Mathematical
Inequalities
& Applications
Volume 15, Number 4 (2012), 835–844
doi:10.7153/mia-15-71

THE INVERSE THEOREM OF APPROXIMATION THEORY IN SMIRNOV–ORLICZ CLASSES

SADULLA Z. JAFAROV

(*Communicated by I. Perić*)

Abstract. Let Γ be a Dini- smooth curve in the complex plane $\mathbb{C}$. In this study we prove inverse theorem of approximation theory by polynomials in Smirnov-Orlicz classes $E_M(G)$.

1. Introduction and main results

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. This curve separates the plane into two domains $G := \operatorname{int}\Gamma$, $G^- := \operatorname{ext}\Gamma$. Without loss of generality we may assume $0 \in G$. Let D be the unit disc, $T := \partial D$, $D^- := \operatorname{ext} T$. We denote by ϕ the conformal mapping of G^- onto D^- normalized by $\phi(\infty) = \infty$ and $\lim\limits_{z\to\infty} \phi(z)/z > 0$. Let $\psi(w)$ be the inverse to $\phi(z)$.

Let h be a continuous function on $[0,2\pi]$. Its modulus of continuity is defined by

$$\omega(t,h) := \sup\{|h(t_1) - h(t_2)| : t_1,t_2 \in [0,2\pi], \quad |t_1 - t_2| \leqslant t\}, \quad t \geqslant 0.$$

DEFINITION 1. The curve Γ is called Dini-smooth if it has a parametrization

$$\Gamma : \phi_0(s), \qquad 0 \leqslant s \leqslant 2\pi$$

such that $\phi_0'(s)$ is Dini-continuous, i.e.

$$\int_0^{\pi} \frac{\omega(t,\phi_0')}{t}\,dt < \infty$$

and $\phi_0'(s) \neq 0$ [25, p. 48].

Note that, when Γ is Dini-smooth curve then the following inequalities hold

$$\begin{aligned} c_1 \leqslant |\psi'(w)| \leqslant c_2, &\qquad |w| \geqslant 1, \\ c_3 \leqslant |\phi'(z)| \leqslant c_4, &\qquad z \in \overline{G}^-, \end{aligned} \tag{1.1}$$

Mathematics subject classification (2010): 30E10, 41A10, 41A25, 46E30.
Keywords and phrases: Dini-smooth curve, Cauchy singular operator, Faber polynomials, inverse theorem, Orlicz space, Boyd indices, modulus of continuity.

Paper MIA-15-71

where the constants c_1, c_2 and c_3, c_4 are independent of w and z respectively [29].

A convex and continuous function $M:[0,\infty)\to[0,\infty)$ for which $M(0)=0,\ M(x)>0$ for $x>0$ and

$$\lim_{x\to 0}\frac{M(x)}{x}=0, \qquad \lim_{x\to\infty}\frac{M(x)}{x}=\infty$$

is called a Young function. The complementary Young function N of M is defined by

$$N(y):=\max_{x\geqslant 0}(xy-M(x)), \qquad y\geqslant 0.$$

We denote by $L_M(\Gamma)$ the linear space of Lebesgue measurable functions $f:\Gamma\to\mathbb{C}$ satisfying the condition

$$\int_\Gamma M[\alpha|f(z)|]\,|dz|<\infty$$

for some $\alpha>0$. Equipped with the norm

$$\|f\|_{L_M(\Gamma)}:=\sup\left\{\int_\Gamma|f(z)g(z)|\,|dz| : g\in L_N(\Gamma),\ \rho(g,N)\leqslant 1\right\},$$

where

$$\rho(g,N):=\int_\Gamma N(|g(z)|)\,|dz|,$$

the space $L_M(\Gamma)$ becomes a Banach space [27, pp. 52–68]. The norm $\|\cdot\|_{L_M(\Gamma)}$ is called Orlicz norm and the Banach space $L_M(\Gamma)$ is called Orlicz space. It is known [27, p. 50], that every function in $L_M(\Gamma)$ is integrable on Γ, i.e

$$L_M(\Gamma)\subset L_1(\Gamma).$$

An N function M satisfies the Δ_2- condition if

$$\limsup_{x\to\infty}\frac{M(2x)}{M(x)}<\infty.$$

The Orlicz space $L_M(\Gamma)$ is reflexive if and only if the N-function M and its complementary function N both satisfy the Δ_2-condition [27, p. 113]. Furter information about Orlicz spaces may be found in [18] and [27].

Let $M^{-1}:[0,\infty)\to[0,\infty)$ be the inverse function of the N function M. The lower and upper indices $\alpha_M,\ \beta_M$ [20]

$$\alpha_M:=\lim_{x\to 0}\frac{\log h(x)}{\log x}, \qquad \beta_M:=\lim_{x\to\infty}\frac{\log h(x)}{\log x}$$

of the function

$$h:(0,\infty)\to(0,\infty], \qquad h(x):=\limsup_{t\to\infty}\frac{M^{-1}(t)}{M^{-1}\left(\frac{t}{x}\right)}, \qquad x>0$$

first considered by W. Matuszewska and W. Orlicz [22], are called the Boyd indices of the Orlicz space $L_M(\Gamma)$. It is known that

$$0 \leqslant \alpha_M \leqslant \beta_M \leqslant 1$$

and

$$\alpha_N + \beta_M = 1, \qquad \alpha_M + \beta_N = 1.$$

The Boyd indices α_M, β_M are called nontrivial if $0 < \alpha_M$ and $\beta_M < 1$. The Orlicz space $L_M(\Gamma)$ is reflexive if and only if $0 < \alpha_M \leqslant \beta_M < 1$, i.e. if the Boyd indices are nontrivial.

DEFINITION 2. ([19]) The analytic function f in domain G will be called a function of the class $E_M(G)$ if

$$\int_{\Gamma_r} M(|f(z)|)\,|dz| < \infty,$$

where Γ_r is the image of the circumference $|w| = r$ with regard to a conformal mapping of the disc $|w| < 1$ onto G.

DEFINITION 3. ([19]) We shall call the $E_M(G)$ class the Smirnov-Orlicz class.

If $M(u) = |u|^p$ $(1 < p < \infty)$, the $E_M(G)-$class coincides with the well-known $E_p(G)$ Smirnov class.

It is evident that any analytic function $f(z)$ belonging to the $E_M(G)$ class will also belong to the $E_1(G)$ class, that is,

$$\int_{\Gamma_r} |f(z)|\,|dz| \leqslant c < \infty,$$

uniformly in r, $0 < r < 1$. Since $E_M(G) \subset E_1(G)$, every function in the class $E_M(G)$ has the nontangential boundary values almost everywhere (a.e.) on Γ and the boundary value function belongs to $L_M(\Gamma)$ [19]. Hence the $E_M(G)$ norm can be defined as:

$$\|f\|_{E_M(G)} := \|f\|_{L_M(\Gamma)}.$$

DEFINITION 4. Let $g \in L_M(T)$. The function

$$\omega_{T,M}(\delta, g) := \sup_{|h| \leqslant \delta} \left\| g\left(e^{i(\theta+h)}\right) - g\left(e^{i\theta}\right) \right\|_{L_M(T)}$$

is called modulus of continuity of g.

We denote $f_r(w) = f^{(r)}[\psi(w)]$. We define the modulus of continuity for $f^{(r)}(z) \in L_M(\Gamma)$ as

$$\omega^*_{\Gamma,M}\left(\delta, f^{(r)}\right) := \omega_{T,M}(\delta, f_r) = \sup_{|h| \leqslant \delta} \left\| f_r\left(e^{i(\theta+h)}\right) - f_r\left(e^{i\theta}\right) \right\|_{L_M(T)}.$$

Throughout this work by $c_1, c_2, \ldots,$ we denote the constants which are different in different places.

The direct and inverse problems of approximation theory in different spaces have been investigated by several authors (see, for example, [1]–[4], [6]–[17], [19], [23], [24], [26], [28]) All of these have been done under different restrictive conditions on $\Gamma = \partial G$.

When $\Gamma = \partial G$ is a Dini–smooth curve, in this work we prove inverse theorem of approximation theory in Smirnov-Orlicz classes $E_M(G)$. This theorem extended to Smirnov-Orlicz classes $E_M(G)$ of the proved theorem for the Smirnov class $E_p(G)$, $p > 1$, in the work given by [1]. Similar results in weighted Smirnov and Smirnov-Orlicz classes were obtained in [8]–[15].

The following inverse theorem holds:

THEOREM 1. (Main) *Let Γ be a Dini-smooth curve, $L_M(\Gamma)$ be a reflexive Orlicz space on Γ. For each natural number n there exists a polynomial $P_n(z)$ of degree n, such that*

$$\|f(\varsigma) - P_n(\varsigma)\|_{L_M(\Gamma)} \leqslant \frac{c_5}{n^{r+\alpha}}, \tag{1.2}$$

where $0 < \alpha \leqslant 1$ and r is a nonegative integer. Then $f \in E_M(G)$ and for the modulus of continuity $\omega^*_{\Gamma,M}\left(\delta, f^{(r)}\right)$ the following inequalities hold:

$$\omega^*_{\Gamma,M}\left(\delta, f^{(r)}\right) \leqslant c_6 \delta^{\alpha}, \qquad 0 < \alpha < 1, \tag{1.3}$$

$$\omega^*_{\Gamma,M}\left(\delta, f^{(r)}\right) \leqslant c_7 \delta\left(1 + |\ln \delta|\right), \qquad \alpha = 1. \tag{1.4}$$

2. Auxiliary results

Let $f \in L_1(\Gamma)$. The functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(s)}{s-z} ds, \qquad z \in G$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(s)}{s-z} ds, \quad z \in G^-$$

are analytic in G and G^- respectively, and $f^-(\infty) = 0$. The Cauchy singular integral of f at a point $z_0 \in \Gamma$ is defined by

$$S_\Gamma(f)(z_0) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma(z_0, \varepsilon)} \frac{f(s)}{s - z_0} ds$$

if limit exists.

According to the Privalov's theorem [5, p. 431], if one of the functions $f^{+}(z)$ and $f^{-}(z)$ has a nontangential limit on Γ, a.e., then $S_{\Gamma}(f)(z)$ exists a.e. on Γ, and also the other one of the functions $f^{+}(z)$ and $f^{-}(z)$ has a nontangential limit on Γ a.e. Conversely, if $S_{\Gamma}(f)(z)$ exists a.e. on Γ, then the functions $f^{+}(z)$ and $f^{-}(z)$ have nontangential limits a.e. on Γ. In both cases, the formulas

$$f^{+}(z)=S_{\Gamma}(f)(z)+\frac{1}{2}f(z), \qquad f^{-}(z)=S_{\Gamma}(f)(z)-\frac{1}{2}f(z)$$

hold a.e. on Γ.

The followings lemmas will play an important role in the proof the main results.

LEMMA 1. ([12]) *$L_M(T)$ be a reflexive Orlicz space and let M be an N function, then for each trigonometric polynomial T_n of degree n the inequality*

$$\left\|T_n'\right\|_{L_M(T)} \leqslant c_8 n \left\|T_n\right\|_{L_M(T,)} \tag{2.1}$$

holds with a constant c independent of n.

LEMMA 2. *Let Γ be a Dini-smooth curve and let $L_M(\Gamma)$ be a reflexive Orlicz space on Γ, then for a polynomial $P_n(z)$ of degree n the inequality*

$$\left\|P_n'(z)\right\|_{L_M(\Gamma)} \leqslant c_9 n \left\|P_n(z)\right\|_{L_M(\Gamma)} \tag{2.2}$$

holds with a constant c_9 independent of n.

Proof. For the trigonometric polynomial T_n this inequality was obtained in the study [12]. For $z=e^{i\theta}$ we obtain

$$P_n(z)=T_n(\theta) \ \text{ and } \ P_n'(z)\, ie^{i\theta}=T_n'(\theta).$$

For polynomial $P_n(z)$ with respect to Faber polynomials the following expansion holds:

$$P_n(z)=\sum_{k=0}^{n} a_k\phi_k(z)=\sum_{k=0}^{n} a_k\left[\phi(z)\right]^k+\frac{1}{2\pi i}\int_{\Gamma}\frac{d\zeta}{\varsigma-z}\sum_{k=0}^{n} a_k\left[\phi(\varsigma)\right]^k, \quad z\in \operatorname{ext}\Gamma. \tag{2.3}$$

Then for $z\in \operatorname{ext}\Gamma$ we have

$$P_n'(z)=\sum_{k=1}^{n} ka_k\left[\phi(z)\right]^{k-1}\phi'(z)+\frac{1}{2\pi i}\int_{\Gamma}\frac{d\zeta}{\varsigma-z}\sum_{k=1}^{n} ka_k\left[\phi(\varsigma)\right]^{k-1}\phi'(\varsigma). \tag{2.4}$$

Consider the function $P_{n,0}(\tau)=P_n\left[\psi(\tau)\right]\in L_M(T)$.

The Cauchy type integral

$$\frac{1}{2\pi i}\int_{T}\frac{P_{n,0}(\tau)}{\tau-w}d\tau$$

represents analytic functions $P_{n,0}^{+}$ and $P_{n,0}^{-}$ in D and D^{-}, respectively. For $w \in T$ we have

$$P_{n,0}^{+}(w)=\sum_{k=0}^{n} a_k w^k.$$

By (1.1)

$$\left\|P_n\left[\psi\left(\tau\right)\right]\right\|_{L_M(T)} \leqslant c_{10}\left\|P_n\left(z\right)\right\|_{L_M(\Gamma)}. \tag{2.5}$$

Using (2.5) and the boundedness of the singular integral we obtain

$$\left\|S_T\left(P_{n,0}\right)\left(w\right)\right\|_{L_M(T)} \leqslant c\left\|P_n\right\|_{L_M(\Gamma)}. \tag{2.6}$$

Applying the inequalities (2.5), (2.6) and Minkowski inequality we get

$$\begin{aligned}\left\|P_{n,0}^{+}(w)\right\|_{L_M(T)} &= \left\|S_T\left(P_{n,0}\right)\left(w\right)-\frac{1}{2}P_{n,0}\left(w\right)\right\|_{L_M(T)} \\ &\leqslant \left\|S_T\left(P_{n,0}\right)\left(w\right)\right\|_{L_M(T)}+\frac{1}{2}\left\|P_{n,0}\left(w\right)\right\|_{L_M(T)} \leqslant c_{11}\left\|P_n(z)\right\|_{L_M(\Gamma)}.\end{aligned} \tag{2.7}$$

Then by (2.1), (2.7) for the nontangential limits on T we obtain

$$\left\|\left(P_{n,0}^{+}\left(w\right)\right)'\right\|_{L_M(T)} \leqslant c_{13} n\left\|P_n(z)\right\|_{L_M(\Gamma)}.$$

By (1.1) we get

$$\left\|\sum_{k=1}^{n} k a_k\left[\phi\left(z\right)\right]^{k-1}\phi'\left(z\right)\right\|_{L_M(\Gamma)} \leqslant c_{14}\left\|\sum_{k=1}^{n} k a_k w^{k-1}\right\|_{L_M(T)} \leqslant c_{15} n\left\|P_n\left(z\right)\right\|_{L_M(\Gamma)}. \tag{2.8}$$

Using (2.8), Minkowski's inequality and the boundedness of the singular operator S_Γ in weighted Orlicz spaces [21] for the non-tangential limits on Γ in (2.4) of the integral we have

$$\begin{aligned}&\left\|\frac{1}{2}\left(\sum_{k=1}^{n} k a_k\left[\phi\left(z\right)\right]^{k-1}\phi'\left(z\right)\right)+S_\Gamma\left[\sum_{k=1}^{n} k a_k\left[\phi\left(z\right)\right]^{k-1}\phi'\left(z\right)\right]\right\|_{L_M(\Gamma)} \\ &\leqslant c_{16}\left\|\sum_{k=1}^{n} k a_k\left[\phi\left(z\right)\right]^{k-1}\phi'\left(z\right)\right\|_{L_M(\Gamma)} \leqslant c_{17} n\left\|P_n\left(z\right)\right\|_{L_M(\Gamma)}.\end{aligned} \tag{2.9}$$

From (2.4), (2.8) and (2.9) we have inequality (2.2).

3. Proof of the main results

Proof of Theorem 1. The following inequality holds:

$$\left\|P_n\left(\varsigma\right)\right\|_{L_M(\Gamma)} \leqslant \left\|P_n\left(\varsigma\right)-f\left(\varsigma\right)\right\|_{L_M(\Gamma)}+\left\|f\left(\varsigma\right)\right\|_{L_M(\Gamma)} \leqslant c_{18}, \tag{3.1}$$

where constant c_{18} independent of n.

The sequence $\{P_n(\varsigma)\}$ converges in $L_M(\Gamma)$. Therefore, the sequence $\{P_n(\varsigma)\}$ converges with respect to a measure. Since, condition (3.1) is satisfied, according to [13], [19] sequence $\{P_n(\varsigma)\}$ converges uniformly within the domain to the function $f(z) \in E_M(G)$ and nontangential boundary values of the function $f(z)$(from inside Γ) coincides with $f(\varsigma)$ a.e. on Γ.

We define the following form polynomials sequence:

$$T_0(z) = P_1(z), \qquad T_k(z) = P_{2^k}(z) - P_{2^{k-1}}(z) \qquad (k = 1,2,...).$$

The series $\bigcup_{k=0}^{\infty} T_k(z)$ converges uniformly to the function $f(z)$ into G. Then the series $\bigcup_{k=0}^{\infty} T_k^{(r)}$ converges uniformly to the function $f^{(r)}(z)$ into G. We define the following form sequence:

$$K_n(s) = \sum_{k=0}^{n} T_k^{(r)}(s).$$

Now show that the sequence $K_n(\varsigma)$ converges in $L_M(\Gamma)$. By (1.2) we obtain

$$\|T_k(\varsigma)\|_{L_M(\Gamma)} \leqslant \|f(\varsigma) - P_{2^k}(\varsigma)\|_{L_M(\Gamma)} + \left\|f(\varsigma) - P_{2^{k-1}}(\varsigma)\right\|_{L_M(\Gamma)} \leqslant \frac{c_{19}}{2^{k(r+\alpha)}}. \tag{3.2}$$

According to (3.2) and (2.2) we get

$$\left\|T_k^{(r)}(\varsigma)\right\|_{L_M(\Gamma)} \leqslant \frac{c_{20}}{2^{k\alpha}}. \tag{3.3}$$

By (3.3) we have

$$\|K_m(\varsigma) - K_n(\varsigma)\|_{L_M(\Gamma)} \leqslant \sum_{k=n+1}^{m} \left\|T_k^{(r)}(\varsigma)\right\|_{L_M(\Gamma)} \leqslant \frac{c_{21}}{2^{n\alpha}}, \qquad (m > n). \tag{3.4}$$

Then according to inequality (3.4) sequence $K_n(s)$ is a Cauchy sequence. Since, $L_M(\Gamma)$ is a Banach space the sequence $\{K_n(\varsigma)\}$ converges in $L_M(\Gamma)$. Therefore, the sequence $K_n(\varsigma)$ converges with respect to a measure.

Since, $\|K_n(\varsigma)\|_{L_m(\Gamma)} \leqslant c_{22}$ the sequence $K_n(\varsigma)$ converges with respect to a measure to $f^{(r)}(\varsigma)$ nontangential boundary values of the function $f^{(r)}(z)$. There exists subsequence $K_{n_i}(s)$ of the sequence $K_n(s)$, such that

$$K_{n_i}(s) \to f^{(r)}(s)$$

a.e on Γ. Then we obtain a.e. on Γ

$$|K_{n_i}(\varsigma) - K_n(\varsigma)| \to \left|f^{(r)}(\varsigma) - K_n(\varsigma)\right|.$$

According to Fatou's Lemma and (3.4) we have

$$\left\|f^{(r)}(\varsigma) - K_n(\varsigma)\right\|_{L_M(\Gamma)} \leqslant \frac{c_{22}}{2^{n\alpha}}, \tag{3.5}$$

where $K_n(\varsigma)$ polynomial of degree 2^n. We fix δ, satisfying the condition $0<\delta \leqslant \frac{1}{2}$ and choose $m \in N$, such that $2^{m-1} \leqslant \frac{1}{\delta} < 2^m$. If we pass on to the complex plane (w), then from the inequality (3.5) we obtain

$$\left\| f_r\left(we^{ih}\right) - f_r(w)\right\|_{L_M(T)} \leqslant \frac{c_{23}}{2^{m\alpha}} + \left\| K_{m-1}\left[\psi\left(we^{ih}\right)\right] - K_{m-1}\left[\psi(w)\right]\right\|_{L_M(T)}. \quad (3.6)$$

We define the sequence of the polynomials in the following form:

$$Q_1(z) = K_1(z), \qquad Q_k(z) = K_k(z) - K_{k-1}(z),$$

where the polynomial $Q_k(z)$ of degree 2^k.

From (3.5) we have

$$\|Q_k(z)\|_{L_M(\Gamma)} = \|K_k(z) - K_{k-1}(z)\|_{L_M(\Gamma)} \leqslant \frac{c_{23}}{2^{k\alpha}}, \qquad k \geqslant 2. \quad (3.7)$$

Putting $w = e^{ix}$ we define $Q_k\left[\psi\left(e^{ix}\right)\right] = \nu_k(x)$. Then we obtain

$$\begin{aligned} &\|\nu_k(x+h) - \nu_k(x)\|_{L_M(T)} \\ &= \sup\left\{\textstyle\int_T |\nu_k(x+h) - \nu_k(x)|\, g(x)dx : g \in L_N(T), \int_T N(|g(x)|)\,dx \leqslant 1\right\} \\ &= \sup\left\{\textstyle\int_T \left|\int_0^h \nu_k'(x+h)dt\right| g(x)dx : g \in L_N(T), \int_T N(|g(x)|)\,dx \leqslant 1\right\} \\ &\leqslant h\left\|\nu_k'(x)\right\|_{L_M(T)}. \end{aligned} \quad (3.8)$$

Since, the curve Γ is Dini-smooth, by (1.1) we have

$$\left\|\nu_k'(x)\right\|_{L_M(T)} \leqslant c_{24}\left\|Q_k'(z)\right\|_{L_M(\Gamma)}. \quad (3.9)$$

By (3.8) and (3.9) we obtain

$$\left\| Q_k\left[\psi\left(we^{ih}\right)\right] - Q_k\left[\psi(w)\right]\right\|_{L_M(T)} \leqslant c_{25} h\left\|Q_k'(z)\right\|_{L_M(\Gamma)}. \quad (3.10)$$

According to (2.2), (3.9) we get

$$\begin{aligned} &\left\| K_{m-1}\left[\psi\left(we^{ih}\right)\right] - K_{m-1}\left[\psi(w)\right]\right\|_{L_M(T)} \\ &\leqslant c_{26} h \sum_{k=1}^{m-1}\left\|Q_k'(z)\right\|_{L_M(\Gamma)} \leqslant c_{27} h \sum_{k=1}^{m-1} 2^k \|Q_k(z)\|_{L_M(\Gamma)}. \end{aligned} \quad (3.11)$$

Using (3.6), (3.11) and (3.7) for $|h| \leqslant \delta$ we have

$$\omega^*_{\Gamma,M}\left(\delta, f^{(r)}\right) \leqslant \frac{c_{28}}{2^{m\alpha}} + c_{29}\delta \sum_{k=2}^{m-1} 2^{k(1-\alpha)} + c_{30}\delta. \quad (3.12)$$

From the inequality (3.12) the required inequalities (1.3) and (1.4) are obtained. Then the proof of Theorem 1 is completed. □

Acknowledgement. I thank the referee for careful reading of my manuscript and for helpful and valuable comments and suggestions.

REFERENCES

[1] S. JA. AL'PER, *Approximation in the Mean of Analytic functions of Class E_p*, Gos. Izdat. Fiz.-Mat. Lit., Moscow, 1960, 272–286 (in Russian).

[2] J. E. ANDERSSON, *On the degree of polynomial approximation in $E^p(D)$*, J. Approx. Theory **19** (1977), 61–68.

[3] M. I. ANDRASKO, *On the approximation in the mean of analytic functions in regions with smooth boundaries*, Problems in mathematical physics and function theory, Izdat. Akad. Nauk. Ukrain. RSR, Kiev, 1963, 1, p. 3 (in Russian).

[4] R. AKGÜN AND D. M. ISRAFLLOV, *Approximation and moduli of fractional orders in Smirnov-Orlicz classes*, Glas. Mat. **43**, 63 (2008), 121–136.

[5] G. M. GOLUZIN, *Geometric Theory of Functions of a Complex Variable*, Translation of Mathematical Monographs vol. 26, R.I.: AMS, Providence,1968.

[6] I. I. IBRAGIMOV, D. I. MAMEDKHANOV, *Constructive characterization of a certain class of functions*, Dokl. Akad. Nauk **16** (1976), 820–823.

[7] D. M. ISRAFILOV, *Approximation properties of generalized Faber series in an integral metric*, Izv. Akad. Nauk Azerb Ser. Fiz.-Tekh. Mat. Nauk **2** (1987), 10–14 (in Russian).

[8] D. M. ISRAFILOV, *Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,w)$ and the Bieberbach polynomials*, Constr. Approx. **17**, 3 (2001), 335–351.

[9] D. M. ISRAFILOV, *Approximation by p-Faber Laurent rational functions in the weighted Lebesque spaces*, Czechoslovak Math. J. **54**, 3(129) (2004), 751–765.

[10] D. M. ISRAFILOV AND A. GUVEN, *Approximation in weighted Smirnov classes*, East J. Approx. **11**, 1 (2005), 91–102.

[11] D. M. ISRAFILOV, B. OKTAY AND R. AKGÜN, *Approximation in Smirnov-Orlicz classes*, Glas. Mat. **40** (60) (2005), 87–102.

[12] D. M. ISRAFILOV, ALI GUVEN, *Approximation by trigonometric polynomials in weighted Orlicz spaces*, Studia Math. **174**, 2 (2006), 147–168.

[13] D. M. ISRAFILOV AND R. AKGÜN, *Approximation in weighted Smirnov-Orlicz classes*, J. Math. Kyoto Univ. **46**, 4 (2006), 775–770.

[14] S. Z. JAFAROV, *Approximation by rational functions in Smirnov-Orlicz classes*, J. Math. Anal. Appl. **379** (2011), 870–877.

[15] S. Z. JAFAROV, *S. M. Nikol'skii type inequality and estimation between the best approximation of a functions in norms of different spaces*, Math. Balkanica (N.S.) **21**, 1-2 (2007), 173–182.

[16] S. Z. JAFAROV, *Approximations of Harmonic Functions Classes with Singularities on Quasiconformal Curves*, Taiwanise J. Math. **12**, 3 (2008), 829–840.

[17] S. Z. JAFAROV, *Approximation by polynomials and rational functions in Orlicz spaces*, J. Comput. Anal. Appl. (JoCAAA) **13**, 5 (2011), 953–962.

[18] M. A. KRASNOSELSKII AND YA. B. RUTICKII, *Convex Functions and Orlicz Spaces*, P. Norrdhoff Ltd., Groningen, 1961.

[19] V. KOKILASHVILI, *On analytic functions of Smirnov-Orlicz classes*, Studia Math. **31** (1968), 43–59.

[20] A. YU. KARLOVICH, *Algebras of singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces*, Math. Nachr. **179** (1996), 187–222.

[21] A. YU. KARLOVICH, *Algebras of singular integral operators with PC coefficients in rearrangement-invariant spaces with Muckernhoupt weights*, J. Operator Theory **47** (2002), 303–323.

[22] W. MATUSZEWSKA AND W. ORLICZ, *On certain properties of ϕ-functions*, Bull. Acad. Polon. Sci., Ser. Math. Aster. et Phys. **8-7** (1960), 439–443.

[23] D. I. MAMEDKHANOV, *Approximation in complex plane and simgular operators with a Cauchy kernel*, Dissertation Doct. Phys-math. nauk. The University of Tblisi, 1984 (in Russian).

[24] H. N. MHASKAR, *Introduction to the Theory of Weighted Polynomial Approximation*, Series in Approximation and Decompositions 7, World Sci., River Edge, NJ, 1996.
[25] CH. POMMERENKE, *Boundary Behavior of Conformal Maps*, Berlin, Springer-Verlag, 1992.
[26] A.-R. K. RAMAZANOV, *On approximation by polynomials and rational functions in Orlicz spaces*, Anal. Math. **10** (1984), 117–132.
[27] M. M. RAO AND Z. D. REN, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991
[28] P. K. SUETIN, *Series of Faber Polynomials*, Gordob and Breack, 1. Reading, 1998.
[29] S. E. WARSCHAWSKI, *Über das Ranverhalten der Ableitung der Abbildunggsfunktion bei konformer Abbildung*, Math. Z. **35** (1932), 321–456.

(Received March 23, 2011)

Sadulla Z. Jafarov
Department of Mathematics
Faculty of Art and Science
Pamukkale University
20017 Denizli, Turkey
e-mail: sjafarov@pau.edu.tr