), \tag{18}$$

$$|\tilde{A}(K)| = \frac{\pi}{2}\sum_{n=2}^{\infty} n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right). \tag{19}$$

3. The proof of our main Theorem

In this section, we prove Theorem 1.4 by using Fourier series.

Proof of Theorem 1.4. By (16), one can easily get

$$\int_0^{2\pi} \rho(\theta)^2 d\theta = 2\left(\pi a_0^2 - \frac{\pi}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)\left(a_n^2+b_n^2\right) + \frac{\pi}{2}\sum_{n=2}^{\infty} n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right)$$
$$= 2\pi\left(a_0^2 + \frac{1}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)^2\left(a_n^2+b_n^2\right)\right),$$

then by (17), (18) and (19) we have

$$\begin{aligned}
\Phi(K,\varepsilon) &= \int_0^{2\pi}\rho(\theta)^2 d\theta - \frac{L^2 - 2\pi\left(A - \varepsilon|\tilde{A}|\right)}{\pi} \\
&= \frac{2\pi^2\left(a_0^2 + \frac{1}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)^2\left(a_n^2+b_n^2\right)\right) - (2\pi a_0)^2}{\pi} \\
&\quad + \frac{2\pi\left(\pi a_0^2 - \frac{\pi}{2}\sum_{n=2}^{\infty}\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right) - 2\pi\varepsilon\left(\frac{\pi}{2}\sum_{n=2}^{\infty} n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right)}{\pi} \\
&= \pi\left(\sum_{n=2}^{\infty}\left(n^2-1\right)^2\left(a_n^2+b_n^2\right) - \sum_{n=2}^{\infty}\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right) \\
&\quad - \pi\left(\varepsilon\sum_{n=2}^{\infty} n^2\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right) \\
&= \pi\sum_{n=2}^{\infty}\left(n^2-1\right)\left((1-\varepsilon)n^2-2\right)\left(a_n^2+b_n^2\right).
\end{aligned}$$

Observing that $\Phi(K,\varepsilon)$ is a linear function with respect to ε such that

$$\Phi\left(K,\frac{1}{2}\right) = \frac{\pi}{2}\sum_{n=3}^{\infty}\left(n^2-1\right)\left(n^2-4\right)\left(a_n^2+b_n^2\right) \geqslant 0,$$

thus for any $0 < \varepsilon \leqslant \frac{1}{2}$ we have

$$\Phi(K,\varepsilon) \geqslant \Phi\left(K,\frac{1}{2}\right) \geqslant 0,$$

which completes the proof of (5).

Note that by the discussion in Remark 2, we actually prove

$$L^2 \leqslant 4\pi A + 2\pi|\tilde{A}|(1-\varepsilon)$$

satisfied for any $\varepsilon \leqslant \frac{1}{2}$. In particular, it follows from the expression of $\Phi\left(K, \frac{1}{2}\right)$ that, for arbitrary $\varepsilon > \frac{1}{2}$, $\Phi(\varepsilon)$ which relies on the Fourier coefficients of $p(\theta)$ can not always be non-negative for an arbitrary curve γ. Thus the inequality

$$L^2 \leqslant 4\pi A + \pi|\tilde{A}|$$

is actually a sharp version of (3). That is to say that the best constant ε in the inequality

$$L^2 \leqslant 4\pi A + \varepsilon|\tilde{A}|$$

is π.

On the other hand, if γ is a circle, equality in (5) holds clearly. Conversely, for $\varepsilon = \frac{1}{2}$, if equality in (5) holds, then it follows from the expression of $\Phi\left(K, \frac{1}{2}\right)$ that the Minkowski support function of γ is of the form

$$p(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta.$$

For arbitrary $\varepsilon < \frac{1}{2}$, if equality holds, then the expression of $\Phi(K, \varepsilon)$ implies that the support function is of the form

$$p(\theta) = a_0 + a_1 \cos\theta + b_1 \sin\theta. \quad \square$$

4. The stability of the isoperimetric inequality

Let K and M be two convex bodies with respective Minkowski support functions p_K and p_M. The most frequently used function to measure the deviation between K and M is the Hausdorff distance

$$h_1(K, M) = \max_u |p_K(u) - p_M(u)|. \tag{20}$$

Another such measure which appears to be of particular value with respect to stability problems is the measure that corresponds to the L^2-metric in function space, which is defined by

$$h_2(K, M) = \left(\int_0^{2\pi} |p_K(\theta) - p_M(\theta)|^2 d\theta\right)^{\frac{1}{2}}. \tag{21}$$

It is obvious that $h_1(K, M) = 0$ or $h_2(K, M) = 0$ if and only if $K = M$.

We now consider the stability property of the reverse isoperimetric inequality (5) with respect to the deviation measures h_1 and h_2.

THEOREM 4.1. *Let K be a convex body enclosed by a C^2 closed and strictly convex plane curve γ with area $A(K)$ and perimeter $L(K)$, and let $\tilde{A}(K)$ denote the oriented area of the domain enclosed by the locus of curvature centers of γ. $S(K)$ denotes the Steiner disc associated with K. Then*

$$h_1(K, S(K))^2 \leqslant \frac{C(\varepsilon)}{\pi}\left(\int_0^{2\pi} \rho(\theta)^2 d\theta - \frac{L^2 - 2\pi\left(A - \varepsilon|\tilde{A}|\right)}{\pi}\right) \tag{22}$$

and

$$h_1(K,S(K))^2 \leqslant \frac{C(\varepsilon)}{\pi^2}\left(4\pi A + 2\pi|\tilde{A}|(1-\varepsilon) - L^2\right)$$
$$\leqslant \frac{C(\varepsilon)}{\pi^2}\left(4\pi\left(A+|\tilde{A}|\right) - L^2\right)$$

for arbitrary $0<\varepsilon<\frac{1}{2}$, *where*

$$C(\varepsilon) = \begin{cases} \frac{\sqrt{\varepsilon}}{2(1-\varepsilon)}\left(\sqrt{\varepsilon} - \pi\cot\frac{\pi}{\sqrt{\varepsilon}} - \frac{2\sqrt{\varepsilon}}{\varepsilon-1} - \frac{3}{2\sqrt{\varepsilon}}\right), & \frac{1}{\sqrt{\varepsilon}} \notin \mathbb{N} \\ \frac{\sqrt{\varepsilon}}{2(1-\varepsilon)}\left(\sum_{n=2-\frac{1}{\sqrt{\varepsilon}}}^{2+\frac{1}{\sqrt{\varepsilon}}} \frac{1}{n} - \frac{3}{2\sqrt{\varepsilon}}\right), & \frac{1}{\sqrt{\varepsilon}} \in \mathbb{N}, \end{cases}$$

and equality in (22) *holds if* γ *is a circle.*

Proof of Theorem 4.1. Without loss of generality, we may assume $\overrightarrow{s}(K)=0$. Because of (13), (14) and (15), the support functions p_K and $p_{S(K)}$ have the following Fourier series:

$$p_K(\theta) = \frac{L(K)}{2\pi} + \sum_{n=2}^{\infty}(a_n\cos n\theta + b_n\sin n\theta) \tag{23}$$

and

$$p_{S(K)}(\theta) = \frac{L(K)}{2\pi}. \tag{24}$$

One can observe that (23) and (24) yield an explicit expression (in terms of the Fourier coefficients) for the quantity

$$\int_0^{2\pi}\rho(\theta)^2 d\theta - \frac{L^2 - 2\pi\left(A-\varepsilon|\tilde{A}|\right)}{\pi} = \pi\sum_{n=2}^{\infty}\left(n^2-1\right)\left((1-\varepsilon)n^2-2\right)\left(a_n^2+b_n^2\right).$$

Since it is easily seen that

$$|a_n\cos n\theta + b_n\sin n\theta| \leqslant \sqrt{a_n^2+b_n^2},$$

it follows that

$$\left|p_K(\theta) - p_{S(K)}(\theta)\right| = \left|\frac{L(K)}{2\pi} + \sum_{n=2}^{\infty}(a_n\cos n\theta + b_n\sin n\theta) - \frac{L(K)}{2\pi}\right|$$
$$\leqslant \sum_{n=2}^{\infty}|a_n\cos n\theta + b_n\sin n\theta|$$
$$\leqslant \sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2}.$$

Recall that if p is not an integer, by Fourier series calculation we have

$$\pi\cot p\pi = \frac{1}{p} - 2p\sum_{n=1}^{\infty}\frac{1}{n^2-p^2}.$$

Together with $0<\varepsilon<\frac{1}{2}$ then we can calculate that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{(\varepsilon n^2-1)(n^2-1)}&=\sum_{n=2}^{\infty}\frac{\varepsilon}{1-\varepsilon}\left(\frac{1}{(\varepsilon n^2-1)}-\frac{1}{\varepsilon(n^2-1)}\right)\\&=\frac{1}{1-\varepsilon}\left(\sum_{n=2}^{\infty}\frac{1}{n^2-\frac{1}{\varepsilon}}-\sum_{n=2}^{\infty}\frac{1}{n^2-1}\right)\\&=\frac{\sqrt{\varepsilon}}{2(1-\varepsilon)}\left(\sqrt{\varepsilon}-\pi\cot\frac{\pi}{\sqrt{\varepsilon}}-\frac{2\sqrt{\varepsilon}}{\varepsilon-1}-\frac{3}{2\sqrt{\varepsilon}}\right)\end{aligned}$$

for any $\frac{1}{\sqrt{\varepsilon}}$ that is not an integer. On the other hand, when $\frac{1}{\sqrt{\varepsilon}}$ is an integer, we can calculate that

$$\sum_{n=2}^{\infty}\frac{1}{(\varepsilon n^2-1)(n^2-1)}=\frac{\sqrt{\varepsilon}}{2(1-\varepsilon)}\left(\sum_{n=2-\frac{1}{\sqrt{\varepsilon}}}^{2+\frac{1}{\sqrt{\varepsilon}}}\frac{1}{n}-\frac{3}{2\sqrt{\varepsilon}}\right).$$

Thus, by using Hölder's inequality, we have

$$\begin{aligned}h_1(K,S(K))^2&\leqslant\left(\sum_{n=2}^{\infty}\frac{1}{(\varepsilon n^2-1)(n^2-1)}\right)\left(\sum_{n=2}^{\infty}\left(\varepsilon n^2-1\right)\left(n^2-1\right)\left(a_n^2+b_n^2\right)\right)\\&=C(\varepsilon)\sum_{n=2}^{\infty}\left(\varepsilon n^2-1\right)\left(n^2-1\right)\left(a_n^2+b_n^2\right)\\&=\frac{C(\varepsilon)}{\pi}\left(\int_0^{2\pi}\rho(\theta)^2d\theta-\frac{L^2-2\pi\left(A-\varepsilon|\tilde{A}|\right)}{\pi}\right)\end{aligned}$$

for arbitrary $0<\varepsilon<\frac{1}{2}$, where

$$C(\varepsilon)=\begin{cases}\frac{\sqrt{\varepsilon}}{2(1-\varepsilon)}\left(\sqrt{\varepsilon}-\pi\cot\frac{\pi}{\sqrt{\varepsilon}}-\frac{2\sqrt{\varepsilon}}{\varepsilon-1}-\frac{3}{2\sqrt{\varepsilon}}\right), & \frac{1}{\sqrt{\varepsilon}}\notin\mathbb{N}\\ \frac{\sqrt{\varepsilon}}{2(1-\varepsilon)}\left(\sum\limits_{n=2-\frac{1}{\sqrt{\varepsilon}}}^{2+\frac{1}{\sqrt{\varepsilon}}}\frac{1}{n}-\frac{3}{2\sqrt{\varepsilon}}\right), & \frac{1}{\sqrt{\varepsilon}}\in\mathbb{N}.\end{cases}$$

On the other hand, since

$$\begin{aligned}\int_0^{2\pi}\rho(\theta)^2d\theta-\frac{L^2-2\pi\left(A-\varepsilon|\tilde{A}|\right)}{\pi}&=2\left(A+|\tilde{A}|\right)-\frac{L^2-2\pi\left(A-\varepsilon|\tilde{A}|\right)}{\pi}\\&=\frac{4\pi A+2\pi|\tilde{A}|(1-\varepsilon)-L^2}{\pi},\end{aligned}$$

together with (22) we derive the second inequality. Furthermore, if γ is a circle, equality in (22) holds clearly. $\square$

THEOREM 4.2. *Under the same assumptions as in Theorem 4.1, we have that*

$$h_2(K,S(K))^2 \leqslant \int_0^{2\pi} \rho(\theta)^2 d\theta - \frac{L^2 - 2\pi\left(A - \varepsilon|\tilde{A}|\right)}{\pi} \tag{25}$$

holds for arbitrary $\varepsilon \leqslant \frac{5}{12}$ *and equality in* (25) *holds if* γ *is a circle. Furthermore, for* $\varepsilon = \frac{5}{12}$*, if equality in* (25) *holds, then the Minkowski support function of the convex body* K *is of the form*

$$p_K(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta.$$

For arbitrary $\varepsilon < \frac{5}{12}$*, if equality holds, then the support function is of the form*

$$p_K(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta.$$

Proof of Theorem 4.2. As in the proof of Theorem 4.1, we use (23), (24) and Parseval's equality to deduce that

$$h_2(K,S(K))^2 = \int_0^{2\pi} \left|p_K(\theta) - p_{S(K)}(\theta)\right|^2 d\theta = \pi\sum_{n=2}^{\infty}\left(a_n^2 + b_n^2\right).$$

Together with the result in section 3 we have

$$\begin{aligned}
\psi(K,\varepsilon) &= \left(\int_0^{2\pi} \rho(\theta)^2 d\theta - \frac{L^2 - 2\pi\left(A - \varepsilon|\tilde{A}|\right)}{\pi}\right) - h_2(K,S(K))^2 \\
&= \pi\sum_{n=2}^{\infty}\left(n^2-1\right)\left((1-\varepsilon)n^2-2\right)\left(a_n^2+b_n^2\right) - \pi\sum_{n=2}^{\infty}\left(a_n^2+b_n^2\right) \\
&= \pi\sum_{n=2}^{\infty}\left(\left(n^2-1\right)\left((1-\varepsilon)n^2-2\right)-1\right)\left(a_n^2+b_n^2\right).
\end{aligned}$$

Therefore, for arbitrary $\varepsilon \leqslant \frac{5}{12}$ we always have $\psi(K,\varepsilon) \geqslant 0$, which is equivalent to

$$h_2(K,S(K))^2 \leqslant \int_0^{2\pi} \rho(\theta)^2 d\theta - \frac{L^2 - 2\pi\left(A - \varepsilon|\tilde{A}|\right)}{\pi}.$$

If γ is a circle, equality in (25) holds clearly. Conversely, for $\varepsilon = \frac{5}{12}$, if equality in (25) holds, then it follows from the expression of $\psi\left(K, \frac{5}{12}\right)$ that the Minkowski support function of γ is of the form

$$p_K(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta.$$

For arbitrarily $\varepsilon < \frac{5}{12}$, if equality holds, then it follows from the expression of $\psi(K,\varepsilon)$ that the support function is of the form

$$p_K(\theta) = a_0 + a_1\cos\theta + b_1\sin\theta. \quad \square$$

REMARK 5. The combination of Theorem 4.1 and 4.2 leads to

$$\max\left\{h_1\left(K,S(K)\right)^2,h_2\left(K,S(K)\right)^2\right\}\leqslant\widetilde{C}(\varepsilon)\left(\int_0^{2\pi}\rho(\theta)^2d\theta-\frac{L^2-2\pi\left(A-\varepsilon\left|\tilde{A}\right|\right)}{\pi}\right)\tag{26}$$

for arbitrary $0<\varepsilon\leqslant\frac{5}{12}$, where

$$\widetilde{C}(\varepsilon)=\max\left\{\frac{C(\varepsilon)}{\pi},1\right\}$$

and $C(\varepsilon)$ is defined as in Theorem 4.1. The estimate (26) states that the isoperimetric inequality (5) we derive does have the good stability property with respect to both the Hausdorff distance and the L^2-metric.

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