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A NOTE ON THE ESTIMATE OF THE BETA DISTRIBUTION

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Abstract. The lower and the upper estimates with explicit coefficients for the beta distribution with $a > 1, b > 1$ are given. Furthermore, using these results, the lower and the upper estimates of the beta distribution of the second kind and F-distribution, and also partial estimates of Student's t-distribution are obtained.

1. Introduction

It is well known that the beta distribution provides the premier family of continuous distributions on bounded support. Recently, attempts have been made to construct new distributions based on the beta distribution. Variety of distributions generated from the beta distribution have extended the original beta distribution. There are many beta-generated families of distributions discussed in the literature. For example, in the paper of Eugene et al. [2], the beta-normal distribution was introduced and its properties were studied. In Nadarajah and Kotz [9], the beta-exponential distribution has been studied wherein the authors gave the detailed presentation. Zografos and Balakrishnan [11] discussed in another point of view that they concerned the entropy characterizations and related properties. On the basis of previous work about the beta-Weibull distribution, Giovana et al. [3] considered the beta modified Weibull distribution with five parameters. The beta-Burr XII distribution has been studied by Patrícia et al. [10], they derived moment generating function, the estimation of parameters and so on. Gittins and Maher [4] studied the incomplete beta function and gave its applications. Kenneth and Lauren [8] concentrated on folded beta random variable.

The beta distribution has received great attention for a long time. However, we find that there are very few estimates with explicit coefficients for the beta distribution in the literature. This is the motivation of our work in this paper.

Returning to our problem, in this paper, the beta ratio is defined as

$$J(s) = \frac{\int_s^1 h(t, a_1, b_1)dt}{h(s, a_1, b_1)}, \tag{1.1}$$

where

$$h(s, a_1, b_1) = \frac{\Gamma(a_1 + b_1)}{\Gamma(a_1)\Gamma(b_1)} s^{a_1-1}(1-s)^{b_1-1}$$

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with $a_1 > 1$, $b_1 > 1$ and $0 < s < 1$, Γ is the usual gamma function. It represents the beta probability beyond a certain point divided by the beta density at that point. (1.1) plays an important role in practice. In insurance, (1.1) is the reciprocal of hazard function where $h(s,a_1,b_1)$ is a probability density function and $\int_s^1 h(t,a_1,b_1)dt$ is a survivor function. We concern the asymptotic of $J(s)$ as $s \to 1$. In another way, by integral transformation, we have

$$J(s) = H(x) = \frac{\int_0^x h(t,a,b)dt}{h(x,a,b)}, \tag{1.2}$$

where $x = 1-s$, $a = b_1$ and $b = a_1$. From (1.2), it is easy to see that the asymptotic of $J(s)$ as $s \to 1$ is equivalent to the asymptotic of $H(x)$ as $x \to 0$. So we only concentrate on $H(x)$ throughout this paper. Using variable substitution, we know that the beta distribution of the second kind, F-distribution and Student's t-distribution have a close connection with the beta distribution. We can apply our results for the beta distribution to these three distributions.

The rest of this paper is arranged as follows. In Section 2, for giving the main result of this paper, some useful propositions are provided. In Section 3, some lower and upper estimates with explicit coefficients for the beta ratio are obtained. In Section 4, in view of the relations among the beta distribution of the second kind, F-distribution and Student's t-distribution, some estimates of these three distributions are given.

2. Useful propositions for the beta ratio

Throughout the paper, the set of natural numbers is denoted by $\mathbb{N}$, the set of positive integers is denoted by $\mathbb{Z}^+$.

From (1.2), the beta ratio is denoted again by

$$H(x) = \frac{\int_0^x t^{a-1}(1-t)^{b-1}dt}{x^{a-1}(1-x)^{b-1}}. \tag{2.1}$$

For the sake of brevity, let

$$f(x) = x^{a-1}(1-x)^{b-1}, \quad g(x) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad \prod_{i=1}^{0} \stackrel{def}{=} 1. \tag{2.2}$$

If $b \in \mathbb{Z}^+$, it is easy to see

$$f(x) = x^{a-1}(1-x)^{b-1} = \sum_{i=0}^{b-1}(-1)^i \frac{\prod_{j=1}^{i}(b-j)}{i!} x^{a+i-1},$$

$$g(x) = \int_0^x t^{a-1}(1-t)^{b-1}dt = \sum_{i=0}^{b-1}(-1)^i \frac{\prod_{j=1}^{i}(b-j)}{i!(a+i)} x^{a+i}.$$

Plugging $f(x)$ and $g(x)$ into (2.1) yields

$$H(x) = \frac{\sum_{i=0}^{b-1} (-1)^i \frac{\prod_{j=1}^{i}(b-j)}{i!(a+i)} x^{a+i}}{\sum_{i=0}^{b-1} (-1)^i \frac{\prod_{j=1}^{i}(b-j)}{i!} x^{a+i-1}}. \tag{2.3}$$

If $b \notin \mathbb{Z}^+$, by power series expansion and Riemann integral, it is also easy to obtain

$$f(x) = \sum_{i=0}^{\infty} f_i(x), \quad g(x) = \sum_{i=0}^{\infty} g_i(x), \tag{2.4}$$

where

$$f_i(x) = (-1)^i \frac{\prod_{j=1}^{i}(b-j)}{i!} x^{a+i-1}, \quad g_i(x) = (-1)^i \frac{\prod_{j=1}^{i}(b-j)}{i!(a+i)} x^{a+i}. \tag{2.5}$$

From the expression above, it is easy to see that the signs of $g_i(x)$ and $f_i(x)$ alternate when i is small. However, $g_i(x)$ and $f_i(x)$ are always negative or positive for a large i. Details are given in the following proposition.

PROPOSITION 2.1. *Let $g_i(x)$ and $f_i(x)$ be defined as above, $b \notin \mathbb{Z}^+$ and $i \in \mathbb{N}$. For $2k_1+1 < b < 2k_1+2$, $k_1 \in \mathbb{N}$, if $i \leqslant 2k_1$ and an even number i, we have*

$$g_i(x) > 0 \quad \text{and} \quad f_i(x) > 0. \tag{2.6}$$

If $i \leqslant 2k_1$ and i is an odd number, we have

$$g_i(x) < 0 \quad \text{and} \quad f_i(x) < 0. \tag{2.7}$$

If $i \geqslant 2k_1+1$, we have

$$g_i(x) < 0 \quad \text{and} \quad f_i(x) < 0. \tag{2.8}$$

Similarly, for $2k_2 < b < 2k_2+1$, $k_2 \in \mathbb{Z}^+$, if $i \leqslant 2k_2-1$ and i is an even number, we have

$$g_i(x) > 0 \quad \text{and} \quad f_i(x) > 0. \tag{2.9}$$

If $i \leqslant 2k_2-1$ and i is an odd number, we have

$$g_i(x) < 0 \quad \text{and} \quad f_i(x) < 0. \tag{2.10}$$

If $i \geqslant 2k_2$, we have

$$g_i(x) > 0 \quad \text{and} \quad f_i(x) > 0. \tag{2.11}$$

Proof. The results are obvious and we omit the details of the proof. □

In fact, we can get more detailed results for $g_i(x)$ and $f_i(x)$. Next we give some properties of $g_i(x)$.

PROPOSITION 2.2. *For $2k_1+1<b<2k_1+2,\ k_1\in\mathbb{N}$. If $x<\frac{1}{b-1}$, we have*

$$\left|\frac{g_{i+1}(x)}{g_i(x)}\right|<1, \quad i=0,1,\ldots,2k_1. \tag{2.12}$$

If $x<\frac{2k_1+1}{b}$, we have

$$g_{2k_1}(x)+\sum_{i=2k_1+1}^{\infty} g_i(x)>0. \tag{2.13}$$

For $2k_2<b<2k_2+1,\ k_2\in\mathbb{Z}^+$. If $x<\frac{1}{b-1}$, we have

$$\left|\frac{g_{i+1}(x)}{g_i(x)}\right|<1, \quad i=0,1,\ldots,2k_2-1. \tag{2.14}$$

If $x<\frac{2k_2}{b}$, we have

$$g_{2k_2-1}(x)+\sum_{i=2k_2}^{\infty} g_i(x)<0. \tag{2.15}$$

Proof. First, let $2k_1+1<b<2k_1+2,\ k_1\in\mathbb{N}$. If $i\leqslant 2k_1$, we have $i<b-1$. Using the expression for $g_i(x)$ in (2.5), we have

$$\left|\frac{g_{i+1}(x)}{g_i(x)}\right|=x\cdot\frac{(b-i-1)(a+i)}{(i+1)(a+i+1)}. \tag{2.16}$$

Plugging $x<\frac{1}{b-1}$ into (2.16), we have

$$\left|\frac{g_{i+1}(x)}{g_i(x)}\right|<\frac{b-i-1}{b-1}\frac{1}{i+1}\frac{a+i}{a+i+1}. \tag{2.17}$$

By assumption, it is easy to verify that

$$\frac{b-i-1}{b-1}\leqslant 1, \quad \frac{1}{i+1}\leqslant 1 \quad \textit{and} \quad \frac{a+i}{a+i+1}<1.$$

Then we obtain (2.12).

Next, using (2.5) again, we have

$$\begin{aligned} &g_{2k_1}(x)+\sum_{i=2k_1+1}^{\infty} g_i(x)\\ &=\frac{\prod_{j=1}^{2k_1}(b-j)}{(2k_1)!(a+2k_1)}x^{a+2k_1}-\frac{\prod_{j=1}^{2k_1+1}(b-j)}{(2k_1+1)!(a+2k_1+1)}x^{a+2k_1+1}\\ &\quad\times\left(1+\sum_{s=2k_1+2}^{\infty}\left(\prod_{j=2k_1+2}^{s}\frac{j-b}{j}\quad\right)\frac{a+2k_1+1}{a+s}x^{s-2k_1-1}\right). \end{aligned} \tag{2.18}$$

Note that, for $j \geqslant 2k_1+2$ and $s \geqslant 2k_1+2$, we have

$$\frac{j-b}{j} < 1 \quad \textit{and} \quad \frac{a+2k_1+1}{a+s} < 1.$$

For the last sum in (2.18), we have

$$1+\sum_{s=2k_1+2}^{\infty}\left(\prod_{j=2k_1+2}^{s}\frac{j-b}{j}\right)\frac{a+2k_1+1}{a+s}x^{s-2k_1-1} < \sum_{m=0}^{\infty}x^m. \tag{2.19}$$

Plugging (2.19) into (2.18), we have

$$\begin{aligned} &g_{2k_1}(x)+\sum_{i=2k_1+1}^{\infty}g_i(x)\\ &> \frac{\prod_{j=1}^{2k_1}(b-j)}{(2k_1)!(a+2k_1)}x^{a+2k_1} - \frac{\prod_{j=1}^{2k_1+1}(b-j)}{(2k_1+1)!(a+2k_1+1)}x^{a+2k_1+1}\times\sum_{m=0}^{\infty}x^m\\ &= \frac{\prod_{j=1}^{2k_1}(b-j)}{(2k_1)!(a+2k_1)}x^{a+2k_1}\left(1-\frac{a+2k_1}{a+2k_1+1}\frac{(b-2k_1-1)x}{(2k_1+1)(1-x)}\right), \end{aligned} \tag{2.20}$$

where we use the fact $\sum_{m=0}^{\infty}x^m = 1/(1-x)$ for $0<x<1$. Since $a>1$, we have $0<\frac{a+2k_1}{a+2k_1+1}<1$. This implies

$$1-\frac{a+2k_1}{a+2k_1+1}\frac{(b-2k_1-1)x}{(2k_1+1)(1-x)} > 1-\frac{(b-2k_1-1)x}{(2k_1+1)(1-x)} > 0, \tag{2.21}$$

where we use the assumption $x<(2k_1+1)/b$ in the last inequality in (2.21).

Hence, combining (2.20) and (2.21), we obtain (2.13).

In the case $2k_2<b<2k_2+1$, $k_2\in\mathbb{Z}^+$, using the same argument, we also obtain (2.14) and (2.15). □

Combing Proposition 2.1 and Proposition 2.2, the following results follow easily. We denote $I_n(x)=\sum_{i=0}^{n}g_i(x)$, $n\in\mathbb{N}$.

PROPOSITION 2.3. *For $1<b<2$ and any $n\in\mathbb{N}$, we have*

$$g(x)<I_n(x). \tag{2.22}$$

For $2k_1+1<b<2k_1+2$, $k_1\in\mathbb{Z}^+$, if $x<\frac{1}{b-1}$, we have

$$I_1(x)<I_3(x)<\cdots<I_{2k_1-1}(x)<g(x)<\cdots<I_{2k_1+1}(x)<I_{2k_1}(x)<\cdots<I_2(x)<I_0(x). \tag{2.23}$$

For $2k_2<b<2k_2+1$, $k_2\in\mathbb{Z}^+$, if $x<\frac{1}{b-1}$, we have

$$I_1(x)<I_3(x)<\cdots<I_{2k_2-1}(x)<I_{2k_2}(x)<\cdots<g(x)<I_{2k_2-2}(x)<\cdots<I_2(x)<I_0(x). \tag{2.24}$$

Proof. In the case $1 < b < 2$, from (2.8), we immediately obtain $g_i(x) < 0$ as $i \geqslant 1$. Hence, we obtain (2.22).

In the case $2k_1+1 < b < 2k_1+2$, $k_1 \in \mathbb{Z}^+$, we can easily get $\frac{1}{b-1} < \frac{2k_1+1}{b}$ by assumption. Hence, (2.12) and (2.13) follow if $x < \frac{1}{b-1}$. From (2.13), we have

$$I_{2k_1-1}(x) < g(x). \tag{2.25}$$

From (2.8), we have

$$g(x) < \cdots < I_{2k_1+1}(x) < I_{2k_1}(x). \tag{2.26}$$

From (2.12), (2.6) and (2.7), we have

$$I_1(x) < I_3(x) < \cdots < I_{2k_1-1}(x) \quad \textit{and} \quad I_{2k_1}(x) < \cdots < I_2(x) < I_0(x). \tag{2.27}$$

Combining the results above, we complete the proof of (2.23).

It is easy to verify (2.24) by the same argument and we omit the proof. □

In addition, using the similar argument, we have the following conclusion. We denote $R_n(x) = \sum_{i=0}^{n} f_i(x)$, $n \in \mathbb{N}$.

PROPOSITION 2.4. *For $1 < b < 2$ and any $n \in \mathbb{N}$, we have*

$$f(x) < R_n(x). \tag{2.28}$$

For $2k_1+1 < b < 2k_1+2$, $k_1 \in \mathbb{Z}^+$, if $x < \frac{1}{b-1}$, we have

$$R_1(x)<R_3(x)<\cdots<R_{2k_1-1}(x)<f(x)<\cdots<R_{2k_1+1}(x)<R_{2k_1}(x)<\cdots<R_2(x)<R_0(x). \tag{2.29}$$

For $2k_2 < b < 2k_2+1$, $k_2 \in \mathbb{Z}^+$, if $x < \frac{1}{b-1}$, we have

$$R_1(x)<R_3(x)<\cdots<R_{2k_2-1}(x)<R_{2k_2}(x)<\cdots<f(x)<R_{2k_2-2}(x)<\cdots<R_2(x)<R_0(x). \tag{2.30}$$

REMARK. From the proofs, it is easy to see that Propositions 2.1-2.4 are also valid for $0 < a \leqslant 1$. However, when giving Theorem 3.1 and 3.2 in Section 3, we only consider the case $a > 1$.

3. Asymptotic estimates for the beta ratio

Propositions 2.1-2.4 are the key steps for Theorem 3.1 and Theorem 3.2. In this section, we give our main results. We denote

$$\widetilde{I}_n(x) = \sum_{i=0}^{n} \tilde{g}_i(x) = \sum_{i=0}^{n} (-1)^i \frac{\prod_{j=1}^{i}(\tilde{b}-j)}{i!(\tilde{a}+i)} x^{\tilde{a}+i},$$

$$\overline{R}_n(x) = \sum_{i=0}^{n} \overline{f}_i(x) = \sum_{i=0}^{n} (-1)^i \frac{\prod_{j=1}^{i}(\overline{b}-j)}{i!} x^{\overline{a}+i-1},$$

$n \in \mathbb{N}$. The expression for $\tilde{g}_i(x)$ is the same as for $g_i(x)$ after substituting the explicit coefficients of $g_i(x)$ with $\tilde{a} = a-1, \tilde{b} = b+1$. The expression for $\overline{f}_i(x)$ is the same as for $f_i(x)$ after substituting the explicit coefficients of $f_i(x)$ with $\overline{a} = a, \overline{b} = b+1$.

THEOREM 3.1. *Let s be any positive integer. For $2k_1+1<b<2k_1+2,\ k_1\in\mathbb{N}$, if $x<\frac{1}{b}$, we have*

$$-\frac{x^{a-1}(1-x)^b}{b}+\frac{a-1}{b}\widetilde{I}_{2k_1+s}(x)<g(x)<I_{2k_1+s}(x). \tag{3.1}$$

For $2k_2<b<2k_2+1,\ k_2\in\mathbb{Z}^+$, if $x<\frac{1}{b}$, we have

$$I_{2k_2+s}(x)<g(x)<-\frac{x^{a-1}(1-x)^b}{b}+\frac{a-1}{b}\widetilde{I}_{2k_2+s}(x). \tag{3.2}$$

Proof. Note that $x<\frac{1}{b}<\frac{1}{b-1}$. For $2k_1+1<b<2k_1+2,\ k_1\in\mathbb{Z}^+$, from (2.23), we know that $I_{2k_1+s}(x)$ is strictly decreasing with respect to s and converges to $g(x)$ as $s\to\infty$.

On the other hand, using integration by parts, we get

$$\int_0^x t^{a-1}(1-t)^{b-1}dt=-\frac{x^{a-1}(1-x)^b}{b}+\frac{a-1}{b}\int_0^x t^{a-2}(1-t)^b dt, \tag{3.3}$$

where $\int_0^x t^{a-2}(1-t)^b dt=\sum_{i=0}^{\infty}\tilde{g}_i(x)$ and the explicit coefficients of $\tilde{g}_i(x)$ are $\tilde{a}=a-1,\tilde{b}=b+1$. We can prove (2.24) is also correct for $0<a<1$. $2k_1+2<\tilde{b}=b+1<2k_1+3$ and $x<\frac{1}{b}=\frac{1}{\tilde{b}-1}$, this satisfies the condition of (2.24). Then, by (2.24), we complete (3.1).

Using the same argument, we also obtain the results under the case $1<b<2$ and $2k_2<b<2k_2+1$. □

Moreover, we can also give some estimates of the beta ratio $H(x)$.

THEOREM 3.2. *Let s be any positive integer. For $2k_1+1<b<2k_1+2,\ k_1\in\mathbb{N}$, if $x<\frac{1}{b}$, we have*

$$-\frac{1-x}{b}+\frac{a-1}{b}\frac{\widetilde{I}_{2k_1+s}(x)}{R_{2k_1+s}(x)}<H(x)<\frac{(1-x)I_{2k_1+s}(x)}{\overline{R}_{2k_1+s}(x)}. \tag{3.4}$$

For $2k_2<b<2k_2+1,\ k_2\in\mathbb{Z}^+$, if $x<\frac{1}{b}$, we have

$$\frac{(1-x)I_{2k_2+s}(x)}{\overline{R}_{2k_2+s}(x)}<H(x)<-\frac{1-x}{b}+\frac{a-1}{b}\frac{\widetilde{I}_{2k_2+s}(x)}{R_{2k_2+s}(x)}. \tag{3.5}$$

Proof. From (2.1) and (3.1), for $2k_1+1<b<2k_1+2,\ k_1\in\mathbb{N},\ x<\frac{1}{b}$, we have

$$-\frac{1-x}{b}+\frac{a-1}{b}\frac{\widetilde{I}_{2k_1+s}(x)}{f(x)}<H(x)<\frac{I_{2k_1+s}(x)}{f(x)}. \tag{3.6}$$

For the first inequality in (3.6), in view of $x<\frac{1}{b}<\frac{1}{b-1}$, from (2.28) and (2.29), we have

$$-\frac{1-x}{b}+\frac{a-1}{b}\frac{\widetilde{I}_{2k_1+s}(x)}{R_{2k_1+s}(x)}<H(x). \tag{3.7}$$

For the second inequality in (3.6), we know that

$$\frac{I_{2k_1+s}(x)}{f(x)} = \frac{(1-x)I_{2k_1+s}(x)}{x^{\bar{a}-1}(1-x)^{\bar{b}-1}}. \tag{3.8}$$

From $2k_1+1 < b < 2k_1+2$, $x < \frac{1}{b}$, we have $2k_1+2 < \bar{b} < 2k_1+3$, $x < \frac{1}{b} = \frac{1}{\bar{b}-1}$. Using (2.30), we have

$$H(x) < \frac{(1-x)I_{2k_1+s}(x)}{\overline{R}_{2k_1+s}(x)}. \tag{3.9}$$

Combining (3.7) and (3.9), we obtain (3.4).

Using the same argument, we can get the result for the case $2k_2 < b < 2k_2+1$, $k_2 \in \mathbb{Z}^+$. □

4. Application in the beta distribution of the second kind, F-distribution and Student's t-distribution

In this section, applying Theorem 3.1 and 3.2 into the beta distribution of the second kind, F-distribution and Student's t-distribution, some estimates of these three distributions are obtained. First, we introduce the definitions of the beta distribution of the second kind as follows.

The probability density function of the beta distribution of the second kind is defined by

$$f_{II}(y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{y^{a-1}}{(1+y)^{a+b}}, \quad a>1, b>1, y>0.$$

Similarly to (1.2), we have

$$H_{sec}(x) = \frac{\int_0^x f_{II}(t)dt}{f_{II}(x)} = \frac{g_{sec}(x)}{f_{sec}(x)},$$

where

$$g_{sec}(x) = \int_0^x \frac{t^{a-1}}{(1+t)^{a+b}}dt = g\left(\frac{x}{1+x}\right),$$

$$f_{sec}(x) = \frac{x^{a-1}}{(1+x)^{a+b}} = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x}\right).$$

Furthermore, in view of the relation between (2.1) and (2.2), we also have

$$H_{sec}(x) = (1+x)^2 H\left(\frac{x}{1+x}\right).$$

Thus, using Theorem 3.1 and Theorem 3.2, we have

COROLLARY 4.1. *Let s be any positive integer. For $2k_1+1 < b < 2k_1+2$, $k_1 \in \mathbb{N}$, if $x < \frac{1}{b-1}$, we have*

$$-\frac{x^{a-1}}{b(1+x)^{a+b-1}} + \frac{a-1}{b}\widetilde{I}_{2k_1+s}\left(\frac{x}{1+x}\right) < g_{sec}(x) < I_{2k_1+s}\left(\frac{x}{1+x}\right).$$

For $2k_2 < b < 2k_2+1,\ k_2 \in \mathbb{Z}^+$, *if* $x < \frac{1}{b-1}$, *we have*

$$I_{2k_2+s}\left(\frac{x}{1+x}\right) < g_{sec}(x) < -\frac{x^{a-1}}{b(1+x)^{a+b-1}} + \frac{a-1}{b}\widetilde{I}_{2k_2+s}\left(\frac{x}{1+x}\right).$$

Moreover, we can also give some estimates of $H_{sec}(x)$.

COROLLARY 4.2. *Let* s *be any positive integer. For* $2k_1+1 < b < 2k_1+2,\ k_1 \in \mathbb{N}$, *if* $x < \frac{1}{b-1}$, *we have*

$$-\frac{1+x}{b} + \frac{(a-1)(1+x)^2}{b}\frac{\widetilde{I}_{2k_1+s}(\frac{x}{1+x})}{R_{2k_1+s}(\frac{x}{1+x})} < H_{sec}(x) < \frac{(1+x)I_{2k_1+s}(\frac{x}{1+x})}{\overline{R}_{2k_1+s}(\frac{x}{1+x})}.$$

For $2k_2 < b < 2k_2+1,\ k_2 \in \mathbb{Z}^+$, *if* $x < \frac{1}{b-1}$, *we have*

$$\frac{(1+x)I_{2k_2+s}(\frac{x}{1+x})}{\overline{R}_{2k_2+s}(\frac{x}{1+x})} < H_{sec}(x) < -\frac{1+x}{b} + \frac{(a-1)(1+x)^2}{b}\frac{\widetilde{I}_{2k_2+s}(\frac{x}{1+x})}{R_{2k_2+s}(\frac{x}{1+x})}.$$

Secondly, we introduce the definitions of F-distribution and Student's t-distribution as follows.

The probability density function of F-distribution and Student's t-distribution are defined by

$$f_F(x) = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} m^{\frac{m}{2}} n^{\frac{n}{2}} x^{\frac{m}{2}-1}(n+mx)^{-\frac{m+n}{2}}, \quad x > 0$$

and

$$f_t(x) = \frac{\Gamma(\frac{d+1}{2})}{\sqrt{d\pi}\,\Gamma(\frac{d}{2})}\left(1+\frac{x^2}{d}\right)^{-\frac{d+1}{2}}, \quad -\infty < x < +\infty,$$

where $m > 0$, $n > 0$ and $d > 0$. In this paper, F ratio is defined as

$$r_F(x) = \frac{\int_0^x f_F(t)dt}{f_F(x)} = \frac{\int_0^x t^{\frac{m}{2}-1}(1+\frac{m}{n}t)^{-\frac{m+n}{2}}dt}{x^{\frac{m}{2}-1}(1+\frac{m}{n}x)^{-\frac{m+n}{2}}}, \tag{4.1}$$

and we define

$$P_t(0 < X < x) = \int_0^x f_t(t)dt.$$

In addition, before applying Theorem 3.2, we introduce a useful Lemma.

LEMMA 4.1. *Let* $H(x)$ *and* $r_F(x)$ *be the beta ratio and F ratio, respectively. Then,*

$$H(x) = \frac{\int_0^x t^{a-1}(1-t)^{b-1}dt}{x^{a-1}(1-x)^{b-1}} = \frac{2\int_0^\alpha \sin^{2a-1}\theta\cos^{2b-1}\theta d\theta}{\sin^{2a-2}\alpha\cos^{2b-2}\alpha}, \tag{4.2}$$

where $x = \sin^2\alpha$.

$$r_F(x) = \frac{\int_0^x t^{\frac{m}{2}-1}(1+\frac{m}{n}t)^{-\frac{m+n}{2}}dt}{x^{\frac{m}{2}-1}(1+\frac{m}{n}x)^{-\frac{m+n}{2}}} = \frac{2n\int_0^{\alpha_1}\sin^{m-1}\theta\cos^{n-1}\theta d\theta}{m\sin^{m-2}\alpha_1\cos^{n+2}\alpha_1}, \tag{4.3}$$

where $x = \frac{n}{m}\tan^2\alpha_1$. *Furthermore,*

$$P_t(0<X<x) = \frac{\Gamma(\frac{d+1}{2})}{\sqrt{d\pi}\Gamma(\frac{d}{2})}\int_0^x\left(1+\frac{t^2}{d}\right)^{-\frac{d+1}{2}}dt = \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2})}\int_0^{\alpha_2}\cos^{d-1}\theta d\theta, \quad (4.4)$$

where $x = \sqrt{d}\tan\alpha_2$.

Proof. For $H(x)$, using variable substitution $t = \sin^2\theta$ and the fact $(\sin^2\theta)' = 2\sin\theta\cos\theta$, we obtain (4.2).

For $r_F(x)$, let $t = \frac{n}{m}\tan^2\theta$, using variable substitution again, we obtain (4.3).

For $P_t(0<X<x)$, we can get (4.4) by using $t = \sqrt{d}\tan\theta$. □

It is easily seen that $H(x)$ and $r_F(x)$ have similar expression. Hence, we can apply Theorem 3.2 to $r_F(x)$.

COROLLARY 4.3. *Let* $r_F(x)$ *be F ratio,* $m>2$ *and* $n>2$. *Then we have*

$$r_F(x) = \frac{n}{m}\left(1+\frac{m}{n}x\right)^2 H\left(\frac{mx}{n+mx}\right), \quad (4.5)$$

where the explicit coefficients of $H(x)$ *are* $a = \frac{m}{2}$, $b = \frac{n}{2}$.

For $4k_1+2<n<4k_1+4$, $k_1\in\mathbb{N}$, *if* $x<\frac{2n}{m(n-2)}$, $H\left(\frac{mx}{n+mx}\right)$ *satisfies (3.4).*

For $4k_2<n<4k_2+2$, $k_2\in\mathbb{Z}^+$, *if* $x<\frac{2n}{m(n-2)}$, $H\left(\frac{mx}{n+mx}\right)$ *satisfies (3.5).*

Proof. Using $\cos^4\alpha_1 = \frac{1}{(1+\tan^2\alpha_1)^2}$ and $\tan^2\alpha_1 = \frac{mx}{n}$, from Lemma 4.1 we get

$$r_F(x) = \frac{n}{m}\left(1+\frac{m}{n}x\right)^2\frac{2\int_0^{\alpha_1}\sin^{m-1}\theta\cos^{n-1}\theta d\theta}{\sin^{m-2}\alpha_1\cos^{n-2}\alpha_1}. \quad (4.6)$$

Let $m-1 = 2a-1$ and $n-1 = 2b-1$, we get $a = \frac{m}{2}$, $b = \frac{n}{2}$. Using variable substitution, we obtain

$$r_F(x) = \frac{n}{m}\left(1+\frac{m}{n}x\right)^2 H\left(\frac{mx}{n+mx}\right). \quad (4.7)$$

From $2k_1+1<b<2k_1+2$, we have $4k_1+2<n<4k_1+4$. From $\frac{mx}{n+mx}<\frac{2}{n}$, we obtain $x<\frac{2n}{m(n-2)}$, then we can use inequality (3.4) in Theorem 3.2 to obtain the estimates.

For the other cases, the proof is similar, so we omit it. □

Next, we give the estimate of $P_t(0<X<x)$.

COROLLARY 4.4. *Let* $d>2$. *Then we have*

$$P_t(0<X<x) = \frac{\Gamma(\frac{d+1}{2})}{2\sqrt{\pi}\Gamma(\frac{d}{2})}g\left(\frac{x^2}{x^2+d}\right), \quad (4.8)$$

where the explicit coefficients of $g(x)$ *are* $a=\frac{1}{2}$, $b=\frac{d}{2}$.

For $2<d<4$, $g\left(\frac{x^2}{x^2+d}\right)$ *satisfies* (2.22).

For $4k_1+2<d<4k_1+4$, $k_1\in\mathbb{Z}^+$, *if* $x<\sqrt{\frac{2d}{d-4}}$, $g\left(\frac{x^2}{x^2+d}\right)$ *satisfies* (2.23).

For $4k_1<d<4k_1+2$, $k_1\in\mathbb{Z}^+$, *if* $x<\sqrt{\frac{2d}{d-4}}$, $g\left(\frac{x^2}{x^2+d}\right)$ *satisfies* (2.24).

Proof. First, we have

$$P_t(0<X<x)=\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2})}\int_0^{\alpha_2}\cos^{d-1}\theta d\theta. \tag{4.9}$$

Let $0=2a-1$ and $d-1=2b-1$, we get $a=\frac{1}{2}$, $b=\frac{d}{2}$. Using variable substitution, we obtain

$$P_t(0<X<x)=\frac{\Gamma\left(\frac{d+1}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}g\left(\frac{x^2}{x^2+d}\right), \tag{4.10}$$

where we use $\tan^2\alpha_2=\frac{x^2}{d}$.

From $2k_1+1<b<2k_1+2$, we have $4k_1+2<d<4k_1+4$. From $\frac{x^2}{x^2+d}<\frac{1}{b-1}$, we obtain $x<\sqrt{\frac{2d}{d-4}}$, then we can use inequality (2.23) in Proposition 2.3 to obtain the estimate of $P_t(0<X<x)$.

For the other two cases, we omit the proofs since they are similar. □

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