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INEQUALITIES FOR PRODUCTS OF ZEROS OF POLYNOMIALS AND ENTIRE FUNCTIONS

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Abstract. Estimates for products of the zeros of polynomials and entire functions are derived. By these estimates, new upper bounds for the counting function are suggested. In appropriate situations we improve the Jensen inequality for the counting functions and the Mignotte inequality for products of the zeros of polynomials.

1. Polynomials

Consider the polynomial

$$P(\lambda) = \sum_{k=0}^{n} c_k \lambda^{n-k} \quad (c_0 = 1) \tag{1.1}$$

with complex coefficients. Enumerate the zeros $z_k(P)$ of P taken with their multiplicities in the descending order: $|z_{k+1}(P)| \leqslant |z_k(P)|$. Set

$$\zeta := \left[1 + \sum_{k=1}^{n} |c_k|^2\right]^{1/2}.$$

The following result is well known [5, p. 129, Theorem (28,4)]. Let

$$|z_1(P)z_2(P)...z_p(P)| > 1 \geqslant |z_{p+1}(P)...z_n(P)|.$$

Then

$$|z_1(P)z_2(P)...z_p(P)| \leqslant \zeta.$$

The product of the moduli of all the zeros of a monic polynomial that lie outside the unit disk is commonly known in the literature as *the Mahler measure of the polynomial*. This term came into common use a bit after the publication of the book [5]. The pointed inequality is often known as *Landau's inequality*. There have been improvements to Landau's inequality in the literature, in particular, see the paper J. Vicente Goncalves

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[3]. This inequality was also discovered by Ostrowski [9]. Mignotte [6] had established the inequality

$$|z_1(P)z_2(P)\ldots z_p(P)|^2 + |z_{p+1}(P)\ldots z_n(P)|^{-2} \leqslant \zeta^2. \tag{1.2}$$

See also Schinzel's book [10, Lemma 13 on p. 244] and Mignotte's book [7], p. 151 and the remark on p. 153.

In the present paper we suggest inequalities for the products of the zeros of P, which improve (1.2) under some restrictions. We also generalize our result to entire functions. Moreover, new bounds for the counting function are suggested. In appropriate situations they improve the Jensen inequality for the counting function.

Now we introduce some notation. Let $\psi_1 = 1$ and ψ_k $(k = 2, \ldots, n)$ be positive numbers having the following property: the sequence

$$m_1 = 1,\ m_j := \frac{\psi_j}{\psi_{j-1}}, \quad j = 2, \ldots, n,$$

is nonincreasing. So

$$\psi_j := \prod_{k=1}^{j} m_k, \quad j = 1, \ldots, n.$$

Take $a_k = c_k/\psi_k$. Then P takes the form

$$P(\lambda) = \sum_{k=0}^{n} a_k \psi_k \lambda^{n-k} \quad (a_0 = \psi_0 = 1). \tag{1.3}$$

Denote

$$\theta(P) := \left[\sum_{k=1}^{n} |a_k|^2\right]^{1/2}.$$

Certainly $\theta(P)$ depends on the choice of the numbers ψ_k.

THEOREM 1.1. *The zeros of the polynomial P defined by (1.3) satisfy the inequalities*

$$\prod_{k=1}^{j} |z_k(P)| \leqslant (m_2 + \theta(P))\psi_j, \quad j = 1, \ldots, n. \tag{1.4}$$

Proof. Introduce the $n \times n$-matrix

$$A = \begin{pmatrix} -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n \\ m_2 & 0 & \ldots & 0 & 0 \\ 0 & m_3 & \ldots & 0 & 0 \\ . & . & \ldots & . & . \\ 0 & 0 & \ldots & m_n & 0 \end{pmatrix}.$$

Enumerate the eigenvalues $\lambda_k(A)$ of A counted with their algebraic multiplicities in the descending order. As it is proved in [1, Section 5.2], $\lambda_k(A) = z_k(P)$ $(k = 1, \ldots, n)$.

We have $A = M + C$, where

$$C = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 0 & 0 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \text{ and } M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ m_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & m_3 & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & m_n & 0 \end{pmatrix}.$$

Let $s_k(A^*) = \sqrt{\lambda_k(AA^*)}$ for $1 \leqslant k \leqslant n$ be the singular values of A^*. By the Weyl inequalities [2, Lemma II.3.3],

$$\prod_{k=1}^{j} |\lambda_k(A)| = \prod_{k=1}^{j} |\lambda_k(A^*)| \leqslant \prod_{k=1}^{j} s_k(A^*). \tag{1.5}$$

By the Ky Fan inequalities [2, Corollary II.2.2],

$$s_{\tau+j-1}(A^*) = s_{\tau+j-1}(M^* + C^*) \leqslant s_\tau(C^*) + s_j(M^*). \tag{1.6}$$

But the matrices MM^* and CC^* are diagonal. Moreover, $s_1(C^*) = \theta(P)$ and $s_k(C^*) = 0$, $k > 1$. In addition, $s_k(M^*) = m_{k+1}$ $(k < n)$; $s_n(M^*) = 0$. By (1.6)

$$s_1(A^*) \leqslant s_1(M^*) + s_1(C^*) = \theta(P) + m_2$$

Taking in (1.6), $\tau = 2$, we obtain

$$s_j(A^*) \leqslant s_2(C^*) + s_{j-1}(M^*) = m_j, \quad j = 2, \dots, n.$$

Hence by (1.5),

$$\prod_{k=1}^{j} |z_k(P)| = \prod_{k=1}^{j} |\lambda_k(A)| \leqslant (m_2 + \theta(P)) \prod_{k=1}^{j} m_k = (m_2 + \theta(P))\psi_j.$$

This completes the proof. □

Taking into account that $|z_{k+1}(P)| \leqslant |z_k(P)|$, by Theorem 1.1 we get

COROLLARY 1.2. *Let P be defined by (1.3). Then $|z_j(P)|^j \leqslant (m_2 + \theta(P))\psi_j$ for $j = 1, \dots, n$. In particular,*

$$\min_j |z_j(P)| \leqslant [(m_2 + \theta(P))\psi_n]^{1/n} \text{ and } \max_j |z_j(P)| \leqslant m_2 + \theta(P).$$

Denote $\Omega(r) = \{z \in \mathbb{C} : |z| \leqslant r\}$ for a positive r. Let $\nu(f, r)$ denote the number of the zeros of f in $\Omega(r)$. Corollary 1.2 implies our next result.

COROLLARY 1.3. *For the polynomial defined by (1.3) we have $\nu(P, r) \leqslant n - j + 1$, provided $r \leqslant [(m_2 + \theta(P))\psi_j]^{1/j}$.*

For example, let $m_k = 1/k$. Then

$$P(\lambda) = \sum_{k=0}^{n} \frac{a_k}{k!}\lambda^{n-k} \ (a_0 = 1). \tag{1.7}$$

Inequality (1.4) takes the form

$$\prod_{k=1}^{j} |z_k(P)| \leqslant \left(\frac{1}{2} + \theta(P)\right)\frac{1}{j!}, \quad j = 1,...,n. \tag{1.8}$$

Hence, $|z_j(P)|^j \leqslant (\frac{1}{2} + \theta(P))\frac{1}{j!} \ (j = 1,...,n)$,

$$\min_j |z_j(P)| \leqslant \left[\left(\frac{1}{2} + \theta(P)\right)\frac{1}{n!}\right]^{1/n} \text{ and } \max_j |z_j(P)| \leqslant \frac{1}{2} + \theta(P).$$

The polynomial defined by (1.7) has in $\Omega(r)$ no more that $n - j + 1$ zeros, provided

$$r \leqslant \left[\left(\frac{1}{2} + \theta(P)\right)\frac{1}{j!}\right]^{1/j}.$$

Let us compare (1.8) with (1.2). Clearly, (1.8) gives us bounds for all j, not only for $j = p$. In addition, in the general case it is hard to determine how many zeros whose absolute values are more than one, a polynomial has. Moreover, for the polynomial defined by (1.7), we have

$$\zeta = \left[1 + \sum_{k=1}^{n} \frac{|a_k|^2}{(k!)^2}\right]^{1/2}.$$

So

$$\zeta \geqslant \left(\frac{1}{2} + \theta(P)\right)\frac{1}{j!} = \left(\frac{1}{2} + \left[\sum_{k=1}^{n} |a_k|^2\right]^{1/2}\right)\frac{1}{j!}$$

for all sufficiently large j. Thus, (1.8) is sharper than (1.2) for all sufficiently large p. A simple example here is $P(z) = (z+2)^n$.

2. Entire functions

Consider the entire function

$$f(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k \ (\lambda \in \mathbb{C})$$

with complex coefficients. Again let $\psi_1 = 1$ and $\psi_k \ (k = 2,3,...)$ be positive numbers, such that the sequence

$$m_1 = 1, \, m_j := \frac{\psi_j}{\psi_{j-1}}, \quad j = 2,3,...,$$

is nonincreasing and tends to zero. Set $a_k = c_k/\psi_k$. Then the entire function takes the form

$$f(\lambda) = 1 + \sum_{k=1}^{\infty} a_k \psi_k \lambda^k \ (\lambda \in \mathbb{C}). \tag{2.1}$$

We assume that

$$\theta(f) := \left[\sum_{k=0}^{\infty} |a_k|^2\right]^{1/2} < \infty. \tag{2.2}$$

We will call *(2.1) the ψ-representation of f*. Obviously,

$$\psi_{k+1}/\psi_k = m_{k+1} \to 0.$$

Since $a_k \to 0$, f is really an entire function. For instance, the function

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{a_k \lambda^k}{k!} \quad (a_0 = 1)$$

has the form (2.2) with $m_k = 1/k \ (k = 1,2,...)$. More generally, the finite order function

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{a_k \lambda^k}{(k!)^\gamma} \ (a_0 = 1\ , \gamma > 0) \tag{2.3}$$

can also be written in the form (2.2) with $m_k = 1/k^\gamma \ (k = 1,2,...)$. Relations (2.3) and (2.2), and Hőlder's inequality imply that function f has order $\rho(f) \leqslant 1/\gamma$. Moreover, for any function f with $f(0) = 1$, whose order is $\rho(f) < \infty$, we can take $\gamma > 1/\rho(f)$, such that representation (2.3) holds with condition (2.2).

Let $z_1(f), z_2(f), ...$ be the zeros of f, taken with multiplicity and enumerated by increasing modulus. If f has $l < \infty$ finite zeros, we set

$$\frac{1}{z_k(f)} = 0 \ (k = l+1, l+2, ...).$$

THEOREM 2.1. *Let f be represented by (2.1) and suppose condition (2.2) holds. Then*

$$\prod_{k=1}^{j} |z_k(f)| > \frac{1}{(m_2 + \theta(f))\psi_j}, \quad j = 1,2,\dots . \tag{2.4}$$

Proof. Consider the polynomial

$$f_n(\lambda) = 1 + \sum_{k=1}^{n} a_k \psi_k \lambda^k. \tag{2.5}$$

Clearly, $\lambda^n f_n(1/\lambda) = P(\lambda)$ is a polynomial. So

$$z_k(P) = 1/z_k(f_n). \tag{2.6}$$

Taking into account that the zeros depend continuously on the coefficients, we obtain the required result, letting in Theorem 1.1 $n \to \infty$. □

Taking into account that $|z_{k+1}(f)| \geqslant |z_k(f)|$, from the previous theorem we obtain the following result

COROLLARY 2.2. *Let f be defined by (2.1) and suppose (2.2) holds. Then*

$$|z_j(f)| > \frac{1}{[(m_2+\theta(f))\psi_j]^{1/j}}, \quad j = 1,2,...,$$

and therefore,

$$\min_{k=1,2,...} |z_k(P)| = |z_1(f)| > \frac{1}{m_2+\theta(f)}.$$

Corollary 2.2 implies the following.

COROLLARY 2.3. *Let f be defined by (2.1) and suppose condition (2.2) holds. Then $\nu(f,r) \leqslant j-1$, provided*

$$r \leqslant \frac{1}{[(m_2+\theta(f))\psi_j]^{1/j}}.$$

In particular, let f be written in the form (2.3), and suppose condition (2.2) hold. Then by Theorem 2.1,

$$\prod_{k=1}^{j} |z_k(f)| > \frac{(j!)^\gamma}{2^{-\gamma}+\theta(f)}, \quad j = 1,2,....$$

Hence,

$$|z_j(f)|^j > \frac{(j!)^\gamma}{2^{-\gamma}+\theta(f)}$$

and $\nu(f,r) \leqslant j-1$, provided

$$r \leqslant \left[\frac{(j!)^\gamma}{2^{-\gamma}+\theta(f)}\right]^{1/j}.$$

Let $M(f,r) = \max_{|z|=r} |f(z)|$. Recall the Jensen inequality

$$\nu(f,r) \leqslant log\, M(f,er), \tag{2.7}$$

provided $f(0) = 1$, cf. [4, p. 13]. Our results can be more convenient than the Jensen inequality in the case when the sums of the Taylor coefficients are simply calculated while for $M(P,r)$ it is difficult to establish sharp estimates. Moreover if $f = P$ is a polynomial of the degree n, for a sufficiently large r, the Jensen inequality gives us the inequality $\nu(P,r) \leqslant n_1$ with $n_1 > n$, which is not useful. Consequently, Corollary 2.3 is sharper than (2.7) in this case.

Furthermore, note that usually, (see for instance Theorem 3.1.2 from [8, p. 244]), in the available literature, lower bounds are presented for the minimal absolute value of the zeros of a polynomial. At the same time (1.4) gives us the upper bound for the minimal absolute value.

REFERENCES

[1] M. I. GIL', *Localization and Perturbation of Zeros of Entire Functions*, Lecture Notes in Pure and Applied Mathematics, 258. CRC Press, Boca Raton, FL, 2010.

[2] I. C. GOHBERG AND M. G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Trans. Mathem. Monographs, v. 18, Amer. Math. Soc., Providence, R. I., 1969.

[3] V. GONÇALVES, *L'inégalité de W. Specht*, Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) **1** (1950), 167–171. (French)

[4] B. YA. LEVIN, *Lectures on Entire Functions*, Trans. of Math. Monographs, v. 150. Amer. Math. Soc., R. I., 1996.

[5] M. MARDEN, *Geometry of Polynomials*, Amer. Math. Soc., Providence, R. I., 1966

[6] M. MIGNOTTE, *An Inequality About Factors of Polynomials*, Mathematics of Computation **28**, 128 (1974), 1153–1157.

[7] M. MIGNOTTE, *Mathematics for Computer Algebra*, Springer-Verlag, New York, 1992.

[8] G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ, AND TH. M. RASSIAS, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific Publ., Singapore, 1994.

[9] A. M. OSTROWSKI, *On an inequality of J. Vicente Gonçalves*, Univ. Lisboa Revista Fac. Ci. A (2) **8** (1960), 115–119.

[10] A. SCHINZEL, *Polynomials with Special Regard to Reducibility*, Encyclopedia of Mathematics and its Applications, 77. Cambridge University Press, Cambridge, 2000.

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