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FUGLEDE–PUTNAM'S THEOREM FOR *W*–HYPONORMAL OPERATORS

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Abstract. An asymmetric Fuglede-Putnam's Theorem for w-hyponormal operators and dominant operators is proved, as a consequence of this result, we obtain that the range of the generalized derivation induced by the above classes of operators is orthogonal to its kernel.

1. Introduction

For complex Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, $L(\mathscr{H}),L(\mathscr{K})$ and $L(\mathscr{H},\mathscr{K})$ denote the set of all bounded linear operators on $\mathscr{H}$, the set of all bounded linear operators on $\mathscr{K}$ and the set of all bounded linear transformations from $\mathscr{H}$ to $\mathscr{K}$ respectively. A bounded operator $A \in L(\mathscr{H})$ is called normal if $A^*A = AA^*$. According to [5, 15], a bounded operator is called dominant if

$$(A-\lambda I)\mathscr{H} \subset (A-\lambda I)^*\mathscr{H}, \text{ for all } \lambda \in \mathbb{C}.$$

This condition is equivalent to the existence of a positive constant M_λ for each $\lambda \in \mathbb{C}$ such that

$$(A-\lambda I)(A-\lambda I)^* \leqslant M_\lambda (A-\lambda I)^*(A-\lambda I).$$

If there exist a constant M such that $M_\lambda \leqslant M$ for all $\lambda \in \mathbb{C}$, then A is called M-hyponormal, and if $M = 1$, A is hyponormal. Easily we see the following inclusion relations

$$\{Normal\} \subseteq \{Hyponormal\} \subseteq \{M - Hyponormal\} \subseteq \{Dominant\}.$$

Also A is called p-hyponormal [1, 6, 7, 18], if $(A^*A)^p \geqslant (AA^*)^p$ for some $0 < p \leqslant 1$, log-hyponormal [16] if A is invertible operator and satisfies $\log(A^*A) \geqslant \log(AA^*)$, w-hyponormal if $|\widetilde{A}| \geqslant |A| \geqslant |(\widetilde{A})^*|$, where $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ is the Aluthge transformation. It was shown in [2] and [3] that the class of w-hyponormal operators contains both the $p-$ and log-hyponormal operators. We have the following inclusion

$$\{Normal\} \subset \{Hyponormal\} \subset \{p - Hyponormal\} \subset \{w - Hyponormal\}.$$

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$$\{inversible - hyponormal\} \subset \{inversible - p - hyponrmal\}$$

$$\subset \{\log - hyponormal\} \subset \{w - hyponormal\}.$$

If an operator A is p-hyponormal, then $\ker A \subset \ker A^*$, and if A is $\log$-hyponormal, then $\ker A = \ker A^*$. However, if A is w-hyponormal, it is not known whether the kernel condition $\ker A \subset \ker A^*$ holds. Nevertheless in ([2, 3]) w-hyponormal operators have many properties similar to those of p-hyponormal operators.

The familiar Fuglede-Putnam's theorem asserts that if $A \in L(\mathscr{H})$ and $B \in L(\mathscr{K})$ are normal operators and $AX = XB$ for some operators $X \in L(\mathscr{K}, \mathscr{H})$, then $A^*X = XB^*$ ([9], [14]). Many authors have extended this theorem for several classes of operators, recently A. Uchiyama and k. Tanahashi [17] proved that Fuglede-Putnam's theorem holds for p-hyponormal or $\log$-hyponormal and dominant operators, B. P. Duggal [8] and I. H. Jeon, K. Tanahashi and A. Uchiyama [13] proved that Fuglede-Putnam's theorem holds for p-hyponormal or $\log$-hyponormal. We say that the pair (A,B) satisfy Fuglede-Putnam's theorem if $AX = XB$ implies $A^*X = XB^*$.

In this work, we prove that if either

1. A is dominant and B^* is w-hyponormal such that $\ker B^* \subset \ker B$ or

2. A is w-hyponormal and B^* is injective w-hyponormal or

3. A is w-hyponormal such that $\ker A \subset \ker A^*$ and B^* is w-hyponormal such that $\ker B^* \subset \ker B$,

then the pair (A,B) satisfy Fuglede-Putnam's theorem at the end of this paper we study the orthogonality of the range and the null space of the generalized derivation for some classes of operators.

Let $A, B \in L(\mathscr{H})$, we define the generalized derivation $\delta_{A,B}$ induced by A and B by

$$\delta_{A,B}(X) = AX - XB, \text{ for all } X \in L(\mathscr{H}).$$

DEFINITION 1.1. [4] Given subspaces $\mathscr{M}$ and $\mathscr{N}$ of a Banach space $\mathscr{V}$ with norm $\|\cdot\|$. $\mathscr{M}$ is said to be orthogonal to $\mathscr{N}$ if $m + n \geqslant \|n\|$ for all $m \in \mathscr{M}$ and $n \in \mathscr{N}$.

J.H. Anderson and C. Foias [4] proved that if A and B are normal, S is an operator such that $AS = SB$, then

$$\| \delta_{A,B}(X) - S \| \geqslant \| S \|, \text{ for all } X \in L(\mathscr{H}).$$

Where $\|\cdot\|$ is the usual operator norm. Hence the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. The orthogonality here is understood to be in the sense of definition [4].

2. Preliminaries

We will recall some known results which will be used in the sequel.

DEFINITION 2.1. [1] Let $A \in L(\mathscr{H})$ and $A = U|A|$ be the polar decomposition of A, the Aluthge transformation of A is $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$.

THEOREM 2.2. [12] *An operator $A \in L(\mathscr{H})$ is w-hyponormal if and only if*

$$(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}})^{\frac{1}{2}} \geqslant |A^*|.$$

LEMMA 2.3. [15] *Let $A \in L(\mathscr{H})$ be dominant and $\mathscr{M}$ an invariant subspace for A, then the restriction of A to $\mathscr{M}$ is dominant.*

LEMMA 2.4. [18] *Let $A \in L(\mathscr{H})$ be p-hyponormal and $\mathscr{M}$ an invariant subspace for A, then the restriction of A to $\mathscr{M}$ is p-hyponormal.*

THEOREM 2.5. [17] *Let $A \in L(\mathscr{H})$ be dominant and $B^* \in L(\mathscr{K})$ be p-hyponormal or $\log$-hyponormal, then the pair (A,B) satisfy Fuglede-Putnam's theorem.*

THEOREM 2.6. [8] *Let $A \in L(\mathscr{H})$ and $B^* \in L(\mathscr{K})$ are p-hyponormal, then the pair (A,B) satisfy Fuglede-Putnam's theorem, $\overline{R(X)}$ reduces A, $\ker(X)^\perp$ reduces B, and $A\,|_{\overline{R(X)}}$, $B\,|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

THEOREM 2.7. [13] *Let $A \in L(\mathscr{H})$ and $B^* \in L(\mathscr{K})$ are p-hyponormal or $\log$-hyponormal operators, then the pair (A,B) satisfy Fuglede-Putnam's theorem, $\overline{R(X)}$ reduces A, $\ker(X)^\perp$ reduces B, and $A\,|_{\overline{R(X)}}$, $B\,|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

3. Main results

Our goal is to investigate the orthogonality of $R(\delta_{A,B})$ (the range of $\delta_{A,B}$) and $\ker(\delta_{A,B})$ (the kernel of $\delta_{A,B}$) for some operators. We prove that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$ when either (1) A is dominant and B^* is w-hyponormal such that $\ker B^* \subset \ker B$ or (2) A is w-hyponormal and B^* is injective w-hyponormal or (3) A is w-hyponormal such that $\ker A \subset \ker A^*$ and B^* is w-hyponormal such that $\ker B^* \subset \ker B$. Before proving these results, we need the following ones.

THEOREM 3.1. *Let $A \in L(\mathscr{H})$ be w-hyponormal and $\mathscr{M}$ an invariant subspace for A, then the restriction of A to $\mathscr{M}$ is w-hyponormal.*

Proof. Let P be the orthogonal projection on $\mathscr{M}$. We have

$$AP = PAP.$$

It is easy to see that $APPA^* \leqslant AA^*$, therefore

$$|(AP)^*|^2 \leqslant |A^*|^2.$$

By Löwner-Heinz theorem [11], we obtain

$$|(AP)^*| \leqslant |A^*|. \tag{3.1}$$

Since $|AP|^2 = PA^*AP = P|A|^2P$, by Hansen's inequality [10] we get

$$|AP| \geqslant P|A|P,$$

hence

$$P|AP|P \geqslant P|A|P.$$

Since $\ker P \subset \ker|AP| \subset \ker|A|$, from [12, Lemme 8] we obtain

$$|AP| \geqslant |A|. \tag{3.2}$$

We have A is w-hyponormal, then

$$(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}})^{\frac{1}{2}} \geqslant |A^*|.$$

By (3.1) and [19, Lemme 5] we obtain

$$(|(AP)^*|^{\frac{1}{2}}|A||(AP)^*|^{\frac{1}{2}})^{\frac{1}{2}} \geqslant |(AP)^*|. \tag{3.3}$$

By (3.2) we obtain

$$|(AP)^*|^{\frac{1}{2}}|A||(AP)^*|^{\frac{1}{2}} \leqslant |(AP)^*|^{\frac{1}{2}}|AP||(AP)^*|^{\frac{1}{2}}.$$

Applying Löwner Heinz theorem [11], we get

$$(|(AP)^*|^{\frac{1}{2}}|A||(AP)^*|^{\frac{1}{2}})^{\frac{1}{2}} \leqslant (|(AP)^*|^{\frac{1}{2}}|AP||(AP)^*|^{\frac{1}{2}})^{\frac{1}{2}}. \tag{3.4}$$

Hence

$$(|(AP)^*|^{\frac{1}{2}}|AP||(AP)^*|^{\frac{1}{2}})^{\frac{1}{2}} \geqslant |(AP)^*|$$

hold by (3.3) and (3.4), and this shows that AP is w-hyponormal. □

LEMMA 3.2. *Let $A \in L(\mathscr{H})$ be w-hyponormal operator, let $\mathscr{M}$ be an invariant subspace for A and a reduced subspace for $\widetilde{A}$ such that $\widetilde{A}\,|_{\mathscr{M}}$ the restriction of $\widetilde{A}$ to $\mathscr{M}$ is an injective normal operator, then $A\,|_{\mathscr{M}} = \widetilde{A}\,|_{\mathscr{M}}$ and $\mathscr{M}$ reduces A.*

Proof. Let

$$\widetilde{A} = \begin{pmatrix} A_0 & 0 \\ 0 & * \end{pmatrix}, A = \begin{pmatrix} A_1 & B \\ 0 & D \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}.$$

Since A is w-hyponormal, then $|\widetilde{A}| \geqslant |A| \geqslant |(\widetilde{A})^*|$. Let P be the orthogonal projection on $\mathscr{M}$, then

$$(A_0^*A_0)^{\frac{1}{2}} = P|\widetilde{A}|P \geqslant P|A|P \geqslant P|(\widetilde{A})^*|P = (A_0A_0^*)^{\frac{1}{2}}.$$

By applying Löwner Heinz theorem [11] we get

$$|A_0|^{\frac{1}{2}} = P|\widetilde{A}|^{\frac{1}{2}}P \geqslant P|A|^{\frac{1}{2}}P \geqslant P|(\widetilde{A})^*|^{\frac{1}{2}}P = |A_0^*|^{\frac{1}{2}}.$$

Since $|A|^{\frac{1}{2}}A = \widetilde{A}|A|^{\frac{1}{2}}$ and $P|A|^{\frac{1}{2}}P = |A_0|^{\frac{1}{2}}$, we deduce that

$$|A_0|^{\frac{1}{2}}A_1 = A_0|A_0|^{\frac{1}{2}}$$

We have A_0 is an injective normal operator, then $A_1 = A\,|_{\mathscr{M}} = A_0 = \widetilde{A}\,|_{\mathscr{M}}$, consequently

$$A = \begin{pmatrix} A_0 & B \\ 0 & D \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}.$$

Hence

$$A^*A = \begin{pmatrix} A_0^*A_0 & A_0^*B \\ B^*A_0 & B^*B + D^*D \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}.$$

So we can write $|A|^{\frac{1}{2}}$ as

$$|A|^{\frac{1}{2}} = \begin{pmatrix} |A_0|^{\frac{1}{2}} & X \\ X^* & Y \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}.$$

Since

$$P|A|^{\frac{1}{2}}|A|^{\frac{1}{2}}P = |A_0|,$$

then $|A_0| = |A_0| + XX^*$, and thus $X = 0$.

It follows that $|A| = |A_0| \oplus Y^2$ implying $A^*A = A_0^*A_0 \oplus Y^4$. Consequently we get $A_0^*B = 0$ and so $B = 0$. This shows that $\mathscr{M}$ reduces A. □

THEOREM 3.3. *Let $A \in L(\mathscr{H})$ be dominant operator and $B^* \in L(\mathscr{K})$ be w-hyponormal such that $\ker B^* \subset \ker B$, then the pair (A,B) satisfy Fuglede-Putnam's theorem.*

Proof. We consider two cases.

Case 1. If B^* is injective. Assume that $AX = XB$ for some $X \in L(\mathscr{K},\mathscr{H})$. Since $\overline{R(X)}$ is invariant for A and $(\ker X)^{\perp}$ is invariant for B^*, we consider the following decompositions

$$\mathscr{H} = \overline{R(X)} \oplus \overline{R(X)}^{\perp}, \mathscr{K} = (\ker X)^{\perp} \oplus \ker X,$$

then we have

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}.$$

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : (\ker X)^{\perp} \oplus \ker X \to \overline{R(X)} \oplus \overline{R(X)}^{\perp}.$$

From $AX = XB$ we get

$$A_1X_1 = X_1B_1. \tag{3.5}$$

Let $B_1^* = U^*|B_1^*|$ be the polar decomposition of B_1^*. Multiply the two members of (3.5) by $|B_1^*|^{\frac{1}{2}}$, we obtain

$$A_1X_1|B_1^*|^{\frac{1}{2}} = X_1B_1|B_1^*|^{\frac{1}{2}},$$

hence

$$A_1X_1|B_1^*|^{\frac{1}{2}} = X_1|B_1^*|^{\frac{1}{2}}(\widetilde{B_1^*})^*.$$

Since A_1 is dominant from Lemma 2.3 and B_1^* is w-hyponormal from Theorem 3.1, then $\widetilde{B_1^*}$ is semi-hyponorml, applying Theorem 2.5 we get the pair $(A_1, \widetilde{B_1^*})$ satisfy Fuglede-Putnam's theorem. Therefore $A_1 \mid_{\overline{R(X_1|B_1^*|^{\frac{1}{2}})}}$ and $\widetilde{B_1^*} \mid_{\ker(X_1|B_1^*|^{\frac{1}{2}})^{\perp}}$ are normal operators.

Since X_1 is injective with dense range and $|B_1^*|^{\frac{1}{2}}$ is injective thus

$$\overline{R(X_1|B_1^*|^{\frac{1}{2}})} = \overline{R(X_1)} = \overline{R(X)},$$

and

$$\ker(X_1|B_1^*|^{\frac{1}{2}}) = \ker(X_1) = \ker(X).$$

Applying Lemma 3.2 we get $B_1^* \mid_{\ker(X)^{\perp}}$ is normal and $\ker(X)^{\perp}$ reduces B^*. Therefore $\overline{R(X)}$ reduces A and $\ker(X)^{\perp}$ reduces B, it follows that $A_2 = B_2 = 0$. Since A_1 and B_1 are normal operators, then $A_1^*X_1 = X_1B_1^*$. Therefore $A^*X = XB^*$.

Case 2. If B^* is not injective, the condition $\ker B^* \subset \ker B$ implies that $\ker B^*$ reduces B^*, since $\ker A$ reduces A, the operators A and B can be written on the following decompositions

$$\mathscr{H} = (\ker A)^{\perp} \oplus \ker A,\ \mathscr{K} = (\ker B^*)^{\perp} \oplus \ker B^*,$$

as follows

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since A_1 is injective dominant operator and B_1^* is injective w-hyponormal operator. Let

$$X : (\ker B^*)^{\perp} \oplus \ker B^* \to (\ker A)^{\perp} \oplus \ker A,$$

and let $X = [X_{ij}]_{i,j=1}^2$ be the matrix representation, then $AX = XB$ implies that $A_1X_{11} = X_{11}B_1$ and $X_{12} = X_{21} = 0$. From the first case we deduce that $A_1^*X_{11} = X_{11}B_1^*$. Thus $A^*X = XB^*$. $\square$

REMARK 3.4. A necessary condition for the pair (A,A^*) to satisfy Fuglede-Putnam's theorem is $\ker A \subset \ker A^*$. Since for a w-hyponormal operator this is not always true, w-hyponormal operators do not satisfy Fuglede-Putnam's theorem. For example, if P is the orthogonal projection onto $\ker A$, with A is w-hyponormal, then $AP = PA^*$ but $A^*P \neq PA$. However, if A, B^* are w-hyponormal operators such that $\ker A$ reduces A and $\ker B^*$ reduces B^*, then the pair (A,B) satisfy Fuglede-Putnam's theorem. The following results prove more.

THEOREM 3.5. *Let $A \in L(\mathscr{H})$ be w-hyponormal operator and $B^* \in L(\mathscr{K})$ be injective w-hyponormal, then the pair (A,B) satisfy Fuglede-Putnam's theorem.*

Proof. Assume that $AX = XB$ for some $X \in L(\mathscr{K},\mathscr{H})$. Since $\overline{R(X)}$ is invariant for A and $(\ker X)^\perp$ is invariant for B^*, we consider the following decompositions

$$\mathscr{H} = \overline{R(X)} \oplus \overline{R(X)}^\perp,\ \mathscr{K} = (\ker X)^\perp \oplus \ker X,$$

then we have

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}.$$

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : (\ker X)^\perp \oplus \ker X \to \overline{R(X)} \oplus \overline{R(X)}^\perp.$$

From $AX = XB$ we get

$$A_1X_1 = X_1B_1. \tag{3.6}$$

Let $A_1 = V|A_1|$ and $B_1^* = U^*|B_1^*|$ be the polar decompositions of A_1 and B_1^*. Multiply the two members of (3.6) by $|A_1|^{\frac{1}{2}}$ and $|B_1^*|^{\frac{1}{2}}$, we obtain

$$|A_1|^{\frac{1}{2}}A_1X_1|B_1^*|^{\frac{1}{2}} = |A_1|^{\frac{1}{2}}X_1B_1|B_1^*|^{\frac{1}{2}},$$

hence

$$\widetilde{A_1}|A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}} = |A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}}(\widetilde{B_1^*})^*,$$

Since A_1 and B_1^* are w-hyponormal from Theorem 3.1, it follows that $\widetilde{A_1}$ and $\widetilde{B_1^*}$ are semi-hyponormal, applying Theorem 2.6 we get $\widetilde{A_1}\,|_{\overline{R(|A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}})}}$ and $\widetilde{B_1^*}\,|_{\ker(|A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}})^\perp}$ are unitarily normal operators.

Since $\widetilde{B_1^*}\,|_{\ker(|A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}})^\perp}$ is injective, then $\widetilde{A_1}\,|_{\overline{R(|A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}})}}$ is also injective. As $X_1|B_1^*|^{\frac{1}{2}}$ is injective with dense range and $|A_1|^{\frac{1}{2}}$ is injective, it follows that

$$\overline{R(|A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}})} = \overline{R(X_1)} = \overline{R(X)},$$

and

$$\ker(|A_1|^{\frac{1}{2}}X_1|B_1^*|^{\frac{1}{2}}) = \ker(X_1) = \ker(X).$$

Then $B_1^*\,|_{\ker(X)^\perp}$ and $A_1\,|_{\overline{R(X)}}$ are injective normal operators by Lemma 3.2. Therefore $\overline{R(X)}$ reduces A and $\ker(X)^\perp$ reduces B it follows that $A_2 = B_2 = 0$. Since A_1 and B_1 are normal operators then $A_1^*X_1 = X_1B_1^*$. Therefore $A^*X = XB^*$. □

THEOREM 3.6. *Let $A \in L(\mathscr{H})$ be w-hyponormal operator such that* $\ker A \subset \ker A^*$ *and $B^* \in L(\mathscr{K})$ be w-hyponormal such that* $\ker B^* \subset \ker B$, *then the pair (A,B) satisfy Fuglede-Putnam's theorem.*

Proof. the conditions $\ker B^* \subset \ker B$ and $\ker A \subset \ker A^*$ implies that $\ker B^*$ reduces B^* and $\ker A$ reduces A, the operators A and B can be written with respect to the following decompositions

$$\mathscr{H} = (\ker A)^\perp \oplus \ker A,\ \mathscr{K} = (\ker B^*)^\perp \oplus \ker B^*,$$

as follows

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since A_1 is injective w-hyponormal operator and B_1^* is injective w-hyponormal operator. Let

$$X : (\ker B^*)^\perp \oplus \ker B^* \to (\ker A)^\perp \oplus \ker A,$$

and let $X = [X_{ij}]_{i,j=1}^2$ be the matrix representation, then $AX = XB$ implies that $A_1X_{11} = X_{11}B_1$ and $X_{12} = X_{21} = 0$. From Theorem 3.5 we get $A_1^*X_{11} = X_{11}B_1^*$. Thus $A^*X = XB^*$. □

In the following theorem we prove the orthogonality of the range and the null space of $\delta_{A,B}$, for some classes of operators

THEOREM 3.7. *Let $A, B^* \in L(\mathscr{H})$. If one of the following assertions*

1. *A is dominant and B^* is w-hyponormal with* $\ker B^* \subset \ker B$.
2. *A is w-hyponormal and B^* is injective w-hyponormal.*
3. *A is w-hyponormal operator with* $\ker A \subset \ker A^*$ *and B^* is w-hyponormal with* $\ker B^* \subset \ker B$.

is verified, then $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.

Proof. The pair (A,B) verify the Fuglede-Putman's theorem by Theorem 3.3, Theorem 3.5 and Theorem 3.6. Let $C \in L(\mathscr{H})$ be such that $AC = CB$. According to the following decompositions of $\mathscr{H}$.

$$\mathscr{H} = \mathscr{H}_1 = \overline{R(C)} \oplus \overline{R(C)}^\perp,\ \mathscr{H} = \mathscr{H}_2 = (\ker C)^\perp \oplus \ker C,$$

We can write A, B, C and X

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}, X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

Where A_1 and B_1 are normal operators and X is an operator from $\mathscr{H}_1$ to $\mathscr{H}_2$. Since $AC = CB$, then $A_1C_1 = C_1B_1$. Hence

$$AX - XB - C = \begin{pmatrix} A_1X_1 - X_1B_1 - C_1 & A_2X_2 - X_2B_2 \\ A_1X_3 - X_3B_1 & A_2X_4 - X_4B_2 \end{pmatrix}.$$

Since $C_1 \in \ker(\delta_{A_1,B_1})$ and A_1, B_1 are normal, it follows by [4]

$$\| AX - XB - C \| \geqslant \| A_1X_1 - X_1B_1 - C_1 \| \geqslant \| C_1 \| = \| C \|, \ \forall X \in L(\mathscr{H}).$$

This implies that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$. □

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