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ON ω–QUASICONVEX FUNCTIONS

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Abstract. In the paper we introduce convexity-like notions based on modification of quasiconvexity.

DEFINITION. Let I be a real interval and $\omega \geqslant 0$ a given number. We say that a function $f : I \to \mathbb{R}$ is ω*-quasiconvex,* ω*-quasiconcave,* respectively, if

$$f(tx+(1-t)y) \leqslant \max(f(x), f(y)) - \omega \min(t, 1-t)|x-y|,$$
$$f(tx+(1-t)y) \geqslant \max(f(x), f(y)) - \omega \max(t, 1-t)|x-y|,$$
$$\text{for } x, y \in I, t \in (0,1).$$

If $f : I \to \mathbb{R}$ is simultaneously ω-quasiconvex and ω-quasiconcave then we say that f is ω*-quasiaffine*.

We characterize these notions, in particular we show that ω-quasiconcave functions coincide with Lipschitz functions with constant ω. We conclude the paper with the following separation type result.

THEOREM. Let $f : I \to \mathbb{R}$ be ω-quasiconvex function and $g : I \to \mathbb{R}$ ω-quasiconcave such that $f \geqslant g$.

Then there exists an ω-quasiaffine function $h : I \to \mathbb{R}$ such that $f \geqslant h \geqslant g$.

1. Introduction

The notion of quasiconvex function is a very far generalization of the convex function. Let I be a real interval. A function $f : I \to \mathbb{R}$ is called *quasiconvex* [1, 4] if

$$f(tx+(1-t)y) \leqslant \max(f(x), f(y)) \qquad \text{for } x, y \in I, t \in (0,1). \tag{1}$$

This notion occured to be very useful in mathematical economics (for more information and further references see [1]). As quasiconvexity is a rather weak assumption, there appeared a natural need to strenghten it. In such a way, in an analogy to strict convexity, there appeared the notion of strict quasiconvexity [1]. A function f is *strictly quasiconvex* if

$$f(tx+(1-t)y) < \max(f(x), f(y)) \qquad \text{for } x, y \in I, x \neq y,\ t \in (0,1).$$

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Following the way from convexity to strong convexity [9] (see also [8, 7]), which relies on subtracting from the right hand side of (1) a nonnegative expression, we introduce the notion of ω-quasiconvexity. Let $\omega \geqslant 0$ be a given number. A function f is ω-*quasiconvex* if

$$f(tx+(1-t)y) \leqslant \max(f(x),f(y)) - \omega\min(t,1-t)|x-y| \text{ for } x,y \in I, x \neq y, t \in (0,1).$$

Observe that for $\omega > 0$ every ω-quasiconvex function is strictly quasiconvex, while for $\omega = 0$ we obtain classical quasiconvexity. The above condition for $t = \frac{1}{2}$ was studied in [10]. It follows from Theorem 2.2 [10] that there are no ω-quasiconvex functions with $\omega > 0$ on convex domain of dimension greater then one (obviously in multidimensional case "$|\ |$" is replaced by "$\|\ \|$").

A similar approach was earlier applied in [5], where the notion of strong quasiconvexity was introduced. A function f is *strongly quasiconvex* if for a certain $\omega > 0$

$$f(tx+(1-t)y) \leqslant \max(f(x),f(y)) - \omega t(1-t)|x-y|^2 \quad \text{for } x,y \in I, t \in [0,1].$$

In our opinion ω-quasiconvexity has a stronger resemblance to the convexity theory than strong quasiconvexity. The reasons behind this assertion are the following:

- ω-quasiconvexity is a local notion, that is a locally ω-quasiconvex function is ω-quasiconvex;
- ω-quasi-convexity/concavity/affinity have a very natural geometric description. In particular, ω-quasiconvex functions are functions which first decrease and then increase with speed not smaller then ω; ω-quasiconcave functions coincide with Lipschitz functions with constant ω; and ω-quasiaffine functions are functions of the form $x \to \omega|x-x_0|+y_0$;
- We can naturally define ω-quasiconcave and ω-quasiaffine functions in such a way that we can separate ω-quasiconvex functions from ω-quasiconcave by ω-quasiaffine ones.

2. Characterization of ω-quasiconvexity

Now we are ready to proceed with formal definition. In the whole paper we assume that I is a nondegenerate[1] subinterval of $\mathbb{R}$ and $\omega \geqslant 0$ is a given number.

DEFINITION 2.1. We say that a function $f : I \to \mathbb{R}$ is

(i) ω-*quasiconvex* if

$$f(tx+(1-t)y) \leqslant \max(f(x),f(y)) - \omega\min(t,1-t)|x-y|$$

for all $x,y \in I$, $t \in (0,1)$;

[1] an interval is degenerate if it is either empty or a singleton.

(ii) ω-*quasiconcave* if

$$f(tx+(1-t)y) \geqslant \max(f(x), f(y)) - \omega \max(t, 1-t)|x-y|$$

for all $x, y \in I$, $t \in (0,1)$;

(iii) ω-*quasiaffine* if it is simultaneously ω-quasiconvex and ω-quasiconcave.

One can directly verify that the maximum of two ω-quasiconvex functions is ω-quasiconvex and that minimum of two ω-quasiconcave functions is ω-quasiconcave.

We introduce the following denotations. Let $f: I \to \mathbb{R}$ be any function. If

$$\frac{f(x)-f(y)}{x-y} \leqslant -\omega \qquad \text{for } x, y \in I,\ x \neq y$$

then we will write that $f \in \searrow(\omega)$ and if

$$\frac{f(x)-f(y)}{x-y} \geqslant \omega \qquad \text{for } x, y \in I,\ x \neq y$$

then we write that $f \in \nearrow(\omega)$. In case if $\omega = 0$ instead of $f \in \searrow(0)$, $f \in \nearrow(0)$ we will write $f \in \nearrow$, $f \in \searrow$ respectively.

We begin our considerations with the case of monotonic functions.

PROPOSITION 2.1. *(i) A nondecreasing function $f: I \to \mathbb{R}$ is ω-quasiconvex if and only if $f \in \searrow(\omega)$.*
(ii) A nondecreasing function $f: I \to \mathbb{R}$ is ω-quasiconvex if and only if $f \in \nearrow(\omega)$.

Proof. One can easily notice that (ii) follows from (i) by applying in the domain the substitution $x \to -x$.

We prove (i). Assume that $f: I \to \mathbb{R}$ is nonincreasing and ω-quasiconvex. We prove that $f \in \searrow(\omega)$. Since f is monotonic it is sufficient to show that $f|_{\text{int}\, I} \in \searrow(\omega)$. Consider an arbitrary $z \in \text{int}\, I$. We can find a neighbourhood I_z of z such that $2I_z - I_z \subset \text{int}\, I$. Obviously $I_z \subset 2I_z - I_z$. Let us consider arbitrary $x, y \in I_z$, $x < y$. Then we have

$$f(y) = f\left(\frac{x+(2y-x)}{2}\right) \leqslant f(x) - \omega(y-x).$$

It proves that $f|_{I_z} \in \searrow(\omega)$. Since it holds for each $z \in \text{int}\, I$ and respective neighbourhood I_z of z, we obtain that $f|_{\text{int}\, I} \in \searrow(\omega)$.

Assume now that $f \in \searrow(\omega)$. We prove that f is ω-quasiconvex. Consider arbitrary $x, y \in I$, $t \in (0,1)$. Without loss of generality we may assume that $x < y$. Then we have

$$\frac{f(x)-f(tx+(1-t)y)}{x-(tx+(1-t)y)} \leqslant -\omega.$$

Whence we obtain

$$\begin{aligned} &f(tx+(1-t)y) \leqslant f(x) - \omega(1-t)(y-x) \\ &\leqslant \max(f(x), f(y)) - \omega t(1-t)|x-y|. \quad \square \end{aligned}$$

Given (possibly empty) sets $I_1, I_2 \subset \mathbb{R}$, we write that $I_1 < I_2$ if $x_1 < x_2$ for all $x_1 \in I_1$, $x_2 \in I_2$.

THEOREM 2.1. *A function $f : I \to \mathbb{R}$ is ω-quasiconvex if and only if there exist (possibly degenerate) intervals I_1, I_2, $I_1 < I_2$ such that $I = I_1 \cup I_2$ and*

$$f|_{I_1} \in \searrow(\omega) \text{ and } f|_{I_2} \in \nearrow(\omega). \tag{2}$$

Proof. Assume that f is ω-quasiconvex. Then it is quasiconvex. By [1, Theorem 2.5.1] there exist intervals $I_1 < I_2$ such that $I_1 \cup I_2 = I$ and $f|_{I_1} \in \searrow$ and $f|_{I_2} \in \nearrow$. Proposition 2.1 proves (2).

Assume now that I_1, I_2 are subintervals of I such that $I_1 < I_2$, $I = I_1 \cup I_2$ and (2) is valid. Consider arbitrary $x, y \in I$, $x < y$, $t \in (0,1)$. If $x, y \in I_1$ or $x, y \in I_2$ then by Proposition 2.1 applied to functions $f|_{I_1}$, $f|_{I_1}$ respectively we obtain that

$$f(tx + (1-t)y) \leqslant \max(f(x), f(y)) - \omega \min(t, 1-t)|x-y|.$$

So assume now that $x \in I_1, y \in I_2$. Two cases may occur.

If $tx + (1-t)y \in I_1$ then

$$\frac{f(tx + (1-t)y) - f(x)}{(1-t)(y-x)} \leqslant -\omega.$$

Whence we obtain

$$\begin{aligned} f(tx + (1-t)y) &\leqslant f(x) - \omega(1-t)(y-x) \\ &\leqslant \max(f(x), f(y)) - \omega \min(t, 1-t)|x-y|. \end{aligned}$$

In the case when $tx + (1-t)y \in I_2$ we obtain that

$$\frac{f(y) - f(tx + (1-t)y)}{t(y-x)} \geqslant \omega,$$

and consequently

$$\begin{aligned} f(tx + (1-t)y) &\leqslant f(y) - \omega t(y-x) \\ &\leqslant \max(f(x), f(y)) - \omega \min(t, 1-t)|x-y|. \quad \square \end{aligned}$$

Theorem 2.1 can be written in a more explicit way if one of the intervals I_1, I_2 is degenerate. If I_2 is a degenerate then (2) takes the form

$$f|_{I \setminus \{\sup I\}} \in \searrow(\omega),$$

while when I_1 is degenerate it takes the form

$$f|_{I \setminus \{\inf I\}} \in \nearrow(\omega).$$

Taking in mind the above remarks Theorem 2.1 can be written as follows.

COROLLARY 2.1. *A function $f : I \to \mathbb{R}$ is ω-quasiconvex if and only if exactly one the following conditions hold:*

(i) $f|_{I\setminus\{\sup I\}} \in \searrow(\omega)$ *or* $f|_{I\setminus\{\inf I\}} \in \nearrow(\omega)$;

(ii) there exists an $x_0 \in \operatorname{int} I$ *such that*

$$f|_{I\cap(-\infty,x_0]} \in \searrow(\omega) \text{ and } f|_{I\cap(x_0,\infty)} \in \nearrow(\omega)$$

or

$$f|_{I\cap(-\infty,x_0)} \in \searrow(\omega) \text{ and } f|_{I\cap[x_0,\infty)} \in \nearrow(\omega).$$

We are going to show that ω-quasiconvexity has a local character. We begin with some new notations. Let I_0 be a subinterval of I. We denote

$$I_0^- := \{x \in I : \{x\} < I_0\}, \quad I_0^+ := \{x \in I : I_0 < \{x\}\}.$$

Then evidently

$$I_0^- < I_0 < I_0^+$$

and

$$I = I_0^- \cup I_0 \cup I_0^+.$$

LEMMA 2.1. *Let $f : I \to \mathbb{R}$ be locally ω-quasiconvex and let I_0 be a nonempty open subinterval of I. If $f|_{I_0} \in \searrow(\omega)$ then $f|_{I_0^-\cup I_0} \in \searrow(\omega)$ and if $f|_{I_0} \in \nearrow(\omega)$ then $f|_{I_0\cup I_0^+} \in \nearrow(\omega)$.*

Proof. Assume that f, I, I_0 have the meaning specified in the Lemma. Let $f|_{I_0} \in \searrow(\omega)$. We fix arbitrarily $x_0 \in I_0$ and consider an arbitrary $x \in I$, $x < x_0$. Then, by the compactness argument, we can find a sequence in I

$$x = x_n < \ldots < x_1 < x_0$$

and their open neighbourhoods I_{x_i}, $i = 1,\ldots,n$; $I_{x_0} := I_0$, such that $f|_{I_{x_i}}$ is ω-quasiconvex for $i = 1,\ldots,n$ and

$$I_{x_i} \cap I_{x_{i-1}} \neq \emptyset \qquad \text{for } i = 1,\ldots,n.$$

We claim that $f|_{I_{x_1}} \in \searrow(\omega)$. Suppose for the proof by contradiction that it is not the case. Then in virtue of Corollary 2.1 there exists an open interval $\tilde{I} \subset I_{x_1}$ such that $\tilde{I} \cap I_{x_0} \neq \emptyset$ and $f|_{\tilde{I}} \in \nearrow(\omega)$. But then $f|_{\tilde{I}\cap I_{x_0}} \in \nearrow(\omega)$, which contradicts to our assumption that $f|_{I_{x_0}} \in \searrow(\omega)$. Hence $f|_{I_{x_1}} \in \searrow(\omega)$ and consequently $f|_{I_{x_1}\cup I_{x_0}} \in \searrow(\omega)$. Continuing this procedure we obtain that $f|_{I_{x_n}\cup\ldots\cup I_{x_0}} \in \searrow(\omega)$. Since $x_n = x < x_0$ was arbitrary it proves that $f|_{I_0^-\cup I_0} \searrow(\omega)$.

The second part of the assertion easily follows from the first applied for the mapping $x \to f(-x)$. $\square$

THEOREM 2.2. *If $f : I \to \mathbb{R}$ is locally ω-quasiconvex then it is ω-quasiconvex.*

Proof. Assume that $f: I \to \mathbb{R}$ is locally ω-quasiconvex. For each $x \in \operatorname{int} I$ we choose an open neighbourhood $I_x \subset \operatorname{int} I$ such that $f|_{I_x}$ is ω-quasiconvex. Three cases may occur.

1^0. $f|_{I_x} \in \searrow(\omega)$ for all $x \in \int \mathscr{I}$.

Then by Lemma 2.1 we obtain that $f|_{I\setminus\{\sup I\}} \in \searrow(\omega)$. By Corollary 2.1 it implies that f is ω-quasiconvex.

2^0. $f|_{I_x} \in \nearrow(\omega)$ for all $x \in \int \mathscr{I}$.

By Lemma 2.1 we conclude that $f|_{I\setminus\{\inf I\}} \in \nearrow(\omega)$ and consequently by Corollary 2.1 that f is ω-quasiconvex.

3^0 Neither 1^0 nor 2^0 is valid. Let

$$I_{\searrow} := \{x \in \operatorname{int} I \mid \exists \delta_x > 0 : f|_{I_x \cap (x-\delta_x, x+\delta_x)} \in \searrow(\omega)\},$$
$$I_{\nearrow} := \{x \in I \mid \exists \delta_x > 0 : f|_{I_x \cap (x-\delta_x, x+\delta_x)} \in \nearrow(\omega)\}.$$

One can easily observe that $I_{\searrow}$ and $I_{\nearrow}$ are open and disjoint subsets of $\operatorname{int} I$. Since $\operatorname{int} I$ is connected, we obtain that $\operatorname{int} I \neq I_{\searrow} \cup I_{\nearrow}$, and therefore there exists an $x \in (\operatorname{int} I) \setminus (I_{\searrow} \cup I_{\nearrow})$. Thus

$$f|_{I_x} \notin \searrow(\omega) \text{ and } f|_{I_x} \notin \nearrow(\omega).$$

Since $f|_{I_x}$ is ω-quasiconvex, by Corollary 2.1 there exists an $x_0 \in I_x$ such that

$$f|_{I_x \cap (-\infty, x_0]} \in \searrow(\omega) \text{ and } f|_{I_x \cap (x_0, \infty)} \in \nearrow(\omega)$$

or

$$f|_{I_x \cap (-\infty, x_0)} \in \searrow(\omega) \text{ and } f|_{I_x \cap [x_0, \infty)} \in \nearrow(\omega).$$

Then by Lemma 2.1 we obtain that

$$f|_{I \cap (-\infty, x_0]} \in \searrow(\omega) \text{ and } f|_{I \cap (x_0, \infty)} \in \nearrow(\omega)$$

or

$$f|_{I \cap (-\infty, x_0)} \in \searrow(\omega) \text{ and } f|_{I \cap [x_0, \infty)} \in \nearrow(\omega).$$

Now by Corollary 2.1 we get that f is ω-quasiconvex. □

3. Characterizations of ω-quasiconcavity and ω-quasiaffinity

In this section we characterize ω-quasiconcave and ω-quasiaffine functions.

THEOREM 3.1. *Let I be open, and let $f: I \to \mathbb{R}$ be ω-quasiconcave. Then f is Lipschitz with constant ω.*

Proof. Consider arbitrary $x, y \in I$, $x < y$. Let $n_0 \in \mathbb{N}$ be such that

$$x - \frac{1}{n_0} \in I, \quad y + \frac{1}{n_0} \in I, \quad \frac{1}{n_0} < y - x.$$

We put

$$x_n := x - \frac{1}{n}, \quad y_n := y + \frac{1}{n} \qquad \text{for } n \in \mathbb{N},\ n \geqslant n_0.$$

Then $x_n, y_n \in I$ for $n \in \mathbb{N}, n \geqslant n_0$. We have for $n \in \mathbb{N},\ n \geqslant n_0$

$$y = \frac{y_n - y}{y_n - x} x + \frac{y - x}{y_n - x} y_n,$$

and hence

$$\begin{aligned} f(y) &\geqslant \max(f(x), f(y_n)) - \omega \max\left(\frac{\frac{1}{n}}{y - x + \frac{1}{n}}, \frac{y - x}{y - x + \frac{1}{n}}\right) \left|x - y - \frac{1}{n}\right| \\ &\geqslant f(x) - \omega \left|x - y - \frac{1}{n}\right|. \end{aligned}$$

Whence we get

$$f(y) - f(x) \geqslant -\omega \left|x - y - \frac{1}{n}\right| \qquad \text{for } n \in \mathbb{N},\ n \geqslant n_0. \tag{3}$$

Similarly we have for $n \in \mathbb{N},\ n \geqslant n_0$

$$f(x) = f\left(\frac{y - x}{y - x_n} x_n + \frac{x - x_n}{y - x_n} y\right)$$

$$\geqslant \max(f(x_n), f(y)) - \omega \max\left(\frac{y - x}{y - x + \frac{1}{n}}, \frac{\frac{1}{n}}{y - x + \frac{1}{n}}\right) \left|x - y - \frac{1}{n}\right| \geqslant f(y) - \omega \left|x - y - \frac{1}{n}\right|,$$

and hence

$$f(x) - f(y) \geqslant -\omega \left|x - y - \frac{1}{n}\right| \qquad \text{for } n \in \mathbb{N},\ n \geqslant n_0. \tag{4}$$

Letting in (3), (4) $n \to \infty$ we obtain that

$$|f(x) - f(y)| \leqslant \omega |x - y|. \quad \square$$

Now we characterize ω-quasiconcave functions.

THEOREM 3.2. *A function $f : I \to \mathbb{R}$ is ω-quasiconcave if and only if $f|_{\operatorname{int} I}$ is Lipschitz with the constant ω and*

$$f(\inf I) \leqslant \lim_{x \to \inf I} f(x) \textit{ if } \inf I \in I, \tag{5}$$

$$f(\sup I) \leqslant \lim_{x \to \sup I} f(x) \textit{ if } \sup I \in I. \tag{6}$$

Proof. Assume that $f : I \to \mathbb{R}$ is ω-quasiconcave. By Theorem 3.1 $f|_{\text{int } I}$ is Lipschitz with the constant ω. Suppose that $\inf I \in I$. Then there exists a finite limit $\lim_{x\to\inf I} f(x)$. We have

$$f\left(\frac{\inf I + y}{2}\right) \geqslant \max(f(\inf I), f(y)) - \frac{\omega}{2}|\inf I - y|$$
$$\geqslant f(\inf I) - \frac{\omega}{2}|\inf I - y| \quad \text{for } y \in I.$$

Letting in this inequality $y \to \inf I$ we get

$$\lim_{x\to\inf I} f(x) \geqslant f(\inf I).$$

Similarly one can show condition (6).

Assume now that $f|_{\text{int } I}$ is Lipschitz with the constant ω and that conditions (5) and (6) are satisfied. We define function $\tilde{f} : I \to \mathbb{R}$ in the following way

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in \text{int } I, \\ \lim\limits_{x\to\inf I} f(x) & \text{if } \inf I \in I, \\ \lim\limits_{x\to\sup I} f(x) & \text{if } \sup I \in I. \end{cases} \tag{7}$$

Then obviously $\tilde{f}$ is Lipschitz with the constant ω. Therefore we have for $x, y \in I$, $t, t' \in [0,1]$

$$-\omega|t - t'|\,|x - y| \leqslant \tilde{f}(tx + (1-t)y) - \tilde{f}(t'x + (1-t')y).$$

Substituting sequentially $t' = 0$ and $t' = 1$ in the above inequality we obtain for $x, y \in I$, $t \in [0,1]$

$$\tilde{f}(tx + (1-t)y) \geqslant \tilde{f}(y) - \omega t|x - y| \geqslant \tilde{f}(y) - \omega \max(t, 1-t)|x - y|,$$
$$\tilde{f}(tx + (1-t)y) \geqslant \tilde{f}(x) - \omega(1-t)|x - y| \geqslant \tilde{f}(x) - \omega \max(t, 1-t)|x - y|.$$

Hence

$$\tilde{f}(tx+(1-t)y) \geqslant \max(\tilde{f}(x), \tilde{f}(y)) - \omega \max(t, 1-t)|x-y| \quad \text{for } x, y \in I,\ t \in [0,1]. \tag{8}$$

Whence and from (7) we obtain

$$f(tx+(1-t)y) \geqslant \max(f(x), f(y)) - \omega \max(t, 1-t)|x-y| \quad \text{for } x, y \in \text{int } I,\ t \in (0,1).$$

To prove that f is ω-quasiconcave we have to consider yet the following cases:

(a) $x = \inf I \in I$, $y \in \text{int } I$,

(b) $x \in \text{int } I$, $y = \sup I \in I$,

(c) $x = \inf I \in I$, $y = \sup I \in I$.

In case (a) by (8), (7) and (5) we get for $t \in (0,1)$

$$f(t\ \inf I + (1-t)y) \geqslant \max(\tilde{f}(\inf I), f(y)) - \omega \max(t, 1-t)|\inf I - y|$$
$$\geqslant \max(f(\inf I), f(y)) - \omega \max(t, 1-t)|\inf I - y|.$$

The case (b) is analogous.

In case (c) by (7), (8), (5) and (6) we obtain for $t \in (0,1)$

$$f(t\ \inf I + (1-t)\sup I) \geqslant \max(\tilde{f}(\inf I), \tilde{f}(\sup I)) - \omega \max(t, 1-t)|\inf I - \sup I|$$
$$\geqslant \max(f(\inf I), f(\sup I)) - \omega \max(t, 1-t)|\inf I - \sup I|. \quad \square$$

The next results gives a characterization of ω-quasiaffine functions.

THEOREM 3.3. *A function $f : I \to \mathbb{R}$ is ω-quasiaffine if and only if it has one of the following forms:*

(i) $f(x) = -\omega x + y_0$ *for* $x \in I \setminus \{\sup I\}$, *where* $y_0 \in \mathbb{R}$ *and*
$$f(\sup I) \leqslant -\omega \sup I + y_0 \text{ if } \sup I \in I;$$

(ii) $f(x) = \omega x + y_0$ *for* $x \in I \setminus \{\inf I\}$, *where* $y_0 \in \mathbb{R}$ *and*
$$f(\inf I) \leqslant \omega \inf I + y_0 \text{ if } \inf I \in I;$$

(iii) $f(x) = \omega|x - x_0| + y_0$ *for* $x \in I$, *where* $x_0 \in \operatorname{int} I,\ y_0 \in \mathbb{R}$.

Proof. It follows from Theorems 2.1 and 3.2 that the functions of the above forms are ω-quasiaffine.

Assume now that $f : I \to \mathbb{R}$ is ω-quasiaffine. Then by Corollary 2.1, either $f|_{I\setminus\{supI\}} \in \searrow(\omega)$ or $f|_{I\setminus\{infI\}} \in \nearrow(\omega)$ or there exists an $x_0 \in \int I$ such that $f|_{I\cap(-\infty,x_0]} \in \searrow(\omega)$ and $f|_{I\cap(x_0,\infty)} \in \nearrow(\omega)$ or $f|_{I\cap(-\infty,x_0)} \in \searrow(\omega)$ and $f|_{I\cap[x_0,\infty)} \in \nearrow(\omega)$.

In the first case by Theorem 3.2 we obtain that

$$\frac{f(x)-f(y)}{x-y} = -\omega \qquad \text{for } x,y \in \operatorname{int} I,\ x \neq y,$$

which implies that there exists a $y_0 \in \mathbb{R}$ such that

$$f(x) = -\omega x + y_0 \qquad \text{for } x \in \operatorname{int} I.$$

Since $f|_{I\setminus\{\sup I\}}$ is nonincreasing we obtain from Theorem 3.2 that if $\inf I \in I$ then

$$f(\inf I) = \lim_{x\to\inf I} f(x) = -\omega \inf I + y_0.$$

Furthermore if $\sup I \in I$ then by Theorem 3.2 we have

$$f(\sup I) \leqslant \lim_{x\to\sup I} f(x) = -\omega x + y_0.$$

By the similar reasoning in the second case we obtain that f has the form (ii).

Consider now the third case. By the same argumentation as in the first and second case we obtain that there exist $y_1, y_2 \in \mathbb{R}$ such that

$$\begin{aligned} f(x) &= -\omega x + y_1 \qquad \text{for } x \in I \cap (-\infty, x_0), \\ f(x) &= \omega x + y_2 \qquad \text{for } x \in I \cap (x_0, \infty). \end{aligned}$$

Since f is continuous on $\operatorname{int} I$, the above conditions implies that f is of the form (iii). $\square$

As the direct corollary from Theorem 3.2 we obtain the following result.

COROLLARY 3.1. *If a function $f : I \to \mathbb{R}$ is locally ω-quasiconcave then it is ω-quasiconcave.*

As the direct consequence of Theorem 2.2 and Corollary 3.1 we get analogous result for ω-quasiaffinity.

COROLLARY 3.2. *If a function $f : I \to \mathbb{R}$ is locally ω-quasiaffine then it is ω-quasiaffine.*

4. Separation

Now we prove "sandwich" type theorem. Such the result is characteristic for convex functions.

THEOREM 4.1. *Let $f : I \to \mathbb{R}$ be ω-quasiconvex, $g : I \to \mathbb{R}$ ω-quasiconcave, and let*

$$g(x) \leqslant f(x) \qquad \text{for } x \in I. \tag{9}$$

Then there exists an ω-quasiaffine function $h : I \to \mathbb{R}$ which separates f and g, i.e.

$$f(x) \geqslant h(x) \geqslant g(x) \qquad \text{for } x \in I. \tag{10}$$

Proof. Consider first the case when f is of the form (iii) from Corollary 2.1. It means that there exists an $x_0 \in \operatorname{int} I$ such that $f|_{I\cap(-\infty,x_0)} \in \searrow(\omega)$, $f|_{I\cap(x_0,\infty)} \in \nearrow(\omega)$. Then there exist the limits:

$$\lim_{x\to x_0^-} f(x), \quad \lim_{x\to x_0^+} f(x).$$

In view of (9) we have

$$g(x_0) \leqslant \lim_{x\to x_0^-} f(x) \text{ and } g(x_0) \leqslant \lim_{x\to x_0^+} f(x).$$

We put

$$y_0 := \min(\lim_{x\to x_0^-} f(x), f(x_0), \lim_{x\to x_0^+} f(x)).$$

Then

$$g(x_0) \leqslant y_0 \tag{11}$$

and

$$f(x) \geqslant y_0 \qquad \text{for } x \in I. \tag{12}$$

We define

$$h(x) := \omega|x - x_0| + y_0 \qquad \text{for } x \in I.$$

By Theorem 3.3 the function h is ω-quasiaffine. By (11) and (12) we have

$$g(x_0) \leqslant y_0 = h(x_0) \leqslant f(x_0). \tag{13}$$

Since $f|_{I\cap(x_0,\infty)} \in \nearrow(\omega)$ and g is Lipschitz with the constant ω, in virtue of (13) we obtain that

$$g(x) \leqslant h(x) \leqslant f(x) \qquad \text{for } x \in I \cap [x_0, \infty).$$

Similarly, since $f|_{I\cap(-\infty,x_0)} \in \searrow(\omega)$ and g is Lipschitz with the constant ω, in view of (13) we get

$$g(x) \leqslant h(x) \leqslant f(x) \qquad \text{for } x \in I \cap (-\infty, x_0].$$

We have proved (10).

Now we assume that f is of the form (ii) from Corollary 2.1, i.e. that $f|_{I\setminus\{\inf\, I\}} \in \nearrow(\omega)$. We are going to prove that

$$\sup_{x\in \text{int}\, I} [g(x) - \omega x] \leqslant \inf_{x\in \text{int}\, I} [f(x) - \omega x]. \tag{14}$$

Let $a := \inf I$. Since $f|_{I\setminus\{a\}} \in \nearrow(\omega)$, the function $I \setminus \{a\} \ni x \mapsto f(x) - \omega x$ is nondecreasing. It follows from Theorem 3.2 that the function $g|_{\text{int}\, I}$ is Lipschitz with the constant ω . Therefore we have for $x, y \in \text{int}\, I$, $x < y$

$$\omega(x - y) \leqslant g(x) - g(y)$$

and consequently that

$$g(x) - \omega x \geqslant g(y) - \omega y,$$

which means that the function $\text{int}\, I \ni x \mapsto g(x) - \omega x$ is nonincreasing. Hence there exist the limits

$$\lim_{x\to a^+} [f(x) - \omega x], \quad \lim_{x\to a^+} [g(x) - \omega x]$$

and

$$\lim_{x\to a^+} [f(x) - \omega x] = \inf_{x\in \text{int}\, I} [f(x) - \omega x], \tag{15}$$

$$\lim_{x\to a^+} [g(x) - \omega x] = \sup_{x\in \text{int}\, I} [f(x) - \omega x]. \tag{16}$$

Obviously we have

$$g(x) - \omega x \leqslant f(x) - \omega x \qquad \text{for } x \in I.$$

From this inequality, (15) and (16) we obtain (14).

We fix an arbitrary $y_0 \in \mathbb{R}$ such that

$$\sup_{x\in \text{int } I} [g(x) - \omega x] \leqslant y_0 \leqslant \inf_{x\in \text{int } I} [f(x) - \omega x]. \tag{17}$$

The existence of such y_0 is guaranteed by (14). We put

$$g(x) = \begin{cases} \omega x + y_0 & \text{for } x \in I \setminus \{a\} \\ g(a) & \text{for } x = a \text{ if } a \in I \cap \mathbb{R}. \end{cases} \tag{18}$$

Assume that $a \in I \cap \mathbb{R}$. Making use of (17), (16) and next Theorem 3.2 we obtain that

$$y_0 \geqslant \lim_{x\to a^+} [g(x) - \omega x] = \lim_{x\to a^+} g(x) - \omega a \geqslant g(a) - \omega a.$$

Whence we have

$$h(a) = g(a) \leqslant \omega a + y_0. \tag{19}$$

In view of Theorem 3.3 this together with (18) mean that h is ω-quasiaffine.

Now we prove that

$$g(x) \leqslant h(x) \leqslant f(x) \qquad \text{for } x \in I. \tag{20}$$

For $x \in \text{int } I$ it follows directly from (17) and (18).

Assume that $a \in I \cap \mathbb{R}$. Since

$$g(x) \leqslant f(x) \qquad \text{for } x \in I,$$

in view of (19) we have

$$g(a) = h(a) \leqslant f(a).$$

It remains to consider the case if $\sup I \in I \cap \mathbb{R}$. We have to prove that then

$$g(\sup I) \leqslant \omega \sup I + y_0 \leqslant f(\sup I). \tag{21}$$

It follows from Theorem 3.2 that

$$g(\sup I) \leqslant \lim_{x\to \sup I} g(x). \tag{22}$$

Since the function $\text{int } I \ni x \mapsto g(x) - \omega x$ is nonincreasing, we obtain from (17)

$$\lim_{x\to \sup I} [g(x) - \omega x] \leqslant \sup_{x\in \text{int } I} [g(x) - \omega x] \leqslant y_0,$$

i.e.

$$\lim_{x\to \sup I} g(x) \leqslant \omega \sup I + y_0.$$

This inequality and (22) yields the first inequality in (21). Now we prove the second one. Since the function $I \setminus \{a\} \ni x \mapsto f(x) - \omega x$ is nondecreasing we obtain from (17)

$$\begin{aligned} y_0 &\leqslant \inf_{x\in \text{int } I} [f(x) - \omega x] \leqslant \lim_{x\to \sup I} [f(x) - \omega x] \\ &= \lim_{x\to \sup I} f(x) - \omega \sup I \leqslant f(\sup I) - \omega \sup I. \end{aligned}$$

Hence

$$\omega \sup I + y_0 \leqslant f(\sup I).$$

We have proved that the function h defined by (18) is ω-affine and that it separates f and g.

In the case when f is of the form (i) from Theorem 2.1 we can get the assertion by a similar reasoning. We can also reduce this case to the previous one by applying the substitution $-I \ni x \mapsto -x$. □

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